QED-correction enhancements from non-local effects

(M. Beneke, C. Bobeth, R. Szafron, Phys. Rev. Lett. 120, 011801)

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Outline

This lecture is a short introduction to method of regions and SCET with the emphasis on some applications to QED in flavour physics

- Two aspects of QED corrections
- Methods of regions: basics
- Methods of regions: power-enhancement in $B_s \to \mu^+\mu^-$
- Soft-Collinear Effective Field Theory (SCET)
- Hadronic matrix elements
- Sudakov resummation

QED corrections

$$\Delta E_n \sim \frac{2e^2}{3\pi\hbar c^3} \int_0^K dk \sum_n \frac{|v_{nm}|^2 (E_n - E_m)}{E_n - E_m + k}$$

QED in Flavor physics

QED effects can be divided into two classes of effects:

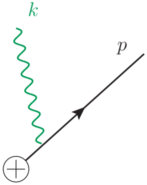
- Ultra-soft photons (sometimes simply called soft photons)
  Based on eikonal approximation, well understood, under the assumption that $\Delta E \ll \Lambda_{\text{QCD}}$
- Non-universal corrections
  Hard, hard-collinear, collinear, and soft

Both effects are important - even with strong cut on real photons ΔE,
the virtual corrections can resolve the structure of the meson!
Virtual photons can couple to initial and final state and may have wave-lengths smaller than the typical meson size $\sim 1/\Lambda_{\text{QCD}}$

We refer to photons with energy $k \sim \Lambda_{\text{QCD}}$ as *soft*
Photons with momentum $k \sim \Delta E$ are *ultra-soft*

Ultra-soft photons

- Numerically important, but very easy to compute
- Based on eikonal approximation: spin universal



$$\varepsilon_\mu(k)\bar{u}(p)\gamma^\mu \frac{\not{p}+\not{k}+m}{(k+p)^2-m^2} \to \frac{\varepsilon_\mu(k)p^\mu}{p\cdot k}\bar{u}(p)$$

Note $k^\mu \ll p^\mu, m$

- General all-order solution is well known
  [see eg. S. Weinberg, The Quantum theory of fields. Vol. 1: Foundations]

$$\Gamma_{\beta\alpha} \to \mathcal{F}(A(\alpha\to\beta))\left(\frac{\Delta E}{\Lambda}\right)^{A(\alpha\to\beta)}\Gamma^{\Lambda}_{\beta\alpha} \approx \left(\frac{\Delta E}{\Lambda}\right)^{A(\alpha\to\beta)}\Gamma_{\beta\alpha}$$

Note that Λ should be at most Λ_{QCD} or m

$$A(\alpha\to\beta) = -\frac{1}{8\pi^2}\sum_{nm}\frac{e_n e_m \eta_n \eta_m}{\beta_{nm}}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right)$$

- Should be included in experimental analysis, but not interesting from theory perspective. It is important to avoid double counting

Virtual corrections above ultra-soft scale

There are several kinematical and dynamical scales relevant to $B_s \to \mu^+\mu^-$:

- m_B – the hard scale given by kinematics
- $m_b \sim m_B$ – heavy quark mass – expansion parameter for the b quark HQET
- $\Lambda_{\rm QCD}$ – soft scale, typical momentum of the quarks in the meson (or inverse radius of the meson)
- $m_\mu \sim \Lambda_{\rm QCD}$ – collinear scale, muon mass acts as a regulator for collinear divergences

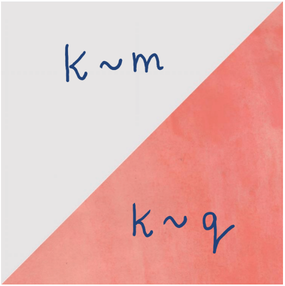
To compute corrections: *expand* the amplitude in $\lambda^2 = \frac{m_\mu}{m_B} \sim \frac{\Lambda_{\rm QCD}}{m_b}$
We need a more systematic approach than eikonal (soft) expansion!
Different logarithms appear

$$\ln \frac{m_\mu}{\Delta E}; \qquad \ln \frac{m_B}{m_\mu}; \qquad \ln \frac{m_B}{\Lambda_{\rm QCD}}; \qquad \ldots$$

Mixed QED-QCD logs are important! Expansion parameter is $\frac{\alpha}{\pi} \times \log^2$ rather than just $\frac{\alpha}{\pi}$

How to go beyond ultra-soft photon approximation in a systematic way? We need the method of regions

Method of Regions



Method of regions and Effective Field Theory

Method of regions can be applied to expand diagrams and it is useful to set-up the effective field theory
Advantages:

- systematic expansion allows for a good control of the theory accuracy
- resulting expressions are simpler than in the full theory
- allows for factorization and resummation of the large logarithms
- allows to exploit perturbative QCD and uniquely define non-perturbative objects

Appropriate EFT is SCET $\otimes$ HQET: needs energetic modes in the low-energy EFT $\rightarrow$ EFT cannot be obtained by integrating out complete fields but only certain modes $\rightarrow$ needs different fields to describe different modes and resulting theory is a non-local QFT

[M. Beneke , V. A. Smirnov, Asymptotic expansion of Feynman integrals near threshold, hep-ph/9711391]

Simple example

Consider a simple (euclidean) integral

$$I = \int d^{4-2\epsilon}k \frac{1}{(k^2+m^2)(k^2+q^2)}$$
$$\sim \int_0^\infty dk \frac{k^{3-2\epsilon}}{(k^2+m^2)(k^2+q^2)}$$

Simple example

Consider a simple (euclidean) integral

$$\begin{aligned} I &= \int d^{4-2\epsilon}k \frac{1}{(k^2+m^2)(k^2+q^2)} \\ &\sim \int_0^\infty dk \frac{k^{3-2\epsilon}}{(k^2+m^2)(k^2+q^2)} \\ &= \frac{1}{m^2-q^2} \int_0^\infty dk\, k^{3-2\epsilon} \left[\frac{1}{k^2+q^2} - \frac{1}{k^2+m^2} \right] \end{aligned}$$

