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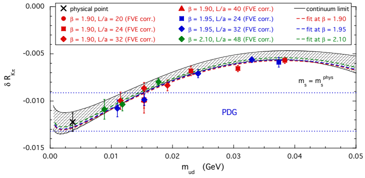
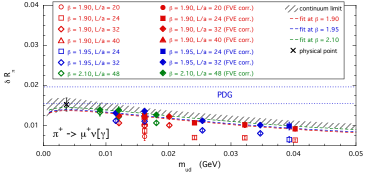
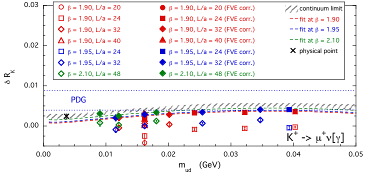
QED corrections to hadronic decays on the lattice

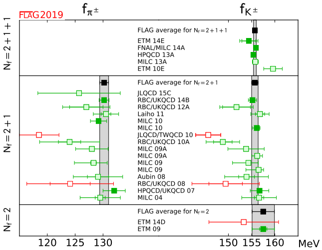
- phenomenological relevance of QED radiative corrections
- QED radiative corrections on the lattice:
  - a consistent definition of QED on a finite volume (backup)
  - a prescription to define QCD
  - extraction of the physical observable from euclidean correlators
  - infrared-safe observable and finite-volume effects
- non-perturbative calculation of the leptonic decay rates of pseudoscalar mesons at $O(\alpha)$
- non-perturbative calculation of the radiative leptonic decay rates of pseudoscalar mesons
- summary & outlooks

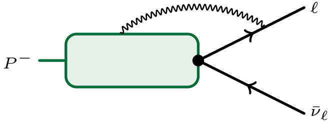






FLAG, arXiv:1902.08191
PDG review, j.rosner, s.stone, r.van de water, 2016
v.cirigliano et al., Rev.Mod.Phys. 84 (2012)





- from the last FLAG review we have

$$f_{\pi^\pm} = 130.2(0.8)\ \text{MeV}\ , \qquad \delta = 0.6\%\ ,$$

$$f_{K^\pm} = 155.7(0.3)\ \text{MeV}\ , \qquad \delta = 0.2\%\ ,$$

$$f_+(0) = 0.9706(27)\ , \qquad \delta = 0.3\%$$

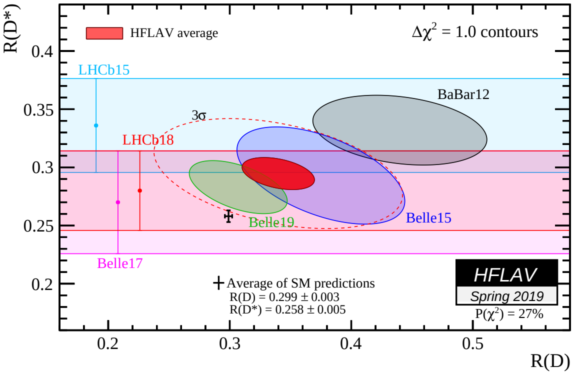
- in the case of pions and kaons, QED corrections can be calculated in χ-pt by estimating the relevant low-energy constants

$$\delta_{QED}\Gamma[\pi^- \to \ell\bar{\nu}(\gamma)] = 1.8\%\ ,$$

$$\delta_{QED}\Gamma[K^- \to \ell\bar{\nu}(\gamma)] = 1.1\%\ ,$$

$$\delta_{QED}\Gamma[K \to \pi\ell\bar{\nu}(\gamma)] = [0.5, 3]\%$$

- **at this level of precision QED radiative corrections must be included!**



- **yes! that's why we are here. . .**

- see the lectures from **m.misiak, r.szafron, z.was, t.kitahara** and all the talks at this workshop

including QED radiative corrections into a non-perturbative lattice calculation is a very challenging problem!

