## Nikolay Gromov

Based on works with V.Kazakov, P.Vieira \& A.Kozak


## Integrability for the exact spectrum of AdSI

Facets of Integrability, CEA Saclay and ENS Paris 2009

## AdS/CFT correspondence

AdS/CFT duality:

$S=\frac{T}{2} \int \partial_{\mu} \vec{u} \cdot \partial^{\mu} \vec{u} d \sigma d \tau+$ fermions
String tension $\quad T=\frac{\sqrt{\lambda}}{4 \pi} \equiv g \quad$ 't Hooft coupling $\quad \lambda=g_{Y}^{2} M^{N}$

Anomalous dimensions $=$ spectrum of 2D integrable field theories
-S-matrix is well defined
-S-matrix factorizes - "only" 256 components

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$$
\begin{aligned}
& \text { Symmetry: } \quad \operatorname{psu}(2,2 \mid 4) \\
& \otimes-\bigcirc-\otimes-\bigcirc-\otimes-\otimes
\end{aligned}
$$

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## Integrability in N=4 SYM

$\mathrm{N}=4$ Super Yang-Mills $S=\frac{1}{g_{Y M}^{2}} \int d^{4} x \operatorname{tr}\left\{\frac{1}{2} F_{\mu \nu}^{2}+\left(D_{\mu} \Phi_{i}\right)^{2}-\frac{1}{2}\left[\Phi_{i}, \Phi_{j}\right]^{2}+\right.$ fermions $\}$
$\mathcal{O}_{i}(x)=\operatorname{tr} \Phi_{1} \Phi_{2} \Phi_{1} \Phi_{1} \Phi_{1} \Phi_{2} \Phi_{2} \Phi_{1} \Phi_{1} \Phi_{1}$

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$\mathcal{O}_{i}^{\text {rem }}=Z_{i j}(\Lambda) \mathcal{O}_{j}^{\text {bare }}$

$$
\Gamma=Z^{-1} \frac{d Z}{d \log \Lambda} \text { integrable Hamiltonian }
$$

## Integrability in N=4 SYM

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$\mathcal{O}_{i}(x)=\operatorname{tr}_{1} \Phi_{2} \Phi_{1} \Phi_{1} \Phi_{1} \Phi_{2} \Phi_{2} \Phi_{1} \Phi_{1} \Phi_{1}$

$\mathcal{O}_{i}^{\text {en }}=Z_{i j}(\Lambda) \mathcal{O}_{j}^{\text {bare }}$ At one loop:

$$
\Gamma=Z^{-1} \frac{d Z}{d \log \Lambda} \text { integrable Hamiltonian }
$$

$$
\left(\frac{u_{j}+i / 2}{u_{j}-i / 2}\right)^{L}=-\prod_{k=1}^{M} \frac{u_{j}-u_{k}+i}{u_{j}-u_{k}-i}
$$

$$
\gamma=\sum_{k=1}^{M} \frac{g}{u_{k}^{2}+1 / 4}
$$

## Clasical integrability



Motion of the string in AdS:

$$
\partial^{2} X_{a}+\left(\partial X_{b} \partial X^{b}\right) X_{a}=0
$$

Infinitely many
Integrals of motion:

$$
Q(z) \equiv \operatorname{tr}\left[\overrightarrow{\mathrm{P}} \exp \left(\int_{0}^{1} \frac{J_{1}+z J_{0}}{1-z^{2}} d \sigma\right)\right]
$$

## Integrable field theory

-No particle creation/anihilation
-Factorizability of S-matrix

- Yang-Baxter equation


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## Asymptotic Spectrum

- For spectral density we need finite volume


$$
\psi\left(x_{1}+L, x_{2}, \ldots\right)=e^{i p_{1} L} S\left(p_{1}, p_{2}\right) \ldots S\left(p_{1}, p_{n}\right) \Psi\left(x_{1}, x_{2}, \ldots\right)
$$

- From periodicity of the wave function

$$
i p_{i} L=2 \pi i n_{i}+\sum_{j} \log S\left(p_{i}, p_{j}\right)
$$

-Bethe equations

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$$
\left(\frac{u_{j}+i / 2}{u_{j}-i / 2}\right)^{L}=-\prod_{k=1}^{M} \frac{u_{j}-u_{k}+i}{u_{j}-u_{k}-i}
$$

## All loop Bethe equations

$$
\begin{aligned}
& \text { 毋 } 1=\prod_{j=1}^{R_{2}} \frac{u_{1, k}=u_{2, j}+\frac{8}{2}}{u_{1, k}-u_{2, j}-\frac{8}{2}} \prod_{j=1}^{K_{1}} \frac{1=1 / x_{1, k} x_{4, j}^{+}}{1-1 / x_{1, k} x_{4, j}^{-}}, \\
& 1=\prod_{j / k}^{K_{2}} \frac{u_{2, k}-u_{2, j}-i}{u_{2, k}-u_{2, j}+i} \prod_{j=1}^{K_{3}} \frac{u_{2, k}-u_{3, j}+\frac{i}{2}}{u_{2, k}-u_{3, g}-\frac{i}{2}} \prod_{j=1}^{K_{1}} \frac{w_{2, k}-u_{1, j}+\frac{g}{2}}{w_{2, k}-u_{1, g}-\frac{i}{2}}, \\
& 1=\prod_{j=1}^{K_{2}} \frac{u_{3, k}-u_{2, j}+\frac{1}{2}}{u_{3, k}-u_{2, j}-\frac{i}{2}} \prod_{j=1}^{K_{1}} \frac{x_{3, k}-x_{4, j}^{+}}{x_{3, k}-x_{4, j}^{-}}, \\
& 1=\left(\frac{x_{4, k}^{-}}{x_{4, k}^{+}}\right)^{L} \prod_{j \neq k}^{K_{1}} \frac{u_{4, k}-u_{4, j}+i}{u_{4, k}-u_{4, j}-i} \prod_{j}^{K_{1}}\left(\frac{1-1 / x_{4, k}^{+} x_{4, j}^{-}}{1=1 / x_{4, k}^{-} x_{4, j}^{+}}\right) \quad \sigma^{2}\left(x_{4, k v} x_{4, j}\right) \\
& \times \prod_{j=1}^{K_{1}} \frac{1-1 / x_{q, k}^{-} x_{1, j}}{1-1 / x_{4, k}^{+} x_{1, j}} \prod_{j=1}^{K_{1}} \frac{x_{4, k}^{-}-x_{3, j}}{x_{4, k}^{+}-x_{3, j}} \prod_{j=1}^{K_{3}} \frac{x_{4, k}^{-}-x_{5, j}}{x_{4, k}^{+}-x_{5, j}} \prod_{j=1}^{K_{7}} \frac{1-1 / x_{4, k}^{-} x_{7, j}}{1-1 / x_{4, k}^{+} x_{7, j}}, \\
& 1=\prod_{j=1}^{K_{n}} \frac{u_{5, k}-u_{6, j}+\frac{8}{2}}{u_{5, k}-u_{6, j}-\frac{8}{2}} \prod_{j=1}^{K_{n}} \frac{x_{9, k}-x_{4, j}^{+}}{x_{3, k}-x_{4, j}^{-}},
\end{aligned}
$$