Simple example

Consider a simple (euclidean) integral

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= & \frac{1}{m^2-q^2} \int_0^\infty dk\, k^{3-2\epsilon} \left[\frac{1}{k^2+q^2} - \frac{1}{k^2+m^2} \right]
\end{aligned}
$$

Use (n is assumed to be odd)

$$
\int_0^\infty dk \frac{k^{n-2\epsilon}}{k^2+M^2} = i^{n+1} M^{n-1} \left(-\frac{1}{2\epsilon} + \ln M + \mathcal{O}(\epsilon) \right)
$$

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$$
I \sim \frac{1}{2\epsilon} + \frac{m^2 \ln m - q^2 \ln q}{q^2 - m^2} = \frac{1}{2\epsilon} - \ln m + \frac{q^2}{m^2} \ln\left(\frac{q}{m}\right) + \mathcal{O}\left(\frac{q^4}{m^4}\right)
$$

for $q^2 \ll m^2$

Expansion in q/m

$q \ll \Lambda \ll m$

$$\int_0^\infty dk \frac{k^{3-2\epsilon}}{(k^2+m^2)(k^2+q^2)}$$
$$= \left(\int_0^\Lambda dk + \int_\Lambda^\infty dk\right) \frac{k^{3-2\epsilon}}{(k^2+m^2)(k^2+q^2)}$$

Expansion in q/m

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$$= \int_0^\Lambda dk \frac{k^{3-2\epsilon}\sum_{n=0}^\infty \left(-\frac{k^2}{m^2}\right)^n}{m^2(k^2+q^2)} + \int_\Lambda^\infty dk \frac{k^{3-2\epsilon}\sum_{n=0}^\infty \left(-\frac{q^2}{k^2}\right)^n}{(k^2+m^2)k^2}$$

Expansion in q/m

$q \ll \Lambda \ll m$

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$$= \int_0^\Lambda dk \frac{k^{3-2\epsilon}\sum_{n=0}^\infty \left(-\frac{k^2}{m^2}\right)^n}{m^2(k^2+q^2)} + \int_\Lambda^\infty dk \frac{k^{3-2\epsilon}\sum_{n=0}^\infty \left(-\frac{q^2}{k^2}\right)^n}{(k^2+m^2)k^2}$$

$$= \left(\int_0^\infty dk - \int_\Lambda^\infty dk\right) \frac{k^{3-2\epsilon}\sum_{n=0}^\infty \left(-\frac{k^2}{m^2}\right)^n}{m^2(k^2+q^2)}$$

$$+ \left(\int_0^\infty dk - \int_0^\Lambda dk\right) \frac{k^{3-2\epsilon}\sum_{n=0}^\infty \left(-\frac{q^2}{k^2}\right)^n}{(k^2+m^2)k^2}$$

Expansion in q/m

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Scaleless contribution

$$\int_0^\infty dk \frac{k^{3-2\epsilon}}{(k^2+m^2)(k^2+q^2)}$$

$$= \int_0^\infty dk \frac{k^{3-2\epsilon}\sum_{n=0}^\infty \left(-\frac{k^2}{m^2}\right)^n}{m^2(k^2+q^2)} - \int_\Lambda^\infty dk \frac{k^{3-2\epsilon}}{m^2k^2}\sum_{n=0}^\infty\left(-\frac{k^2}{m^2}\right)^n \sum_{m=0}^\infty\left(-\frac{q^2}{k^2}\right)^m$$

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After exchanging summation and integration, the last integral is of the from

$$\int_0^\infty dk k^{2m-2n+1-2\epsilon} = 0$$

Scaleless integrals vanish in dimensional regularization!

Regions and scaling

Expanded integrals have homogeneous scaling

Region I: $k \sim q$

$$\sum_{n=0}^{\infty} \int_0^{\infty} dk \frac{k^{3-2\epsilon} \left(-\frac{k^2}{m^2}\right)^n}{m^2(k^2+q^2)} \sim \frac{q \times q^3 \times q^{2n}/m^{2n}}{m^2 \times q^2} \sim \frac{q^{2n+2}}{m^{2n+2}}$$

Region II: $k \sim m$

$$\sum_{n=0}^{\infty} \int_0^{\infty} dk \frac{k^{3-2\epsilon} \left(-\frac{q^2}{k^2}\right)^n}{(k^2+m^2)k^2} \sim \frac{m \times m^3 \times q^{2n}/m^{2n}}{m^2 \times m^2} \sim \frac{q^{2n}}{m^{2n}}$$

Recovering the full result

Reminder

$$\int_0^\infty dk \frac{k^{n-2\epsilon}}{k^2+M^2} = i^{n+1} M^{n-1} \left(-\frac{1}{2\epsilon} + \ln M + \mathcal{O}(\epsilon) \right)$$

$$I \sim \frac{1}{2\epsilon} + \frac{m^2 \ln m - q^2 \ln q}{q^2 - m^2} = \frac{1}{2\epsilon} - \ln m + \frac{q^2}{m^2} \ln\left(\frac{q}{m}\right) + \mathcal{O}\left(\frac{q^4}{m^4}\right)$$

Leading power $\left(\frac{q^2}{m^2}\right)^0$, only region $k \sim m$ contributes :

$$\int_0^\infty dk \frac{k^{3-2\epsilon}}{(k^2+m^2)k^2} = \frac{1}{2\epsilon} - \ln m$$

Next-to-leading power $\left(\frac{q^2}{m^2}\right)^1$:

$$\int_0^\infty dk \frac{-q^2 k^{3-2\epsilon}}{(k^2+m^2)k^4} = \frac{q^2}{m^2} \left(\frac{1}{2\epsilon} - \ln m \right)$$