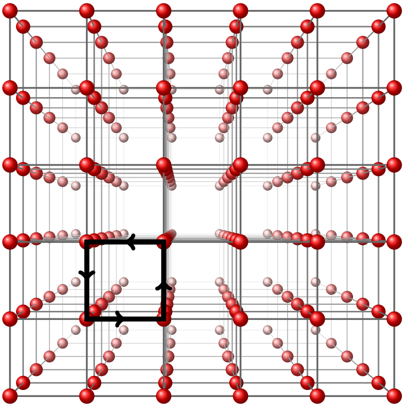
- **QED** is a long-range unconfined interaction that **needs to be consistently defined on a finite volume**: this is a very subtle issue that I'll not discuss in this lecture (see backup slides)

- from the numerical point of view it is difficult to disentangle QED radiative corrections from the leading QCD contributions but, first of all, **what is QCD?**

- as for any other observable on the lattice, QED radiative corrections have to be extracted from **euclidean correlators**

- **finite-volume effects** are potentially **very large**, e.g. of $O(1/L)$ in the case of the masses of stable hadrons

- **in the case of decay rates the problem is much more involved** because of the appearance of infrared divergences, $O(\log(L))$, at intermediate stages of the calculation: **the infrared problem**!



- in order to compare results for QED radiative corrections **we must first agree on what we call QCD**...

- indeed, when electromagnetic interactions are taken into account **the physical theory is QCD+QED**

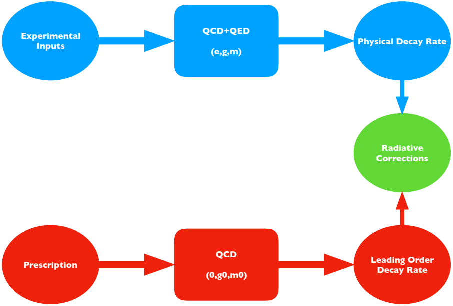
- **the QCD action is no longer expected to reproduce the physics** and, consequently, its renormalization becomes **prescription dependent**

- a **natural prescription** is to use again **physical experimental inputs** to set the QCD parameters

- **another prescription** (**j.gasser, a.rusetsky and i.scimemi, EPJ C32 (2003)**) consists in imposing the condition that the **renormalized couplings of the full theory and QCD are the same**, say in the $\bar{M}S$ scheme at $\mu = 2$ GeV

- in **RM123+SOTON, PRL 120 (2018), arXiv:1904.08731** we have compared the two approaches and found that **the difference, nowadays, is smaller than the statistical uncertainties**

- this will rapidly became **an important issue** on which we should find an agreement

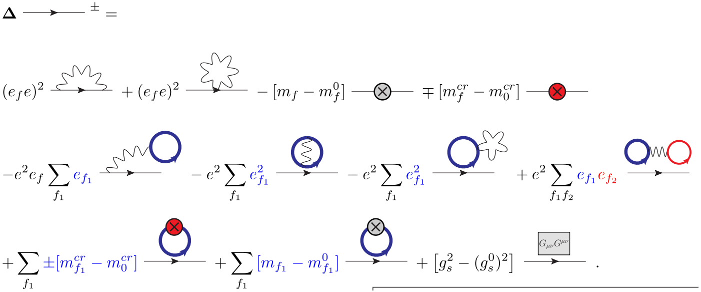


RM123, JHEP 1204 (2012)
RM123, PRD 87 (2013)

- once QCD has been defined, **QED radiative corrections** can be calculated directly or **by expanding the lattice path-integral** with respect to $\alpha \sim (m_d - m_u)/\Lambda_{QCD}$

$$\mathcal{O}(\boldsymbol{g}_s) = \frac{\langle e^{-S^{full}}\, O \rangle}{\langle e^{-S^{full}} \rangle} = \frac{\langle\, e^{-S^{QCD}} \left(e^{-\Delta S}\, O\right)\, \rangle}{\langle\, e^{-S^{QCD}}\, \left(e^{-\Delta S}\,\right) \rangle} = \mathcal{O}(\boldsymbol{g}_s^0) + \Delta\mathcal{O}$$