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$$
\begin{aligned}
& 1=\prod_{j=1}^{K_{2}} \frac{u_{1, k}-u_{2, j}+\frac{2}{2}}{u_{1, k}-u_{2, j}-\frac{1}{2}} \prod_{j=1}^{K_{1}} \frac{1-1 / x_{1, k} x_{4, j}^{+}}{1-1 / x_{1, k} x_{4, j}^{-}}, \\
& 1=\prod_{j \neq k}^{K_{2}} \frac{u_{2, k}-u_{2, j}-i}{u_{2, k}-u_{2, j}+i} \prod_{j=1}^{K_{3}} \frac{u_{2, k}-u_{3, j}+\frac{i}{2}}{u_{2, k}-u_{3, g}-\frac{i}{2}} \prod_{j=1}^{K_{1}} \frac{w_{2, k}-u_{1, j}+\frac{i}{2}}{u_{2, k}-u_{1, g}-\frac{i}{2}}, \\
& 1=\prod_{j=1}^{K_{2}} \frac{u_{3, k}-u_{2, j}+\frac{i}{2}}{u_{3, k}-u_{2, j}-\frac{4}{2}} \prod_{j=1}^{K_{1}} \frac{x_{3, k}-x_{4,}^{+}}{x_{3, k}-x_{4, j}^{-}}, \\
& 1=\left(\frac{x_{4, k}^{-}}{x_{4, k}^{+}}\right)^{L} \prod_{j \neq k}^{K_{1}} \frac{u_{4, k}-u_{4, k}+i}{u_{4, k}=u_{4, j}-i} \prod_{k}^{K_{5}}\left(\frac{1-1 / x_{4, k}^{+} x_{4, j}^{-}}{1=1 / x_{4, k}^{-} x_{4, \lambda}^{+}}\right) \sigma^{2}\left(x_{4, k v} x_{4, j,}\right) \\
& \times \prod_{j=1}^{K_{1}} \frac{1-1 / x_{4, k}^{-} x_{1, j}}{1-1 / x_{4, k}^{+} x_{1, j}} \prod_{j=1}^{K_{1}} \frac{x_{4, k}^{-}-x_{3, j}}{x_{4, k}^{+}-x_{3, j}} \prod_{j=1}^{K_{3}} \frac{x_{4, k}^{-}-x_{5, j}}{x_{4, k}^{+}-x_{5, j}} \prod_{j=1}^{K_{7}} \frac{1-1 / x_{4, k}^{-} x_{7, j}}{1-1 / x_{4, k}^{+} x_{7, j}}, \\
& 1=\prod_{j=1}^{K_{n}} \frac{u_{5, k}-u_{6, j}+\frac{g}{2}}{u_{5, k}-u_{6, j}-\frac{8}{2}} \prod_{j=1}^{K_{n}} \frac{x_{9, k}-x_{4, j}^{+}}{x_{9, k}-x_{4, j}^{-}}, \\
& 1=\prod_{j \neq k}^{K_{0}} \frac{u_{6, k}-u_{6, j}-i}{u_{6, k}-u_{6, g}+i} \prod_{j=1}^{K_{5}} \frac{u_{6, k}-u_{5, g}+\frac{i}{2}}{u_{6, k}-u_{5, g}-\frac{1}{2}} \prod_{j=1}^{K_{7}} \frac{u_{6, k}-u_{7, g}+\frac{i}{2}}{u_{6, k}-u_{\bar{\gamma}, \delta}-\frac{i}{2}},
\end{aligned}
$$

$$
E=\sum_{k} \epsilon_{k}=\sum_{k} 2 g i\left(\frac{1}{x_{4, k}^{+}}-\frac{1}{x_{4, k}^{-}}\right)
$$

## All loop Bethe equations



$$
x+\frac{1}{x}=\frac{u}{g}, \quad x^{ \pm}+\frac{1}{x^{ \pm}}=\frac{u \pm i / 2}{g}
$$

$$
E=\sum_{k} \epsilon_{k}=\sum_{k} 2 g i\left(\frac{1}{x_{4, k}^{+}}-\frac{1}{x_{4, k}^{-}}\right)
$$