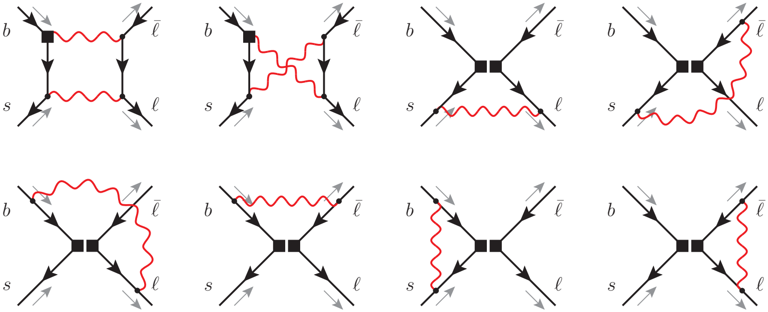
$$\int_0^\infty dk \frac{k^{3-2\epsilon}}{m^2(k^2+q^2)} = \frac{q^2}{m^2} \left(-\frac{1}{2\epsilon} - \ln q \right)$$

Sum of the regions:

$$\frac{q^2}{m^2} \ln\left(\frac{q}{m}\right)$$

How to apply this technique to leptonic decay of B_s?

QED corrections for $B_s \to \mu^+\mu^-$: regions and origin of power-enhancement



$B_s \to \mu^+\mu^-$

In the SM the process is

- loop suppressed (FCNC)

$$Br(B_s \to \mu^+\mu^-) = \frac{G_F^2\alpha^2}{64\pi^3} f_{B_s}^2 \tau_{B_s} m_{B_s}^3 |V_{tb}V_{ts}^*|^2 \sqrt{1 - \frac{4m_\mu^2}{m_{B_s}^2}} \times \left| \frac{2m_\mu}{m_{B_s}} C_{10} \right|^2$$

[see talk by Christoph for more comprehensive discussion]

$B_s \to \mu^+\mu^-$

In the SM the process is

- loop suppressed (FCNC)
- helicity suppressed (scalar meson decaying into energetic muons, vector interaction)

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- purely leptonic final state allows for a precise SM prediction, QCD contained in the meson decay constant f_{B_s}

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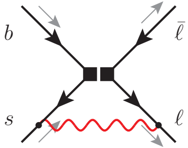
Let use use method of regions to compute



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Kinematics

The two-body decay $B_s(p_B) \to \ell(p_\ell)\bar{\ell}(p_{\bar{\ell}})$ implies lepton energies $E_\ell = E_{\bar{\ell}} = m_B/2$

Auxiliary light-cone vectors

$$n_+^\mu = (1,0,0,1) \qquad n_-^\mu = (1,0,0,-1)$$

$$n_+^2 = n_-^2 = 0 \qquad n_+ \cdot n_- = 2$$

$$p^\mu = (n_+p)\frac{n_-{}^\mu}{2} + p_\perp^\mu + (n_-p)\frac{n_+{}^\mu}{2}$$

$$p^2 = n_+p\, n_-p + p_\perp^2$$

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Partonic level:

$$b(p_b) + q(l_q) \quad \to \quad \ell(p_\ell) + \bar{\ell}(p_{\bar{\ell}})$$

$$p_b = m_b v + l_b \qquad l_b \sim \Lambda_{\rm QCD} \qquad l_q \sim \Lambda_{\rm QCD}$$

$$n_+p_\ell \sim m_b \qquad n_-p_\ell \sim \frac{m_\ell^2}{m_b} \qquad p_\ell^\perp \sim m_\ell$$

$$n_-p_{\bar{\ell}} \sim m_b \qquad n_+p_{\bar{\ell}} \sim \frac{m_\ell^2}{m_b} \qquad p_{\bar{\ell}}^\perp \sim m_\ell$$

Regions

Unlike in the previous example, the loop integrals are not spherically symmetric $\to$ different components can have different scaling!

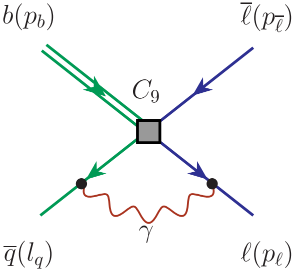
Scaling parameter $\lambda^2 = \frac{\Lambda_{\mathrm{QCD}}}{m_b} \sim \frac{m_\mu}{m_b}$

$$k = (n_+ k, k_\perp, n_- k)$$

mode	relative scaling	absolute scaling	virtuality k^2
hard	$(1,1,1)$	(m_b, m_b, m_b)	m_b^2
hard-collinear	$(1,\lambda,\lambda^2)$	$(m_b, \sqrt{m_b \Lambda_{\mathrm{QCD}}}, \Lambda_{\mathrm{QCD}})$	$m_b \Lambda_{\mathrm{QCD}}$
anti-hard-collinear	$(\lambda^2,\lambda,1)$	$(\Lambda_{\mathrm{QCD}}, \sqrt{m_b \Lambda_{\mathrm{QCD}}}, m_b)$	$m_b \Lambda_{\mathrm{QCD}}$
collinear	$(1,\lambda^2,\lambda^4)$	$(m_b, m_\mu, m_\mu^2/m_b)$	m_μ^2
anticollinear	$(\lambda^4,\lambda^2,1)$	$(m_\mu^2/m_b, m_\mu, m_b)$	m_μ^2
soft	$(\lambda^2,\lambda^2,\lambda^2)$	$(\Lambda_{\mathrm{QCD}}, \Lambda_{\mathrm{QCD}}, \Lambda_{\mathrm{QCD}})$	Λ_{QCD}^2

Note: collinear+soft = hard-collinear!