- the building-blocks for the graphical notation, used as a device to do calculations, are the corrections to the quark propagator



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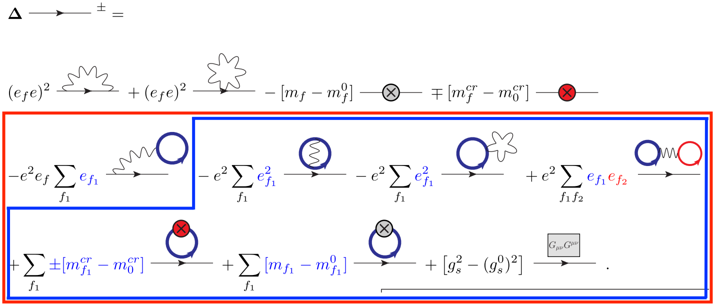
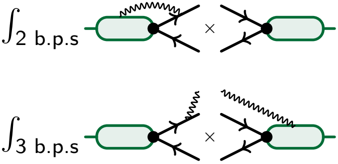
- vacuum polarization effects are the numerical issue with our method



f.bloch, a.nordsieck, Phys.Rev. 52 (1937)
t.d.lee, m.nauenberg, Phys.Rev. 133 (1964)
p.p.kulish, l.d.faddeev, Theor.Math.Phys. 4 (1970)

- **the infrared problem** has been analyzed by many authors over the years
- electrically-charged asymptotic states are *not* eigenstates of the photon-number operator
- the perturbative expansion of decay-rates and cross-sections with respect to e^2 is cumbersome because of the degeneracies
- the **block & nordsieck approach** consists in lifting the degeneracies by introducing an infrared regulator, say m_γ, and in computing infrared-safe observables
- at any fixed order in e^2, infrared-safe observables are obtained by adding the appropriate number of photons in the final states and by integrating over their energy in a finite range, say $[0, E]$
- in this framework, **infrared divergences** appear at intermediate stages of the calculations and **cancel in the sum of** the so-called **virtual and real contributions**



$$(p+k)^2 + m_P^2 = 2p\cdot k + k^2 \sim 2p\cdot k \ ,$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m_\gamma^2)\,(2p\cdot k)\,(2p_\ell\cdot k)} \sim c_{IR} \log\left(\frac{m_P}{m_\gamma}\right) ,$$

$$c_{IR}\left\{\log\left(\frac{m_P}{m_\gamma}\right) + \log\left(\frac{m_\gamma}{E}\right)\right\} = c_{IR}\,\log\left(\frac{m_P}{E}\right)$$

- it is always a good idea to address the issue of analytical continuation by starting from correlators, it is usually more cumbersome to locate singularities in the amplitudes

- the reason is that **correlators** (Schwinger's functions) **can always be Wick rotated** without any problem

- **euclidean reduction formulae work straightforwardly only for the lightest states**, i.e. the leading exponentials appearing in the correlators, because the corresponding integrals are convergent

- **problems arise** when one is interested in processes corresponding to **non-leading exponentials** (notice that at finite L the spectrum of H is discrete)

- **the first step in a lattice calculation** of a new observable is to **understand if the leading exponentials correspond to the external states for the process of interest**