$$
\begin{aligned}
& 1=\prod_{j=1}^{K_{2}} \frac{u_{1, k}-u_{2, j}+\frac{\frac{2}{2}}{u_{1, k}}-u_{2, j}-\frac{4}{2}}{\tilde{K}_{j=1}} \frac{1-1 / x_{1, k} x_{4,}^{+}}{1-1 / x_{1, k} x_{4, j}^{-}}, \\
& 1=\prod_{j / k}^{K_{2}} \frac{u_{2, k}-u_{2, j}-i}{u_{2, k}-u_{2, j}+i} \prod_{j=1}^{K_{3}} \frac{u_{2, k}-u_{3, j}+\frac{i}{2}}{u_{2, k}-u_{2, g}-\frac{i}{2}} \prod_{j=1}^{K_{1}} \frac{w_{2, k}-u_{1, j}+\frac{d}{2}}{w_{2, k}-u_{1, g}-\frac{i}{2}}, \\
& 1=\prod_{j=1}^{K_{2}} \frac{u_{3, k}-u_{2, j}+\frac{i}{2}}{u_{3, k}-u_{2, j}-\frac{4}{2}} \prod_{j=1}^{K_{1}} \frac{x_{3, k}-x_{4, j}^{+}}{x_{3, k}-x_{4, j}^{-}}, \\
& 1=\left(\frac{x_{4, k}^{-}}{x_{4, k}^{+}}\right)^{L} \prod_{j \neq k}^{K_{1}} \frac{u_{4, k}=u_{4, k}+i}{u_{4, k}-u_{4, j}-i} \prod_{k}^{K}\left(\frac{1-1 / x_{4, k}^{+} x_{4, j}^{-}}{1=1 / x_{4, k}^{-} x_{4, j}^{+}}\right) \sigma^{2}\left(x_{4, k,} x_{4, j,}\right) \\
& \times \prod_{j=1}^{K_{1}} \frac{1-1 / x_{q, k}^{-} x_{1, j}}{1-1 / x_{4, k}^{+} x_{1, k}} \prod_{j=1}^{K_{1}} \frac{x_{4, k}^{-}-x_{3, j}}{x_{4, k}^{+}-x_{3, j}} \prod_{j=1}^{K_{3}} \frac{x_{4, k}^{-}-x_{5, j}}{x_{4, k}^{+}-x_{5, j}} \prod_{j=1}^{K_{7}} \frac{1-1 / x_{4, k}^{-} x_{7, j}}{1-1 / x_{4, k}^{+} x_{7, j}}, \\
& 1=\prod_{j=1}^{K_{n}} \frac{u_{5, k}-u_{6, j}+\frac{g}{2}}{u_{5, k}-u_{6, j}-\frac{R}{2}} \prod_{j=1}^{K_{n}} \frac{x_{j, k}-x_{4, j}^{+}}{x_{j, k}-x_{4, j}^{-}}, \\
& 1=\prod_{j \neq k}^{K_{0}} \frac{u_{6, k}-u_{6, j}-i}{u_{6, k}-u_{6, g}+i} \prod_{j=1}^{K_{5}} \frac{u_{6, k}-u_{5, g}+\frac{i}{2}}{u_{6, k}-u_{5, g}-\frac{k}{2}} \prod_{j=1}^{K_{7}} \frac{u_{6, k}-u_{7, Q}+\frac{i}{2}}{u_{6, k}-u_{\bar{\gamma}, g}-\frac{i}{2}},
\end{aligned}
$$

## Vacuum



## $Z(\tau, \sigma)=Z(\sigma, \tau)$

## Vacuum



$$
\begin{aligned}
& Z(\tau, \sigma)=Z(\sigma, \tau) \\
& \sum e^{-e^{-\sigma_{n}(L) R}} \sum \sum \sum^{-E_{n}(R) L L}
\end{aligned}
$$

## Vacuum



$$
\begin{aligned}
& Z(\tau, \sigma)=Z(\sigma, \tau) \\
& \sum e^{-E_{n}(L) R} \quad \sum e^{-E_{n}(R) L} \\
& e^{-E_{0}(L) R}
\end{aligned}
$$

## Vacuum



$$
\begin{aligned}
& Z(\tau, \sigma)=Z(\sigma, \tau) \\
& \sum e^{-E_{n}(L) R} \quad \sum e^{-E_{\sigma_{n}}(R) L} \\
& e^{-E_{0}(L) R}
\end{aligned}
$$

I.e. from the asymptotical spectrum (infinite $R$ ) we can compute the Ground state energy for ANY finite volume!

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$$

I.e. from the asymptotical spectrum (infinite R) we can compute the Ground state energy for ANY finite volume.

$$
E_{0}(L)=-\lim _{R \rightarrow \infty} \frac{\log \sum e^{-E_{n}(R) L}}{R}
$$

## Vacuum



$$
\begin{gathered}
Z(\tau, \sigma)=Z(\sigma, \tau) \\
\sum e^{-E_{n}(L) R} \quad \sum e^{-E_{n}(R) L} \\
e^{-E_{0}(L) R}
\end{gathered}
$$

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E_{0}(L)=-\lim _{R \rightarrow \infty} \frac{\log \sum e^{-E_{n}(R) L}}{R}
$$

In our case the wick rotated theory is different - "mirror"
theory
Ambjorn, Janik, Kristjansen; Arutynov, Frolov

## AdS/CFT bound states

$$
\left(\frac{u_{j}+i / 2}{u_{j}-i / 2}\right)^{L}=\prod_{k=1(k \neq j)}^{J} \frac{u_{j}-u_{k}+i}{u_{j}-u_{k}-i}
$$

Till Bargheer, Niklas Beisert, N. G.


## AdS/CFT bound states

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$$

${ }_{2}\left[\begin{array}{c}0 \\ 0 \\ 0\end{array}\right]-n$


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\left(\frac{u_{j}+i / 2}{u_{j}-i / 2}\right)^{L}=\prod_{k=1(k \neq j)}^{J} \frac{u_{j}-u_{k}+i}{u_{j}-u_{k}-i}
$$



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## AdS/CFT bound states



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$0000000+$
$Y_{1}(u)$

## AdS/CFT bound states

| $i^{0}$ | $n$ | $0 \mathrm{OO} \mathrm{O}^{\circ} \mathrm{O} \mathrm{O}$ 0000000 000000000 |
| :---: | :---: | :---: |
|  |  |  |
|  |  |  |
|  |  | 000000 |
| 0 |  | O000000 |

$0000000+\frac{0000000}{0000000}+$

$$
Y_{1}(u) \quad Y_{2}(u)
$$

## AdS/CFT bound states



00000000
$0000000 \rightarrow \frac{0000000}{0000000} \rightarrow+\begin{array}{r}00000000 \\ 00000000 \\ Y_{1}(u) \\ Y_{2}(u)\end{array}$
$Y_{3}(u)$