Q_9 operator insertion



Regions

- hard-collinear
- collinear
- soft (no power enhancement)

$$\int \frac{d^d k}{(2\pi)^d} \overline{u}\,(p_{\overline{\ell}})\,(-ieQ_\ell)\,\gamma_\mu i \frac{\not{k} + \not{p}_\ell + m_\ell}{(k+p_\ell)^2 - m_l^2 + i0} \gamma_\nu v\,(p_{\overline{\ell}})$$

$$\times \frac{-i}{k^2 + i0}$$

$$\times\, \overline{v}\,(l_q)\,(-ieQ_s)\,\gamma^\mu i \frac{-\not{l}_q - \not{k} + m_q}{(l_q+k)^2 - m_q^2 + i0} \gamma^\nu P_L u\,(p_b)$$

Simplifications and expansions for collinear and hard-collinear regions

We choose $p_\ell^\perp = 0$ and use equation of motion

$$\overline{u}\left(p_{\overline{\ell}}\right)\gamma^\mu\left(\not{k}+\not{p}_\ell+m_\ell\right)=\overline{u}\left(p_{\overline{\ell}}\right)\left(\gamma^\mu\not{k}+2p_\ell^\mu\right)$$

$$\overline{v}\left(l_q\right)\gamma^\mu\left(-\not{l}_q-\not{k}+m_q\right)=\overline{v}\left(l_q\right)\left(-\gamma^\mu\not{k}-2l_q^\mu\right)$$

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Expansion of the spinor

$$\bar{u}(p_\ell) = \bar{u}_c(p_\ell)\left[1 + \frac{\not{n}_+}{2}\frac{m_\ell}{n_+ p_\ell}\right]$$

Projection property of collinear spinor

$$\bar{u}_c \frac{\not{n}_-}{2} = 0 \qquad\qquad \bar{u}_c \frac{\not{n}_+ \not{n}_-}{4} = \bar{u}_c$$

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Dirac matrices

$$\not{k} = n_+ k \frac{\not{n}_-}{2} + \not{k}_\perp + n_- k \frac{\not{n}_+}{2} =$$

$$\gamma^\mu = \frac{\not{n}_-}{2} n_+{}^\mu + \gamma_\perp^\mu + \frac{\not{n}_+}{2} n_-{}^\mu$$

$$\not{n}_\pm^2 = 0 \qquad \not{n}_\pm \gamma_\perp^\mu = -\gamma_\perp^\mu \not{n}_\pm$$

Simplifications and expansions for collinear and hard-collinear regions

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$$\not{k}=n_+k\frac{\not{n}_-}{2}+\not{k}_\perp+n_-k\frac{\not{n}_+}{2}=n_+k\frac{\not{n}_-}{2}+\mathcal{O}(\lambda)$$

$$\gamma^\mu=\frac{\not{n}_-}{2}{n_+}^\mu+\gamma_\perp^\mu+\frac{\not{n}_+}{2}{n_-}^\mu$$

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Expansions of the numerator

For the power-enhanced parts, numerators in both regions are the same

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Lepton part – we are looking for terms with an odd number of $\gamma_\perp$

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$$= -\frac{m_\ell n_+ k}{n_+ p_\ell}\bar{u}_c(p_\ell)\,\frac{\not{n}_+}{2}\frac{\not{n}_-}{2}\gamma_\perp^\mu \gamma_\perp^\nu v_{\bar{c}}(p_{\bar{\ell}})$$

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Quark Part – we take the leading part

$$\overline{v}(l_q)\left(\gamma^\mu \not{k} + 2l_q^\mu\right)\gamma^\nu P_L u(p_b) \rightarrow -n_+k\,\overline{v}(l_q)\gamma_\perp^\mu\gamma_\perp^\nu\frac{\not{n}_-}{2}P_L u(p_b)$$

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Let us define

$$\frac{m_\ell(n_+k)^2}{n_+p_\ell}\left[\overline{u}_c\left(p_\ell\right)\gamma_\perp^\mu\gamma_\perp^\nu v_{\overline{c}}\left(p_{\overline{\ell}}\right)\right]\left[\overline{v}\left(l_q\right)\gamma_\perp^\mu\gamma_\perp^\nu\frac{\not{n}_-}{2}P_L u\left(p_b\right)\right] \equiv \frac{m_\ell(n_+k)^2}{n_+p_\ell}\times T$$

C_9 diagram – expansion: hard-collinear region

Denominators expansion

$$(l_q + k)^2 - m_q^2 = k^2 + n_+ k n_- l_q + \dots$$
$$(p_\ell + k)^2 - m_l^2 = k^2 + n_- k n_+ p_\ell + \dots$$

Note that both denominators have hard-collinear virtuality $\sim \lambda^2 m_b^2$

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$$= \frac{i}{(4\pi)^2} \frac{n_+p_\ell}{2n_-l_q} \left[\frac{1}{\epsilon} + 2 - \ln\left(\frac{n_-l_q n_+p_\ell}{\mu^2} \right) \right]$$

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The only difference to the hard-collinear case is the expansion of the denominators

$$(l_q + k)^2 - m_q^2 = n_+ k n_- l_q + \dots$$
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C_9 diagram – result

Add both regions

$$i\frac{\alpha}{4\pi}Q_\ell Q_s \frac{m_\ell}{2n_- l_q} T \times \left\{ \left[\frac{1}{\epsilon} + 2 - \ln\left(\frac{n_- l_q n_+ p_\ell}{\mu^2} \right) \right] + \left[-\frac{1}{\epsilon} - 1 + \ln\left(\frac{m_\ell^2}{\mu^2} \right) \right] \right\}$$
$$= i\frac{\alpha}{4\pi}Q_\ell Q_s \frac{m_\ell}{2n_- l_q} T \times \left[1 - \ln\left(\frac{n_- l_q n_+ p_\ell}{m_\ell^2} \right) \right]$$

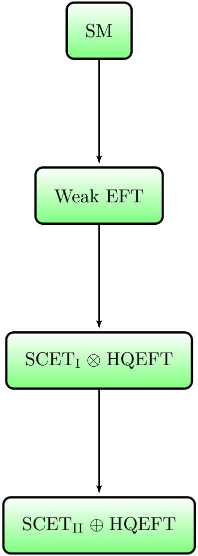
- Poles cancel – collinear contribution is UV-divergent but IR finite; hard-collinear is IR divergent but UV finite
- We get a logarithm of the ratio of hard-collinear scale to the collinear scale
- Explicit dependence on the soft quark momentum – correction is sensitive to the structure of the meson
- Factor $\frac{m_\ell}{2n_- l_q}$ is responsible for power-enhancement, comes from the hard-collinear quark propagator