- the lightest state appearing in a correlator is readily found by using the quantum numbers of the theory (in p.t. by using the quantum numbers of the *full* theory)

in minkowsky time:

$$C(t) = \mathtt{T}\langle 0| \cdots \bar{O}(t)\, O(0)|0\rangle$$
$$= \langle 0| \cdots e^{-it(H-i\epsilon)}\, O|0\rangle + \text{o.t.o.}$$

$$\mathcal{A}(E) = 2E(p^0 - E)\int_0^\infty dt\, e^{ip^0 t}\, C(t) + \text{o.t.o.}$$

in euclidean time:

$$C_E(\tau) = \langle 0| \cdots e^{-\tau H}\, O|0\rangle + \text{o.t.o.}$$

$$\mathcal{A}(E) = -2iE(p^0 - E)\int_0^\infty d\tau\, e^{p^0 \tau}\, C_E(\tau) + \text{o.t.o.}$$

from the spectral decomposition of correlators at $O(\alpha)$ one gets **expressions that are rather involved but their structure is easy to understand** and somehow illuminating

$$C(t) = e^{-tE(\boldsymbol{p})} \int \frac{d^4 q}{(2\pi)^4} A^{virt}(q)$$

$$+ \int \frac{d^3 q}{(2\pi)^3} A^{real}(\boldsymbol{q}) \; e^{-t[E(\boldsymbol{p}-\boldsymbol{q})+E_\gamma(\boldsymbol{q})]}$$

$$+ \cdots$$

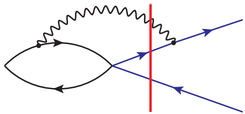
when the spatial momentum $\boldsymbol{q}$ of the photon goes to zero we have

$$|\boldsymbol{q}| \mapsto 0$$

$$E(\boldsymbol{p}-\boldsymbol{q}) + E_\gamma(\boldsymbol{q}) \mapsto E(\boldsymbol{p})$$

$$A^{virt}(q) \mapsto c^{virt} - c_{IR} \, \log \frac{|\boldsymbol{q}|}{m}$$

$$A^{real}(q) \mapsto c^{real} + c_{IR} \, \log \frac{|\boldsymbol{q}|}{m}$$



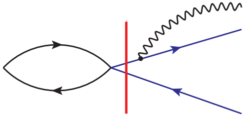
for each charged particle emitting a photon one has the exponential corresponding to the charged particle itself as an external state (the virtual photon contribution)

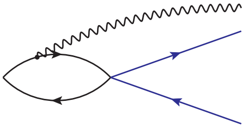
but also the exponential corresponding to the external states with the photon on-shell (the real photon contribution)

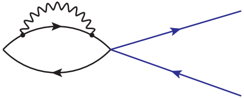
since

$$|\boldsymbol{q}| + \sqrt{M^2 + |\boldsymbol{p}-\boldsymbol{q}|^2} \quad \geq \quad \sqrt{M^2 + |\boldsymbol{p}|^2}$$

with an infrared regulator the blue exponentials are sub-leading and, if one is interested in the virtual contribution, there is **no problem of analytical continuation** in the case of the $O(e^2)$ QED radiative corrections to the **leptonic decays of pseudoscalar mesons**







since as we have seen

$$|\boldsymbol{q}| + \sqrt{M^2 + |\boldsymbol{p} - \boldsymbol{q}|^2} \quad \geq \quad \sqrt{M^2 + |\boldsymbol{p}|^2}$$

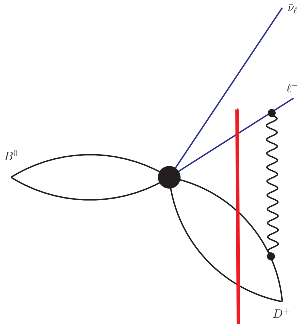
at fixed total momentum and with an infrared regulator the pseudoscalar meson is the lightest state in QED+QCD with the given quantum numbers

here there is a problem of analytical continuation! but this diagram can be factorized and the leptonic part can be computed analytically

therefore, **no problems of analytical continuation** arise in the self-energy diagrams and in the diagram in which the real photon is emitted from the meson!

notice that this is true **for a pion but also in the case of flavoured pseudoscalar mesons such as K, B, D!**

- **problems of analytical continuation do arise** in the case of **semileptonic decays** because of **electromagnetic final state interactions**
- the internal meson-lepton pair, and eventually multi-hadrons-lepton internal states, can be lighter than the external meson-lepton state
- this is a big issue, particularly in the case of B decays because of the presence of many kinematically-allowed multi-hadron states