## All loop Bethe equations



$$
\begin{aligned}
& 1=\prod_{j=1}^{K_{2}} \frac{u_{1, k}-u_{2, j}+\frac{i}{2}}{u_{1, k}-u_{2 j j}-\frac{1}{2}} \prod_{j=1}^{K} \frac{1-1 / x_{1, k} x_{4, j}^{+}}{1-1 / x_{1, k} x_{4, j}^{-}}, \\
& 1=\prod_{j \neq k}^{K_{2}} \frac{u_{2, k}-u_{2, j}-i}{u_{2, k}-u_{2, j}+i} \prod_{j=1}^{K_{3}} \frac{w_{2, k}-u_{3, j}+\frac{i}{2}}{u_{2, k}-u_{3, j}-\frac{i}{2}} \prod_{j=1}^{K_{1}} \frac{w_{2, k}-u_{1, j}+\frac{i}{2}}{u_{2, k}-u_{1, j}-\frac{i}{2}}, \\
& 1=\prod_{j=1}^{K_{2}} \frac{u_{3, k}-u_{2, j}+\frac{1}{2}}{u_{3, k}-u_{2, j}-\frac{1}{2}} \prod_{j=1}^{K} \frac{x_{3, k}-x_{4, j}^{+}}{x_{3, k}-x_{4, j}^{-}}, \\
& 1=\left(\frac{x_{4, k}^{-}}{x_{4, k}^{+}}\right)^{L} \prod_{j \neq k}^{K_{1}} \frac{u_{4, k}=u_{4, j}+i}{u_{4, k}=u_{4, k}-i} \prod_{k}^{K_{3}}\left(\frac{1-1 / x_{4, k}^{+} x_{4, j}^{-}}{1=1 / x_{4, k}^{-} x_{4, k}^{+}}\right) \sigma^{2}\left(x_{4, k, k} x_{4, j}\right) \\
& \times \prod_{j=1}^{K_{1}} \frac{1-1 / x_{4, k}^{-} x_{1, j}}{1-1 / x_{4, k}^{+} x_{1, j}} \prod_{j=1}^{K_{1}} \frac{x_{4, k}^{-}-x_{3, j}}{x_{4, k}^{+}-x_{3, j}} \prod_{j=1}^{K_{3}} \frac{x_{4, k}^{-}-x_{5, j}}{x_{i, k}^{+}-x_{5, j}} \prod_{j=1}^{K_{7}} \frac{1-1 / x_{4, k}^{-} x_{7, j}}{1-1 / x_{4, k}^{+} x_{7, j}}, \\
& 1=\prod_{j=1}^{K_{6}} \frac{u_{5, k}-u_{6, j}+\frac{8}{2}}{u_{5, k}-u_{6, j}-\frac{g}{2}} \prod_{j=1}^{K_{n}} \frac{x_{5, k}-x_{4, j}^{+}}{x_{s, k}-x_{4, j}^{-}}, \\
& 1=\prod_{j \neq k}^{K_{0}} \frac{u_{6, k}-u_{6, j}-i}{u_{6, k}-u_{6, j}+i} \prod_{j=1}^{K_{5}} \frac{u_{6, k}-u_{5, j}+\frac{i}{2}}{u_{6, k}-u_{5, j}-\frac{i}{2}} \prod_{j=1}^{K_{7}} \frac{u_{6, k}-u_{7, j}+\frac{i}{2}}{u_{6, k}-u_{7, j}-\frac{i}{2}}, \\
& 1=\prod_{j=1}^{K_{6}} \frac{u_{7, k}=u_{6,9}+\frac{i}{2}}{u_{7, k}-u_{6,9}-\frac{1}{2}} \prod_{j=1}^{K} \frac{1-1 / x_{\gamma, k} x_{4, j}^{4}}{1-1 / x_{7, k} x_{4,9}^{-}} .
\end{aligned}
$$

## All loop Bethe equations

$$
\begin{aligned}
& 1=\prod_{j=1}^{F_{2}} \frac{u_{1, k}-u_{2, j}+\frac{2}{2}}{u_{1, k}-u_{2, j}=\frac{K_{4}}{2}} \prod_{g=1}^{1} \frac{1-1 / x_{1, k} x_{4,}^{4}}{1-1 / x_{1, k} x_{4,}^{-}}
\end{aligned}
$$

$$
\begin{aligned}
& 1=\prod_{j=1}^{R_{2}} \frac{u_{3, k}=u_{2,}+\frac{1}{2}}{u_{3, k}=u_{2,}=\frac{4}{2} \prod_{y=1}^{N_{3}} \frac{x_{3, k}=x_{4, j}^{+}}{x_{3,}=x_{4, j}^{-}}}
\end{aligned}
$$

## All loop Bethe equations




$$
\times \prod_{j=1}^{K_{1}} \frac{1-1 / x_{4, k}^{-} x_{1 j}}{1-1 / x_{4, k}^{+} x_{1, j}} \prod_{j=1}^{K_{5}} \frac{x_{4, k}^{-}-x_{3, j}}{x_{4, k}^{+}-x_{3, j}} \prod_{j=1}^{K_{5}} \frac{x_{4, k}^{-}-x_{5, j}}{x_{4, k}^{+}-x_{5, j}} \prod_{j=1}^{K_{7}} \frac{1-1 / x_{4, k}^{-} x_{7 j}}{1-1 / x_{4, k}^{+} x_{7 j}}
$$

$$
1=\prod_{j=1}^{K_{6}} \frac{u_{5, k}-u_{6, j}+\frac{q}{2}}{u_{5, k}-u_{6, j}-\frac{R}{2}} \prod_{j=1}^{K_{j}} \frac{x_{5, k}-x_{4, j}^{+}}{x_{5, k}-x_{4, j}^{-}}
$$

$$
1=\prod_{j / k}^{K_{0}} \frac{u_{6, k}-u_{6, j}-i}{u_{6, k}-u_{6, j}+i} \prod_{j=1}^{K_{5}} \frac{u_{6, k}-u_{5, j}+\frac{1}{2}}{u_{6, k}-u_{5, j}-\frac{i}{2}} \prod_{j=1}^{K_{9}} \frac{u_{6, k}-u_{7, j}+\frac{i}{2}}{u_{6, k}-u_{7, j}-\frac{i}{2}}
$$

$$
1=\prod_{j=1}^{K_{6}} \frac{u_{7, k}=u_{6,9}+\frac{1}{2}}{u_{7, k}=u_{6,9}-\frac{i}{2}} \prod_{\Omega=1}^{K_{9}} \frac{1-1 / x_{7, k} x_{4, j}^{4}}{1-1 / x_{T, k} x_{4,9}^{-}}
$$