How to deal with $n_- l_q$? We need an EFT

SCET approach to QED corrections

$$\frac{1}{in_+\partial}\phi(x) = -i\int_{-\infty}^{0} du\,\phi(x+un_+)$$

Tower of EFTs



$$m_W^2 \to \infty$$

$$m_b^2 \to \infty$$

$$m_b \Lambda_{QCD} \to \infty$$

EFT approach to systematically integrate-out different scales

- Operatorial definitions allow to separate non-perturbative input from perturbative corrections

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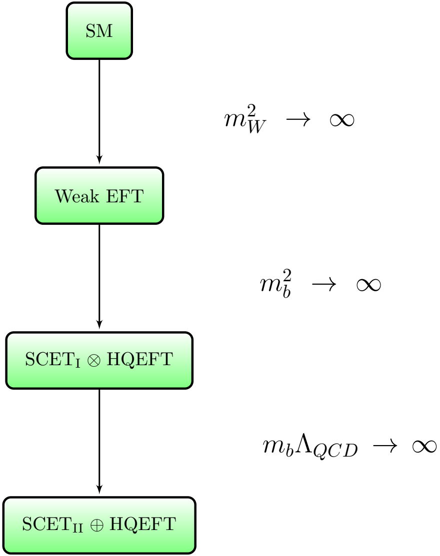

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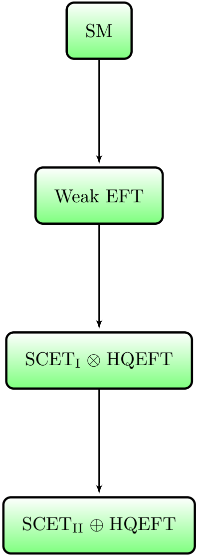

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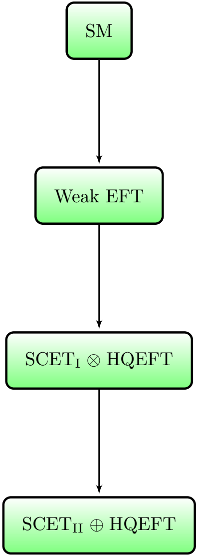

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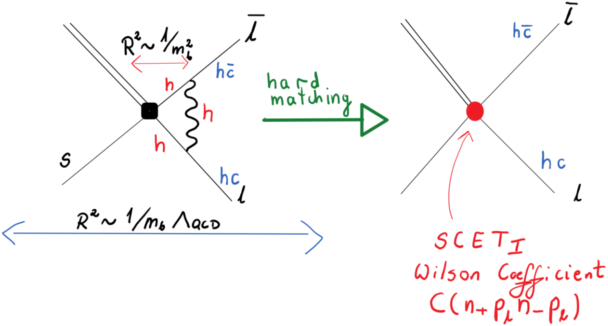
EFT approach to systematically integrate-out different scales

- Operatorial definitions allow to separate non-perturbative input from perturbative corrections
- Renormalization Group technique can be used to perform resummation
- Objects have well-defined counting in λ and their computation is typically simpler than in the full theory
- It is more intuitive than the full theory

Short and long-distance contributions: $\mathrm{SCET_I}$

Hard modes have typical fluctuations at distances $\sim 1/m_b \ll 1/\sqrt{m_b \Lambda_{\mathrm{QCD}}}$ typical size of fluctuations of the hard-collinear modes

They modify Wilson coefficients of the $\mathrm{SCET_I}$ operators



Note that $n_+ p_{\mathrm{hc}} \sim n_+ p_{\mathrm{h}}$ thus $\mathrm{SCET_{II}}$ is non-local along n_+ direction

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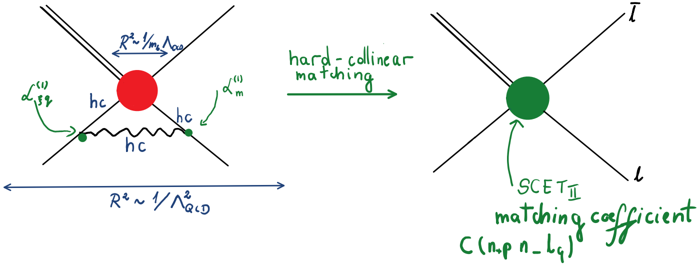

Note that $n_- p_{\mathrm{s}} \sim n_- p_{\mathrm{hc}}$ thus $\mathrm{SCET_{II}}$ soft sector is non-local along n_- direction

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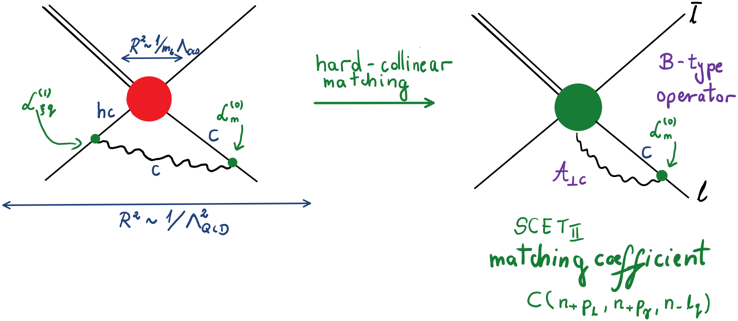

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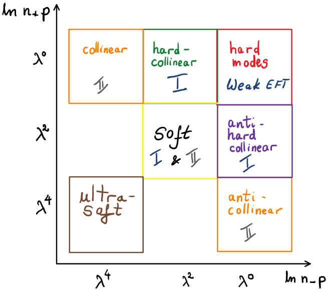
What is SCET?