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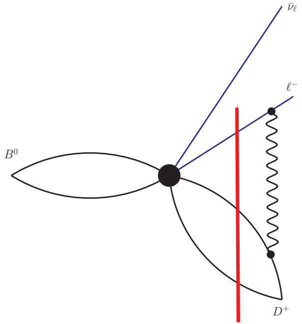
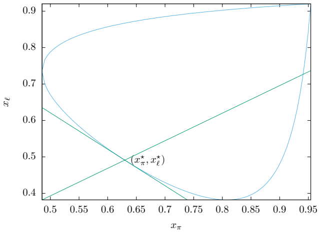
- this is a big issue, particularly in the case of B decays because of the presence of many kinematically-allowed multi-hadron states

- the problem does not arise at the point (on the boundary of the allowed phase-space)

$$s_\nu = (p_B - p_\nu)^2 = (p_D + p_\ell)^2 = (m_D + m_\ell)^2$$

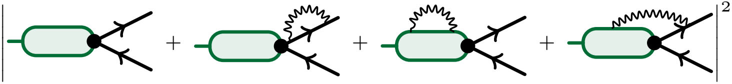
- **in this particular kinematical configuration**, by calling $s_D = (p_B - p_D)^2$, **the calculation of the QED radiative corrections** to the double-differential decay rate $d\Gamma/ds_D ds_\nu$ **might be feasible!**



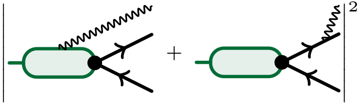


RM123+SOTON collaboration: m.di carlo, d.giusti, v.lubicz, g.martinelli, c.t.sachrajda, f.sanfilippo, s.simula, c.tarantino, n.t.

$\Gamma(E) = \int_{2\text{ b.p.s.}}$



$+ \int_{3\text{ b.p.s.}}^{E_\gamma < E}$



- I'm now going to describe in some details the non-perturbative lattice calculation of the $O(\alpha)$ QED radiative corrections to the decay rates $P \mapsto \ell\bar{\nu}(\gamma)$
- both the theoretical and numerical results discussed below are the outcome of a big effort of the **RM123+SOTON** collaboration started in 2015

RM123+SOTON, PRD 91 (2015)

- let's consider the infrared-safe observable: at $O(\alpha)$ this is obtained by considering the real contributions with a single photon in the final state

$$\Gamma(E) = \Gamma_0 + e^2 \lim_{L\to\infty} \{\Gamma_V(L) + \Gamma_R(L, E)\}$$

- **the finite-volume calculation of the real contribution is the issue: momenta are quantized!**

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- for this reason, by relying on the **universality of infrared divergences**, we have rewritten the previous formula as

$$\Gamma(E) = \Gamma_0 + e^2 \lim_{L\to\infty} \left\{ \Gamma_V(L) \overbrace{-\Gamma_V^{pt}(L) + \Gamma_V^{pt}(L) + \Gamma_R^{pt}(L,E) - \Gamma_R^{pt}(L,E)}^{=0} + \Gamma_R(L,E) \right\}$$

where $\Gamma^{pt}_{V,R}$ are evaluated in the **point-like effective theory**: these have **the same infrared behaviour** of $\Gamma_{V,R}$

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where $\Gamma_{V,R}^{pt}$ are evaluated in the **point-like effective theory**: these have **the same infrared behaviour** of $\Gamma_{V,R}$

- therefore, the different terms can be separated and eventually evaluated with different infrared regulators

$$\Gamma(E) = \Gamma_0 + e^2 \lim_{L\to\infty} \Gamma_V^{SD}(L) + e^2 \lim_{m_\gamma\to 0} \left\{ \Gamma_V^{pt}(m_\gamma) + \Gamma_R^{pt}(m_\gamma, E) \right\} + e^2 \lim_{m_\gamma\to 0} \Gamma_R^{SD}(m_\gamma, E)$$