$$
\begin{aligned}
& 1=\prod_{j=1}^{K_{2}} \frac{u_{1, k}-u_{2, j}+\frac{g}{2}}{u_{1, k}-u_{2, j}-\frac{R}{2}} \prod_{j=1}^{K_{2}} \frac{1-1 / x_{1, k} x_{4,2}^{+}}{1-1 / x_{1, k} x_{4, j}^{-}}, \\
& 1=\prod_{j=k}^{K_{2}} \frac{u_{2, k}-u_{2, j}-i}{u_{2, k}-u_{2, j}+i} \prod_{j=1}^{K_{3}} \frac{u_{2, k}-u_{3, j}+\frac{i}{2}}{u_{2, k}-u_{3, j}-\frac{i}{2}} \prod_{j=1}^{K_{1}} \frac{u_{2, k}-u_{1, j}+\frac{i}{2}}{u_{2, k}-u_{1, j}-\frac{i}{2}} \\
& 1=\prod_{j=1}^{K_{2}} \frac{u_{3, k}-u_{2, ~}+\frac{i}{2}}{u_{3, k}-u_{2, j}-\frac{i}{2}} \prod_{g=1}^{K_{2}} \frac{x_{3, k}=x_{4, j}^{+}}{x_{3, k}-x_{4, j}^{-}} \\
& 1=\left(\frac{x_{4, k}^{-}}{x_{4, k}^{+}}\right)^{L} \prod_{i / k}^{K_{1}} \frac{u_{4, k}=u_{4, j}+i}{u_{4, k}=u_{4, j}-i} \prod_{k}^{K_{1}}\left(\frac{1-1 / x_{4, k}^{+} x_{4, g}^{-}}{1-1 / x_{4, k}^{-} x_{4, j}^{+}}\right) \quad \sigma^{2}\left(x_{4, k,} x_{4, j}\right)_{0}
\end{aligned}
$$

## All loop Bethe equations

$$
\times \prod_{j=1}^{K_{1}} \frac{1-1 / x_{4, k}^{-} x_{1, j}}{1-1 / x_{4, k}^{+} x_{1, j}} \prod_{j=1}^{K_{i}} \frac{x_{4, k}^{-}-x_{3_{j}}}{x_{4, k}^{+}-x_{3, j}} \prod_{j=1}^{K_{5}} \frac{x_{4, k}^{-}-x_{5, j}}{x_{4, k}^{+}-x_{5, j}} \prod_{j=1}^{K_{7}} \frac{1-1 / x_{4, k}^{-} x_{7 j}}{1-1 / x_{4, k}^{+} x_{7 j}}
$$

$$
1=\prod_{j=1}^{K_{6}} \frac{u_{5, k}-u_{6, j}+\frac{q}{2}}{u_{5, k}-u_{6, j}-\frac{R}{2}} \prod_{j=1}^{K_{j}} \frac{x_{5, k}-x_{4, j}^{+}}{x_{5, k}-x_{4, j}^{-}}
$$

$$
1=\prod_{j / k}^{K_{0}} \frac{u_{6, k}-u_{6, j}-i}{u_{6, k}-u_{6, j}+i} \prod_{j=1}^{K_{5}} \frac{u_{6, k}-u_{5, j}+\frac{1}{2}}{u_{6, k}-u_{5, j}-\frac{i}{2}} \prod_{j=1}^{K_{9}} \frac{u_{6, k}-u_{7, j}+\frac{i}{2}}{u_{6, k}-u_{7, j}-\frac{i}{2}}
$$

$$
1=\prod_{j=1}^{K_{6}} \frac{u_{7, k}=u_{6,9}+\frac{1}{2}}{u_{7, k}=u_{6,9}-\frac{K_{2}}{2}} \prod_{\Omega=1}^{1-1 / x_{7, k} x_{4, j}^{+}} \frac{1-1 / x_{T, k} x_{4,9}^{-}}{1-}
$$

$$
\begin{aligned}
& 1=\prod_{j=1}^{K_{2}} \frac{u_{1, k}-u_{2, j}+\frac{g}{2}}{u_{1, k}-u_{2, j}-\frac{R}{2}} \prod_{j=1}^{K_{2}} \frac{1-1 / x_{1, k} x_{4,2}^{+}}{1-1 / x_{1, k} x_{4, j}^{-}}, \\
& 1=\prod_{j+k}^{K_{2}} \frac{u_{2, k}-u_{2, j}-i}{u_{2, k}-u_{2, j}+i} \prod_{j=1}^{K_{3}} \frac{u_{2, k}-u_{3, j}+\frac{i}{2}}{u_{2, k}-u_{3, j}-\frac{i}{2}} \prod_{j=1}^{K_{1}} \frac{u_{2, k}-u_{1, j}+\frac{i}{2}}{u_{2, k}-u_{1, j}-\frac{i}{2}} \\
& 1=\prod_{j=1}^{K_{2}} \frac{u_{3, k}-u_{2, ~}+\frac{i}{2}}{u_{3, k}-u_{2, j}-\frac{i}{2}} \prod_{g=1}^{K_{2}} \frac{x_{3, k}=x_{4, j}^{+}}{x_{3, k}-x_{4, j}^{-}} \\
& 1=\left(\frac{x_{4, k}^{-}}{x_{4, k}^{+}}\right)^{L} \prod_{i / k}^{K_{1}} \frac{u_{4, k}=u_{4, j}+i}{u_{4, k}=u_{4, j}-i} \prod_{k}^{K_{1}}\left(\frac{1-1 / x_{4, k}^{+} x_{4, g}^{-}}{1-1 / x_{4, k}^{-} x_{4, j}^{+}}\right) \quad \sigma^{2}\left(x_{4, k,} x_{4, j}\right)_{0}
\end{aligned}
$$

## All loop Bethe equations

$0-0.0-0.0$

$$
\times \prod_{j=1}^{K_{1}} \frac{1-1 / x_{4, k}^{-} x_{1 j}}{1-1 / x_{4, k}^{+} x_{1, j}} \prod_{j=1}^{K_{5}} \frac{x_{4, k}^{-}-x_{3_{j}}}{x_{4, k}^{+}-x_{3, j}} \prod_{j=1}^{K_{5}} \frac{x_{4, k}^{-}-x_{5, j}}{x_{4, k}^{+}-x_{5, j}} \prod_{j=1}^{K_{7}} \frac{1-1 / x_{4, k}^{-} x_{7 j}}{1-1 / x_{4, k}^{+} x_{7 j}}
$$

$$
1=\prod_{j=1}^{K_{0}} \frac{u_{5, k}-u_{6, j}+\frac{q}{2}}{u_{5, k}-u_{6, j}-\frac{2}{2}} \prod_{j=1}^{K_{n}} \frac{x_{5, k}-x_{4, j}^{+}}{x_{5, k}-x_{4, j}^{-}}
$$

$$
1=\prod_{j / k}^{K_{0}} \frac{u_{6, k}-u_{6, j}-i}{u_{6, k}-u_{6, j}+i} \prod_{j=1}^{K_{5}} \frac{u_{6, k}-u_{5, j}+\frac{1}{2}}{u_{6, k}-u_{5, j}-\frac{i}{2}} \prod_{j=1}^{K_{9}} \frac{u_{6, k}-u_{7, j}+\frac{i}{2}}{u_{6, k}-u_{7, j}-\frac{i}{2}}
$$