SCET is an EFT which describes energetic particle. Here we need SCET_I and SCET_II

- Each mode has its own field, e.g. in SCET_I we have hard-collinear and soft modes
- Modes are separated in virtuality in SCET_I but not in SCET_II
- Lagrangian has expansion in λ such that it reproduces expansion by regions



$$\mathcal{L}_C^{(0)} = \bar{\xi}_C \left[in_- D + i\not{D}_\perp \frac{1}{in_+ D} i\not{D}_\perp \right] \frac{\not{n}_+}{2} \xi_C$$

$\mathcal{L} = \mathcal{L}_C^{(0)} + \mathcal{L}_{\bar{C}}^{(0)} + \ldots$ – only soft fields mediate interactions between collinear and anticollinear sectors

$$\xi_C \sim \lambda$$

$$n_+ D = n_+ \partial - ieQ_\xi n_+ A_C \sim 1$$

$$D_\perp^\mu = \partial_\perp^\mu - ieQ_\xi n_+ A_{\perp C}^\mu \sim \lambda$$

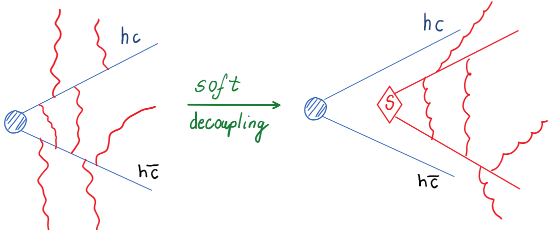
$$n_- D = n_- \partial - ieQ_\xi n_- A_C - ieQ_\xi n_- A_s(x_-) \sim \lambda^2$$

Soft modes are multipole expanded

$\phi_C(x)\phi_s(x) \to$

$\phi_C(x)\phi_s(x_-) + \ldots; \; x_- = \frac{n_-^\mu}{2} n_+ x$

Decoupling transformation – physical picture



Hard-collinear modes and soft modes do not interact at leading power in λ expansion – soft radiation is described by soft Wilson lines

Factorization of the amplitude:

[hard – collinear] × [anti – hard – collinear] × [soft]

Wilson lines and soft decoupling transformation

Soft Wilson line

$$Y_{\xi\pm}(x) = \exp\left[ie\,Q_\xi \int_{-\infty}^{0} ds\, n_\mp A_s(x+sn_\mp)\right]$$

allows to remove LP soft interactions.

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$$n_-\partial Y_{\xi+}(x) = n_-\partial \exp\left[ie\, Q_\xi \int_{-\infty}^{0} ds\, n_- A_s(x + s n_-)\right]$$

$$= \exp\left[ie\, Q_\xi \int_{-\infty}^{0} ds\, n_- A_s(x + s n_{n_-})\right] ie\, Q_\xi \int_{-\infty}^{0} ds n_-\partial\, n_- A_s(x + s n_-)$$

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$$\xi_C \to Y_{\xi+}(x_-)\xi_C$$

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$$\bar{\xi}_C n_- D \xi_C \to \bar{\xi}_C Y^\dagger_{\xi+}(x_-) n_- D Y_{\xi+}(x_-)\xi_C$$

Wilson lines and soft decoupling transformation

Soft Wilson line

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$$= \bar{\xi}_C \left[n_-\partial - ieQ_\xi n_- A_C\right]\xi_C$$

SCET$_{\mathrm{I}}$ operators

We have seen that hard-collinear quark leads to power-enhancement
Hence, we match weak EFT operators on SCET operator with
hard-collinear quark field

$$\widetilde{\mathcal{O}}_9 = \left(\bar{q}\gamma^\mu P_L b\right) \sum_\ell \left(\bar{\ell}\gamma_\mu \ell\right)$$

$$\widetilde{\mathcal{O}}_9^{\mathrm{I}}(s,t) = g_{\mu\nu}^{\perp} \left[\overline{\chi}_C(sn_+)\, \gamma_\perp^\mu P_L\, h_v(0)\right]\left[\overline{\ell}_C(tn_+)\, \gamma_\perp^\nu\, \ell_{\overline{C}}(0)\right]$$

Matching equation

$$Q_9 = \int dsdt H_9(s,t) \widetilde{\mathcal{O}}_9^{\mathrm{I}}(s,t)$$

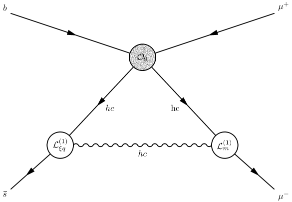
At leading order

$$H_9\,(s,t) = \delta\,(s)\,\delta\,(t)$$

Note that hard-collinear and hard-anticollinear field interact only through soft interaction which can be removed at leading power through decoupling transformation

SCET$_{\mathrm{I}}$ diagrams

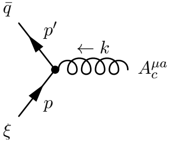
Computing following loops we can reproduce the result obtained by expansion by regions



Soft-collinear power-suppressed interaction in SCET$_{\mathrm{I}}$

$$\mathcal{L}_{\xi q}^{(1)} = \bar{q}\, W_c^{\dagger} i \not{D}_{\perp}\, \xi - \bar{\xi}\, i \overleftarrow{\not{D}}_{\perp} W_c q$$

[M. Beneke, M. Garny, R. Szafron, J. Wang, JHEP 1811 (2018) 112 contains Feynman rules for SCET]



$$i g_s t^a \begin{cases} 0 & \mathcal{O}(\lambda^0) \\ \gamma_{\perp}^{\mu} - \frac{\not{p}_{\perp}}{n_+ p} n_+{}^{\mu} & \mathcal{O}(\lambda) \end{cases}$$