RM123+SOTON, PRD 91 (2015), PRD 95 (2017), arXiv:1612.00199

- **infrared divergences can be computed in the so called point-like effective theory**

$$\mathcal{L}_{pt} = \phi_P^\dagger \left\{ -D_\mu^2 + m_P^2 \right\} \phi_P + \left\{ 2iG_F V_{CKM}\, f_P D_\mu \phi_P^\dagger\, \bar{\ell}\gamma^\mu \nu + \text{h.c.} \right\} , \qquad D_\mu = \partial_\mu - ieA_\mu$$

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- properly matched effective field theories have, by definition, the same infrared structure of the fundamental theory: at leading order the matching is obtained by using Γ_0

$$\Gamma_0^{pt} = \Gamma_0 = \frac{G_F^2\, |V_{CKM}|^2 f_P^2}{8\pi}\, m_P^3 r_\ell^2 \left(1 - r_\ell^2\right)^2 \;, \qquad r_\ell = \frac{m_\ell}{m_P}$$

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- structure-dependent terms can also be understood in the effective field theory language by adding to the lagrangian all the operators that are compatible with the symmetries of the full- theory, e.g.

$$\mathcal{O}_V(x) = F_V\, \epsilon^{\mu\nu\rho\sigma} D_\mu \phi_P\, F_{\nu\rho}\, \bar{\ell}\gamma_\sigma \nu \;, \qquad F_{\nu\rho} = \partial_\nu A_\rho - \partial_\nu A_\rho \;, \qquad \text{subleading in } \frac{E_\gamma}{m_\pi}$$

- **by exploiting** the full set of **constraints coming from the WIs and from the e.o.m one can rigorously show** that in the expansion around vanishing photon energies both **the leading (infrared divergent) and the next-to-leading terms are universal: fixed from the amplitude without QED!**

- by using this remarkable result (an application of Low's theorem, see also backup) we managed to perform an analytical calculation of the leading $O(1/L)$ finite volume effects, see below

- we are now going to look a bit more in details to the different terms entering our **master formula**

$$\Gamma(E) = \Gamma_0 + e^2 \lim_{m_\gamma \to 0} \left\{ \Gamma_V^{pt}(m_\gamma) + \Gamma_R^{pt}(m_\gamma, E) \right\} + e^2 \lim_{L \to \infty} \Gamma_V^{SD}(L) + e^2 \lim_{m_\gamma \to 0} \Gamma_R^{SD}(m_\gamma, E)$$

RM123+SOTON, PRD 91 (2015)

- concerning the perturbative point-like calculation in infinite volume, we have generalized the results obtained in the early days of quantum field theory by **berman 58, kinoshita 59**

$$\Gamma^{pt}(E) = e^2 \lim_{m_\gamma \to \infty} \left\{ \Gamma_V^{pt}(m_\gamma) + \Gamma_R^{pt}(m_\gamma, E) \right\}$$

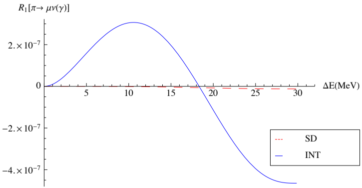
$$\begin{aligned}
= \Gamma_0 \, \frac{\alpha_{em}}{4\pi} \Bigg\{ & 3 \log\left(\frac{m_P^2}{m_W^2}\right) + \log(r_\ell^2) - 4\log(r_E^2) + \frac{2 - 10 r_\ell^2}{1 - r_\ell^2} \log(r_\ell^2) \\
& - 2\frac{1 + r_\ell^2}{1 - r_\ell^2} \log(r_E^2) \log(r_\ell^2) - 4\frac{1 + r_\ell^2}{1 - r_\ell^2} \mathrm{Li}_2(1 - r_\ell^2) - 3 \\
& + \frac{3 + r_E^2 - 6 r_\ell^2 + 4 r_E(-1 + r_\ell^2)}{(1 - r_\ell^2)^2} \log(1 - r_E) + \frac{r_E(4 - r_E - 4 r_\ell^2)}{(1 - r_\ell^2)^2} \log(r_\ell^2) \\
& - \frac{r_E(-22 + 3 r_E + 28 r_\ell^2)}{2(1 - r_\ell^2)^2} - 4\frac{1 + r_\ell^2}{1 - r_\ell^2} \mathrm{Li}_2(r_E) \Bigg\} ,
\end{aligned}$$