$$
1=\prod_{j=1}^{K_{6}} \frac{u_{T, k}=u_{6,9}+\frac{1}{2}}{u_{T, k}=u_{6,9}-\frac{i}{2}} \prod_{\Omega=1}^{K_{9}} \frac{1-1 / x_{7, k} x_{4, j}^{4}}{1-1 / x_{T, k} x_{4, j}^{-}}
$$

$$
\begin{aligned}
& 1=\prod_{j=1}^{K_{2}} \frac{u_{1, k}-u_{2 j}+\frac{q}{2}}{u_{1, k}-u_{2, j}-\frac{i}{2}} \prod_{j=1}^{K} \frac{1-1 / x_{1, k} x_{4,2}^{+}}{1-1 / x_{1, k} x_{4, j}^{-}} g \\
& 1=\prod_{j, k}^{K_{2}} \frac{u_{2, k}-u_{2, j}-i}{u_{2, k}-u_{2, j}+i} \prod_{j=1}^{K_{3}} \frac{u_{2, k}-u_{3, j}+\frac{i}{2}}{u_{2, k}-u_{3, g}-\frac{i}{2}} \prod_{j=1}^{K_{1}} \frac{u_{2, k}-u_{1, g}+\frac{i}{2}}{u_{2, k}-u_{1, g}-\frac{i}{2}} \\
& 1=\prod_{j=1}^{K_{2}} \frac{u_{3, k}-u_{2, ~}+\frac{i}{2}}{u_{3, k}-u_{2, j}-\frac{i}{2}} \prod_{g=1}^{K_{2}} \frac{x_{3, k}=x_{4, j}^{+}}{x_{3, k}-x_{4, j}^{-}} \\
& 1=\left(\frac{x_{4, k}^{-}}{x_{4, k}^{+}}\right)^{L} \prod_{i / k}^{K_{1}} \frac{u_{4, k}=u_{4, j}+i}{u_{4, k}=u_{4, j}-i} \prod_{j}^{K_{N}}\left(\frac{1-1 / x_{4, k}^{+} x_{4, j}^{-}}{1-1 / x_{4, k}^{-} x_{4, j}^{+}}\right) \sigma^{2}\left(x_{4, k v} x_{4, j}\right)
\end{aligned}
$$

## All loop Bethe equations

$0,0-0.0$

$$
\begin{aligned}
& 1=\prod_{j=1}^{F_{2}} \frac{u_{1, k}-u_{2, j}+\frac{2}{2}}{u_{1, k}-u_{2, j}-\frac{K_{4}}{2}} \prod_{g=1}^{1} \frac{1-1 / x_{1, k} x_{4,}^{4}}{1-1 / x_{1, k} x_{4,}^{-}}
\end{aligned}
$$

$$
\begin{aligned}
& 1=\prod_{j=1}^{R_{2}} \frac{u_{3, k}=u_{2,}+\frac{1}{2}}{u_{3, k}=u_{2 J}=\frac{1}{2}} \prod_{g=1}^{\kappa_{3}} \frac{x_{3, k}=x_{4, j}^{t}}{x_{3, k}=x_{4, j}^{-}}
\end{aligned}
$$



## AdSICFT Y-system



$$
E=\sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{d u}{2 \pi i} \frac{\partial \epsilon_{a}}{\partial u} \log \left(1+Y_{a, 0}(u)\right)
$$

## AdS/CFT Y-system



$$
E=\sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{d u}{2 \pi i} \frac{\partial \epsilon_{a}}{\partial u} \log \left(1+Y_{a, 0}(u)\right)+\sum_{a} \epsilon_{j}\left(u_{4, j}\right)
$$

...Destri de Vega, Bytsko, Teschner.

## AdS/CFT Y-system



$$
\frac{Y_{a, s}^{+} Y_{a, s}^{-}}{Y_{a+1, s} Y_{a-1, s}}=\frac{\left(1+Y_{a, s+1}\right)\left(1+Y_{a, s-1}\right)}{\left(1+Y_{a+1, s}\right)\left(1+Y_{a-1, s}\right)}
$$

$$
Y_{1,0}\left(u_{4, j}\right)=-1
$$

$$
E=\sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{d u}{2 \pi i} \frac{\partial \epsilon_{a}}{\partial u} \log \left(1+Y_{a, 0}(u)\right)+\sum_{a} \epsilon_{j}\left(u_{4, j}\right)
$$

..Destri de Vega, Bytsko, Teschner.

-     -         -             - 

SU(2) Chiral Gross-Neveu


- 1


Sine-Gordon with $\beta^{2}=8 \pi(1-1 / v)$

## Super-sine-Gordon




Current-current perturbation of $\mathrm{SU}_{\mathbf{k}}(2) \mathrm{WZW}$


Sausage model


SU(N+1) Principal Chiral Field

SO(2N) o-model

WZW SU $\mathbf{U}_{\mathbf{k}}(2)$ off-critical
$O(3) \sigma$-model with $\pi-\theta$ term

$$
\frac{Y_{a, s}^{+} Y_{a, s}^{-}}{Y_{a+1, s} Y_{a-1, s}}=\frac{\left(1+Y_{a, s+1}\right)\left(1+Y_{a, s-1}\right)}{\left(1+Y_{a+1, s}\right)\left(1+Y_{a-1, s}\right)}
$$

## Tests of the proposal

## Large volume

N.G., Kazakov, Vieira


## Large volume

N.G., Kazakov, Vieira


## Large volume

N.G., Kazakov, Vieira


## Use Firota equation:

$$
T_{a, s}^{+} T_{a, s}^{-}=T_{a+1, s} T_{a-1, s}+T_{a, s+1} T_{a, s-1}
$$

where $Y_{a, s}=\frac{T_{a, s+1} T_{a, s-1}}{T_{a+1, s} T_{a-1, s}}$.