Quark propagator

$$\frac{i n_+ k}{k^2 + i\varepsilon} \frac{\not{n}_-}{2}$$

SCET$_{\text{II}}$

Hard-collinear modes are integrated-out → soft and collinear modes do not interact (because $p_{\text{c}} + p_{\text{s}} \sim p_{\text{hc}}$)

$$\widetilde{\mathcal{J}}^{A1}_{m\chi}(v) = \left[\overline{q}_s(vn_-)Y(vn_-,0)\frac{\not{n}_-}{2}P_L h_v(0)\right]\left[Y_+^\dagger Y_-\right](0)\left[\overline{\ell}_c(0)(4\,m_\ell P_R)\,\ell_{\overline{c}}(0)\right]$$

$$\widetilde{\mathcal{J}}^{B1}_{\mathcal{A}\chi}(v,t) = \left[\overline{q}_s(vn_-)Y(vn_-,0)\frac{\not{n}_-}{2}P_L h_v(0)\right]\left[Y_+^\dagger Y_-\right](0)\left[\overline{\ell}_c(0)(2\not{\mathcal{A}}_{c\perp}(tn_+)P_R)\ell_{\overline{c}}(0)\right]$$

In practice it is more convenient to work with Fourier-transformed operator

$$\mathcal{J}^{A1}_i(\omega) = \int \frac{dv}{2\pi}\, e^{i\,\omega\, v}\, \widetilde{\mathcal{J}}^{A1}_i(v),$$

- Matching coefficient depend on ω which is just $n_- l_q$ in the expansion by regions!
- Soft Wilson lines $\left[Y_+^\dagger Y_-\right](0)$ appear after decoupling soft photons from the leptons

Matching $\mathrm{SCET_I}$ on $\mathrm{SCET_{II}}$

Evaluate matrix element in $\mathrm{SCET_I}$

$$\langle A\left(p'\right)\ell\left(p\right)\overline{\ell}\left(p_{\overline{\ell}}\right)|\int d^4xT\left\{\mathcal{O}_9^{\mathrm{I}}\left(u\right),\mathcal{L}_{\xi_q}^{(1)}\left(x\right)\right\}|b\left(p_b\right)q\left(l_q\right)\rangle$$

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which gives

$$\sim\left[\overline{u_c}\left(p\right)\gamma_\mu^\perp v\left(p_{\overline{\ell}}\right)\right]\left[\overline{v}\left(l_q\right)\left(ieQ_q\right)\not\epsilon_\perp\left(p'\right)\frac{\not n_-}{2}\frac{-i}{n_-l_q}\gamma_\perp^\mu P_L u_h\left(p_b\right)\right]$$

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On the other hand, the same matrix element for $\mathrm{SCET_{II}}$ operator gives

$$\sim\delta(\omega-n_-l_q)\left[\overline{u_c}\left(p\right)\gamma_\mu^\perp v\left(p_{\overline{\ell}}\right)\right]\left[\overline{v}\left(l_q\right)\left(ie\right)\not\epsilon_\perp\left(p'\right)\frac{\not n_-}{2}\gamma_\perp^\mu P_L u_h\left(p_b\right)\right]$$

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Evaluate matrix element in $\mathrm{SCET_I}$

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$$\sim\left[\overline{u_c}(p)\,\gamma_\mu^\perp v(p_{\bar{\ell}})\right]\left[\bar{v}(l_q)(ieQ_q)\,\not{\epsilon}_\perp(p')\,\frac{\not{n}_-}{2}\frac{-i}{n_-l_q}\gamma_\perp^\mu P_L u_h(p_b)\right]$$

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Thus matching coefficient is

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Note that the quark current is the same as in the collinear region. The rest of the diagram can be reproduced by taking the collinear matrix element

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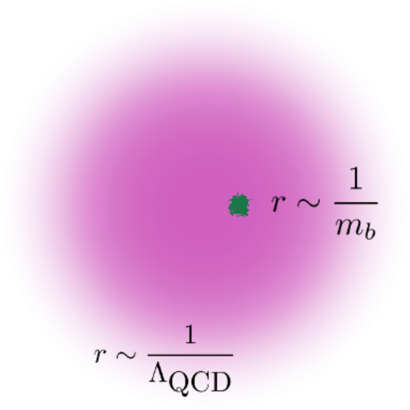
We turned n_-l_q into ω. Where are the hadronic effects?

Hadronic matrix elements and double logarithmic resummation



Factorization of the amplitude in $\mathrm{SCET}_{\mathrm{II}}$

The full amplitude is a convolution of a matching coefficient $\sim 1/\omega$ with the matrix element of $\mathrm{SCET}_{\mathrm{II}}$ operator

$$\mathcal{A} \sim \int \frac{d\omega}{\omega} \langle \ell_c \bar{\ell}_{\bar{c}} | \mathcal{J}_i(\omega) | \overline{B}_q(p) \rangle$$

$$= \langle \ell_c | \, [\text{collinear}] \, | 0 \rangle \langle \bar{\ell}_{\bar{c}} | \, [\text{anticollinear}] \, | 0 \rangle \int \frac{d\omega}{\omega} \langle 0 | \, [\text{soft}] \, (\omega) \, | \overline{B}_q(p) \rangle$$

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In QCD we define the B-meson light-cone distribution amplitude (LCDA) $\phi_+(\omega)$ which is a universal, non-perturbative object ("wave-function of the quark") defined as a non-local soft matrix element

$$\langle 0 | \bar{q}_s(vn_-) Y(vn_-, 0) \not{n}_- \gamma_5 \, h_v(0) \, | \overline{B}_q(p) \rangle \equiv -i f_{B_q} m_{B_q} \int_0^\infty d\omega \, e^{-i\omega v} \phi_+(\omega)$$

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Neglecting QED corrections of LCDA, the amplitude is proportional to (logarithmic) moments of LCDA

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This is justified for power-enhanced corrections since they are already α suppressed. What if we want to go beyond leading order in α or consider non–enhanced corrections?