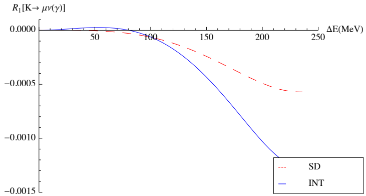
where

$$r_E = \frac{2E}{m_P} , \qquad r_\ell = \frac{m_\ell}{m_P} .$$

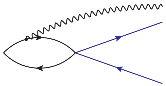
the structure dependent real contribution: $\Gamma_R^{SD}(E)$

RM123+SOTON, PRD 91 (2015)





- concerning the real structure dependent contributions, the relevant hadronic quantity is



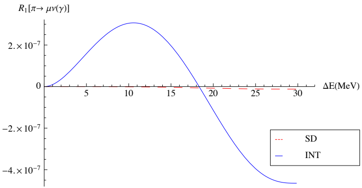
$$\mapsto \quad H^\nu(k,p) = \varepsilon^*_\mu(k) \int d^4x\, e^{ikx}\, T\langle 0|j^\mu(x)\, J^\nu_W(0)|P\rangle \,, \qquad k^2 = 0$$

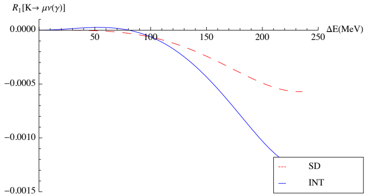
that can be expressed in terms of (two if $\varepsilon^* \cdot k = 0$) hadronic form–factors (see below)

- by using the χpt results (**v.cirigliano and i.rosell, PRL 99 (2007)**) for these quantities, we have estimated the structure dependent real contribution to be, nowadays, phenomenologically irrelevant for $P = \{\pi, K\}$ and $\ell = \mu$

$$\Gamma_R^{SD}(E) = \lim_{m_\gamma \to 0} \left\{ \Gamma_R(m_\gamma, E) - \Gamma_R^{pt}(m_\gamma, E) \right\} < 0.002\, \frac{\Gamma(E) - \Gamma_0}{e^2}$$

RM123+SOTON, PRD 91 (2015)





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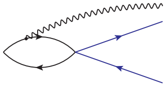

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that can be expressed in terms of (two if $\varepsilon^*\cdot k=0$) hadronic form–factors (see below)

- **in the last part of the lecture** I will show the preliminary **results of a fully non-perturbative calculation of the structure dependent real contribution**: these confirm the phenomenological analysis for $P=\{\pi,K\}$ and open the possibility of calculating $D_{(s)}\mapsto \ell\bar{\nu}\gamma$ and $B\mapsto \ell\bar{\nu}\gamma$