## Hirota - large L

$\widehat{T}_{r e p}\left(u \mid \theta_{1}, \theta_{2}, \ldots\right)=\operatorname{Tr}_{r e p}\left(S\left(u, \theta_{1}\right) S\left(u, \theta_{2}\right) \ldots\right)$


## Generating functional

$$
\begin{aligned}
\mathcal{W} & =\left[1-\frac{Q_{1}^{-} B^{+(+)} R^{-(+)}}{Q_{1}^{+} B^{+(-)} R^{-(-)}} D\right]\left[1-\frac{Q_{1}^{-} Q_{2}^{++} R^{-(+)}}{Q_{1}^{+} Q_{2} R^{-(-)}} D\right]^{-1} \times \\
& \times\left[1-\frac{Q_{2}^{--} Q_{3}^{+} R^{-(+)}}{Q_{2} Q_{3}^{-} R^{-(-)}} D\right]^{-1}\left[1-\frac{Q_{3}^{+}}{Q_{3}^{-}} D\right], D=e^{-i \partial_{u}}
\end{aligned}
$$

## Hirota - large L

## $\widehat{T}_{r e p}\left(u \mid \theta_{1}, \theta_{2}, \ldots\right)=\operatorname{Tr}_{r e p}\left(S\left(u, \theta_{1}\right) S\left(u, \theta_{2}\right) \ldots\right)$

The eigevalues solve Hirota


## Generating functional

$$
\begin{aligned}
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\end{aligned}
$$

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The eigevalues solve Hirota! for rep $=a$ 㬰

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$$
\begin{aligned}
\mathcal{W} & =\left[1-\frac{Q_{1}^{-} B^{+(+)} R^{-(+)}}{Q_{1}^{+} B^{+(-)} R^{-(-)}} D\right]\left[1-\frac{Q_{1}^{-} Q_{2}^{++} R^{-(+)}}{Q_{1}^{+} Q_{2} R^{-(-)}} D\right]^{-1} \times \quad \mathcal{W}=\sum_{s=0}^{\infty} T_{1, s}^{[1-s]} D^{s} \\
& \times\left[1-\frac{Q_{2}^{--} Q_{3}^{+} R^{-(+)}}{Q_{2} Q_{3}^{-} R^{-(-)}} D\right]^{-1}\left[1-\frac{Q_{3}^{+}}{Q_{3}^{-}} D\right], D=e^{-i \partial_{u}} \quad \mathcal{W}^{-1}=\sum_{a=0}^{\infty}(-1)^{a} T_{a, 1}^{[1-a]} D^{a}
\end{aligned}
$$

## Hirota - large L

$\widehat{T}_{r e p}\left(u \mid \theta_{1}, \theta_{2}, \ldots\right)=\operatorname{Tr}_{r e p}\left(S\left(u, \theta_{1}\right) S\left(u, \theta_{2}\right) \ldots\right)$

The eigevalues solve Hirota! for rep $=a$ 囲

## Generating functional

$$
\begin{aligned}
\mathcal{W} & =\left[1-\frac{Q_{1}^{-} B^{+(+)} R^{-(+)}}{Q_{1}^{+} B^{+(-)} R^{-(-)}} D\right]\left[1-\frac{Q_{1}^{-} Q_{2}^{++} R^{-(+)}}{Q_{1}^{+} Q_{2} R^{-(-)}} D\right]^{-1} \times \quad \mathcal{W}=\sum_{s=0}^{\infty} T_{1, s}^{[1-s]} D^{s} \\
& \times\left[1-\frac{Q_{2}^{--} Q_{3}^{+} R^{-(+)}}{Q_{2} Q_{3}^{-} R^{-(-)}} D\right]^{-1}\left[1-\frac{Q_{3}^{+}}{Q_{3}^{-}} D\right], D=e^{-i \partial_{u}} \quad \mathcal{W}^{-1}=\sum_{a=0}^{\infty}(-1)^{a} T_{a, 1}^{[1-a]} D^{a}
\end{aligned}
$$

The ABA equations follows then from
$Y_{1,0}\left(u_{j}\right)=-1$

## Strong coupling



$$
Y_{a s}\left(z+\frac{i}{4 g}\right) Y_{a s}\left(z-\frac{i}{4 g}\right)=\frac{\left(1+Y_{a, s+1}(z)\right)\left(1+Y_{a, s-1}(z)\right)}{\left(1+1 / Y_{a+1, s}(z)\right)\left(1+1 / Y_{a-1, s}(z)\right)}
$$

## Strong coupling



$$
Y_{a s}\left(z+\frac{i}{4 g}\right) Y_{a s}\left(z-\frac{i}{4 g}\right)=\frac{\left(1+Y_{a, s+1}(z)\right)\left(1+Y_{a, s-1}(z)\right)}{\left(1+1 / Y_{a+1, s}(z)\right)\left(1+1 / Y_{a-1, s}(z)\right)}
$$

## We can rewrite it in terms of $T$ 's:

$$
Y_{a, s}=\frac{T_{a, s+1} T_{a, s-1}}{T_{a+1, s} T_{a-1, s}}
$$

## Strong coupling



$$
Y_{a s}\left(z+\frac{i}{4 g}\right) Y_{a s}\left(z-\frac{i}{4 g}\right)=\frac{\left(1+Y_{a, s+1}(z)\right)\left(1+Y_{a, s-1}(z)\right)}{\left(1+1 / Y_{a+1, s}(z)\right)\left(1+1 / Y_{a-1, s}(z)\right)}
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$$

$T_{a, s}^{2}=T_{a+1, s} T_{a-1, s}+T_{a, s+1} T_{a, s-1}$

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Y_{a s}\left(z+\frac{i}{4 g}\right) Y_{a s}\left(z-\frac{i}{4 g}\right)=\frac{\left(1+Y_{a, s+1}(z)\right)\left(1+Y_{a, s-1}(z)\right)}{\left(1+1 / Y_{a+1, s}(z)\right)\left(1+1 / Y_{a-1, s}(z)\right)}
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## Strong coupling



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$$

## We can rewrite it in terms of T's:

$$
Y_{a, s, s}=\frac{T_{a, s+1} T_{a, s-1}}{T_{a+1,1,} s_{a-1, s}}
$$

$T_{a, s}^{2}=T_{a+1, s} T_{a-1, s}+T_{a, s+1} T_{a, s-1}$

## Strong coupling

- For GL(n) characters the general expression are known

$$
s_{\lambda}(y)=\frac{\operatorname{det}\left(y_{i}^{\lambda_{j}+4-j}\right)_{1 \leq i, j \leq 4}}{\operatorname{det}\left(y_{i}^{4-j}\right)_{1 \leq i, j \leq 4}}
$$