Hadronic matrix element

Soft matrix element

$$\frac{\langle 0|\bar{q}_s(vn_-)Y(vn_-,0)\not{n}_-\gamma_5\, h_v(0)\,[Y_+^\dagger\, Y_-](0)|\overline{B}_q(p)\rangle}{\langle 0|\,[Y_+^\dagger\, Y_-](0)|0\rangle}$$

$$\equiv -i\mathscr{F}_{B_q} m_{B_q} \int_0^\infty d\omega\, e^{-i\omega v}\Phi_+(\omega)$$

- $\Phi_+(\omega)$ is the B-meson light-cone distribution amplitude generalized to QED for $B_s \to \ell^+\ell^-$ C_9 contribution
- Wilson lines $[Y_+^\dagger\, Y_-](0)$ are process dependent, consequence of soft photon decoupling
- $Y(vn_-,0)$ is a gauge link generalized to QED. It ensures gauge invariance and is constructed from soft Wilson lines
- We also need to define new, process dependent decay constant

$$\frac{\langle 0|\bar{q}_s(0)\gamma^\mu\gamma_5\, h_v(0)\,[Y_+^\dagger\, Y_-](0)|\overline{B}_q(p)\rangle}{\langle 0|\,[Y_+^\dagger\, Y_-](0)|0\rangle} = i\mathscr{F}_{B_q} m_{B_q} v^\mu,$$

Factor $\langle 0|\,[Y_+^\dagger\, Y_-](0)|0\rangle$ is introduced to properly subtract IR divergences.

Hadronic matrix element process dependence

If we consider $B_s \to \gamma\gamma$ or $B_s \to \nu\bar{\nu}$

$$\langle 0 | \bar{q}_s(vn_-) Y(vn_-,0) \not{n}_- \gamma_5 \, h_v(0) | \overline{B}_q(p) \rangle \equiv -i \mathscr{F}^0_{B_q} m_{B_q} \int_0^\infty d\omega \, e^{-i\omega v} \, \Phi^0_+(\omega)$$

no extra Wilson lines appear

For charged meson decay, we would define

$$\frac{\langle 0 | \bar{q}'_s(vn_-) \widetilde{Y}(vn_-,0) \not{n}_- \gamma_5 \, h_v(0) Y_+(0)^\dagger | \overline{B}_u(p) \rangle}{\langle 0 | Y_v(0) Y_+^\dagger(0) | 0 \rangle} \equiv -i \mathscr{F}^\pm_{B_u} m_{B_u} \int_0^\infty d\omega \, e^{-i\omega v} \Phi^\pm_+(\omega),$$

where $Y_v(0)$ carries total charge of the quarks.

At leading logarithmic accuracy we can still resum the QED logs without knowing QED corrections to the LCDA

$$\frac{d}{d\ln\mu} \left[\mathscr{F}_{B_q}(\mu) \, \Phi_+(\omega,\mu) \right] = -\mathscr{F}_{B_q} \int_0^\infty d\omega' \, \Gamma^s(\omega,\omega') \, \Phi_+(\omega')$$

$$\Gamma^s(\omega,\omega') = \left[-\Gamma_s \ln\frac{\omega}{\mu} - 5\left(\frac{\alpha_s}{4\pi} C_F + \frac{\alpha_{\rm em}}{4\pi} Q_q^2 \right) \right] \delta(\omega-\omega') - 4\left[\frac{\alpha_s}{4\pi} C_F + \frac{\alpha_{\rm em}}{4\pi} Q_q (Q_q + Q_\ell) \right] F(\omega,\omega')$$

Cusp anomalous dimension and resummation

For non-enhanced amplitude perform tree-level matching on $\mathrm{SCET_I}$ operator

$$\widetilde{\mathcal{O}}_m = m_\ell \left[\bar{q}_s(0)\, P_R\, h_v(0)\right] \left[\bar{\ell}_C(0)\, \gamma_5\, \ell_{\overline{C}}(0)\right],$$

with matching coefficient

$$H_m(\mu_b) = \mathcal{N}\, \frac{2\, C_{10}(\mu_b)}{m_b}.$$

and then we match on $\mathrm{SCET_{II}}$ operator

$$\widetilde{\mathcal{J}}_m^{A1} = m_\ell \left[\bar{q}_s(0) P_R h_v(0)\right] \left[Y_+^\dagger Y_-\right](0) \left[\bar{\ell}_c(0)\gamma_5\, \ell_{\bar{c}}(0)\right]$$

which has the same Wilson coefficient
SCET operators typically have cusp anomalous dimension

$$\frac{d}{d\ln\mu} H_m(\mu) = \Gamma_{\rm cusp} \ln\frac{m_{B_q}}{\mu} H_m(\mu)$$

Resummation

The RGE at LL reads

$$\frac{d}{d\ln\mu}H_m(\mu) = \frac{\alpha_{\rm em}}{\pi}2Q_\ell^2 \ln\frac{m_{B_q}}{\mu}H_m(\mu)$$

Similar cusp anomalous dimension appear also for power-enhanced operators

$$H_m(\mu) = H_m(\mu_b)\exp\left[-\frac{\alpha_{\rm em}}{\pi}Q_\ell^2\ln^2\frac{\mu_b}{\mu_c}\right]$$

This can be combined with ultra-soft photon correction (evolved to the collinear scale!), at the level of the decay width to give

$$\Gamma[B_q\to\mu\bar{\mu}](\Delta E) = \Gamma^{(0)}[B_q\to\mu\bar{\mu}]\left(\frac{2\Delta E}{m_{B_q}}\right)^{-\frac{2\alpha_{\rm em}}{\pi}\left(1+\ln\frac{m_\mu^2}{m_{B_q}^2}\right)}$$

Summary

- Mesons are not point-like, eikonal approximation is not enough because there can be large virtual corrections. We need to include QED corrections which depend on the structure of the meson.
- Method of regions allows to identify relevant physical degrees of freedom and determine that size of the corrections
- EFT is needed to properly define hadronic matrix elements, and allow to perform resummation (both QED and QCD)
- Method of regions and EFT approach allow to quantify the error related to yet unevaluated corrections