## Strong coupling

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$$
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$$

$$
T_{a, s}^{2}=T_{a+1, s} T_{a-1, s}+T_{a, s+1} T_{a, s-1}
$$



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## Strong coupling

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s_{\lambda}(y)=\frac{\operatorname{det}\left(y_{i}^{\lambda_{j}+4-j}\right)_{1 \leq i, j \leq 4}}{\operatorname{det}\left(y_{i}^{4-j}\right)_{1 \leq i, j \leq 4}}
$$

$$
T_{a, s}^{2}=T_{a+1, s} T_{a-1, s}+T_{a, s+1} T_{a, s-1}
$$



## Boundary condifions and solution

Solving Hirota equation with the boundary conditions:

$$
\begin{aligned}
& Y_{a, 0} \simeq Y_{a, 0}^{L=\infty} \simeq\left(\Delta f^{2}\right)^{a} \\
& Y_{a, \pm 1} \simeq Y_{a, \pm 1}^{L=\infty} \simeq(\bar{f} / f)^{a}
\end{aligned}
$$

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& Y_{a, \pm 1} \simeq Y_{a, \pm 1}^{L=\infty} \simeq(\bar{f} / f)^{a} \\
& \Delta=\exp \left(-\frac{L}{2 g \sqrt{1-z^{2}}}\right), f(z)=\exp \left(-i \sum \frac{1}{x-x_{j}}\right)
\end{aligned}
$$

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$$
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& Y_{a, \pm 1} \simeq Y_{a, \pm 1}^{L=\infty} \simeq(\bar{f} / f)^{a} \\
& \Delta=\exp \left(-\frac{L}{2 g \sqrt{1-z^{2}}}\right), f(z)=\exp \left(-i \sum \frac{1}{x-x_{j}}\right)
\end{aligned}
$$

For $Y_{1,0}$ we get

$$
\begin{aligned}
& \left(\Delta\left(f^{2} \Delta\left(\left(\Delta^{2}-4 \Delta-1\right) \bar{f}+2 \Delta \bar{f}^{2}+2\right)+\left(\Delta^{2}-4 \Delta-1\right) \bar{f}+f\left(\Delta^{3}(\bar{f}-2) \bar{f}+\Delta^{2}\left(-4 \bar{f}^{2}+4 \bar{f}+1\right)-\Delta\left(\bar{f}^{2}-6 \bar{f}+4\right)-1\right)+2 \Delta \bar{f}^{2}+2\right)^{2}\right) / \\
& \left(( \Delta - 1 ) ^ { 2 } \left(-f^{2} \Delta^{5} \bar{f}^{2}(\bar{f}+f-2)^{2}+\Delta\left(2 f(\bar{f}-2)+(\bar{f}-4) \bar{f}+f^{2}\right)+\right.\right. \\
& \quad f \Delta^{4} \bar{f}\left(f^{3} \bar{f}\left(\bar{f}^{2}-4 \bar{f}+6\right)-2 f^{2}\left(2 \bar{f}^{3}-8 \bar{f}^{2}+8 \bar{f}+1\right)+2 f\left(3 \bar{f}^{3}-8 \bar{f}^{2}+4 \bar{f}+2\right)-2(\bar{f}-2) \bar{f}\right)+ \\
& \Delta^{3}\left(f^{4}(\bar{f}-4) \bar{f}+2 f^{3} \bar{f}\left(\bar{f}^{2}-6 \bar{f}+8\right)+f^{2}\left(\bar{f}^{4}-12 \bar{f}^{3}+28 \bar{f}^{2}-16 \bar{f}-1\right)-2 f \bar{f}\left(2 \bar{f}^{3}-8 \bar{f}^{2}+8 \bar{f}+1\right)-\bar{f}^{2}\right)+ \\
& \left.\left.\Delta^{2}\left(2 f^{3}(\bar{f}-2)+2 f^{2}\left(2 \bar{f}^{2}-6 \bar{f}+3\right)+2 f \bar{f}\left(\bar{f}^{2}-6 \bar{f}+8\right)+\bar{f}^{2}\left(\bar{f}^{2}-4 \bar{f}+6\right)+f^{4}\right)+1\right)\right)
\end{aligned}
$$

## Strong coupling solution

$$
E=\sum_{i=1}^{M} \epsilon\left(u_{4, i}\right)+\sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{d u}{2 \pi i} \frac{\partial \epsilon_{a}}{\partial u} \log \left(1+Y_{a, 0}(u)\right)
$$

## Strong coupling solution

$$
\begin{gathered}
E=\sum_{i=1}^{M} \epsilon\left(u_{4, i}\right)+\sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{d u}{2 \pi i} \frac{\partial \epsilon_{a}}{\partial u} \log \left(1+Y_{a, 0}(u)\right) \\
E=\sum_{i=1}^{M} \frac{x_{i}^{2}+1}{x_{i}^{2}-1}+\int_{-1}^{1} \frac{d z}{2 \pi} \frac{z}{\sqrt{1-z^{2}}} \partial_{z} \log \frac{(f \Delta-1)^{4}(\bar{f} \Delta-1)^{4}}{(\Delta-1)^{4}(f \bar{f} \Delta-1)^{2}\left(f^{2} \Delta-1\right)\left(\bar{f}{ }^{2} \Delta-1\right)}
\end{gathered}
$$

## Strong coupling solution

$$
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## Konishi operator

The simplest operator
$\mathcal{O}=\operatorname{tr}(Z Z W W)-\operatorname{tr}(Z W Z W)$

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g \rightarrow 0
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\begin{gathered}
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y_{a}(u)=9 a^{4}-36 a^{3}+72 u^{2} a^{2}+60 a^{2}-144 u^{2} a-48 a+144 u^{4}+48 u^{2}+16
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$$
\left(324+864 \zeta_{3}-1440 \zeta_{5}\right) g^{8}
$$

In agreement with perturbation theory!!
4-loops!

## Integral form of Y-system for exitations

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\log Y_{n}(u)=\int K_{n m}(u, v) \log \left(1+Y_{m}(v)\right) d v
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Bazhanov, Lukyanov, Zamolodchikov, P.Dorey, Totteo

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Fit:

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\Delta_{K}=2 \lambda^{1 / 4}\left(1.0002+\frac{0.994}{\lambda^{1 / 2}}-\frac{1.30}{\lambda}+\frac{3.1}{\lambda^{3 / 2}}+\ldots\right)
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## ABJM Theory



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N.G., Kazakov, Vieira


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### 0901.3753 <br> $E_{\text {wrapping }}=32-16 \zeta(2)$

### 0908.2463

J.A. Minahan, O.Ohlsson Sax, C. Sieg

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- $Y$-system is proposed for the exact AdS/CFT spectrum


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- YY? Hidden structures in the perturbation theory from the gauge side