- we performed an analytical calculation of $\Gamma_V^{pt}(L)$

RM123+SOTON, PRD 95 (2017), arXiv:1612.00199

$$\frac{\Gamma_V^{pt}(L)-\Gamma_V^{\ell\ell}(L)}{\Gamma_0} = c_{IR}\log(L^2 m_P^2) + c_0 + \frac{c_1}{(m_P L)} + O\left(\frac{1}{L^2}\right)$$

where

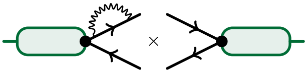
$$c_{IR} = \frac{1}{8\pi^2}\left\{\frac{(1+r_\ell^2)\log(r_\ell^2)}{(1-r_\ell^2)}+1\right\} ,$$

$$c_0 = \frac{1}{16\pi^2}\left\{2\log\left(\frac{m_P^2}{m_W^2}\right) + \frac{(2-6r_\ell^2)\log(r_\ell^2)+(1+r_\ell^2)\log^2(r_\ell^2)}{1-r_\ell^2} - \frac{5}{2}\right\} + \frac{\zeta_C(\mathbf{0})-2\zeta_C(\boldsymbol{\beta}_\ell)}{2},$$

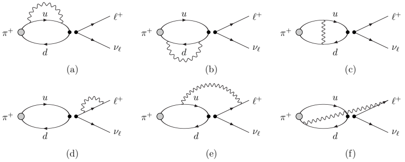
$$c_1 = -\frac{2(1+r_\ell^2)}{1-r_\ell^2}\zeta_B(\mathbf{0}) + \frac{8r_\ell^2}{1-r_\ell^4}\zeta_B(\boldsymbol{\beta}_\ell)$$

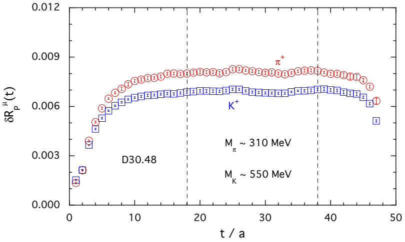
and we have shown that **c_{IR}, c_0 and c_1 are universal**, i.e. they are the same in the point-like and in the full theories! this means that in $\Gamma_V^{SD}(L) = \Gamma_V(L) - \Gamma_V^{pt}(L)$ we subtract exactly, together with the infrared divergence, the leading $O(1/L)$ terms and **we have $O(1/L^2)$ finite size effects**

- notice: the lepton wave-function contribution, $\Gamma_V^{\ell\ell}(L)$, does not contribute to $\Gamma_V^{SD}(L)$



RM123+SOTON, PRL 120 (2018), arXiv:1904.08731

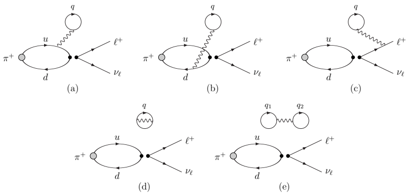


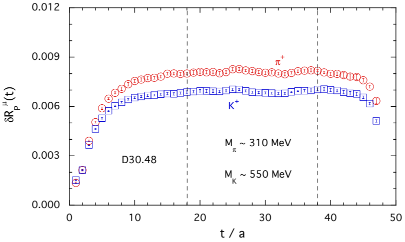


- we have performed the lattice calculation by using the previously mentioned **RM123 method**, i.e. by expanding the lattice path-integral with respect to α and the up-down quark mass difference

- by using this method we managed to obtain **excellent numerical signals** for the correlators corresponding to the diagrams shown in the figure and for the associated counter-terms

RM123+SOTON, PRL 120 (2018), arXiv:1904.08731





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- by using this method we managed to obtain **excellent numerical signals** for the correlators corresponding to the diagrams shown in the figure and for the associated counter-terms

- we have *not* computed the contributions corresponding to charged sea-quarks; this is the so called **electroquenched approximation**: although we have estimated the associated uncertainty, there is certainly room for improvement here. . .

our result for $\Gamma[K^- \to \mu\bar{\nu}_\mu(\gamma)]/\Gamma[\pi^- \to \mu\bar{\nu}_\mu(\gamma)]$

RM123+SOTON, PRL 120 (2018), arXiv:1904.08731

- by defining

$$\Gamma_P(E) = \Gamma_P^0 \left\{1 + \delta R_P(E)\right\} \ ,$$

$$\delta R_{K\pi} = \delta R_K(E_K^{max}) - \delta R_\pi(E_\pi^{max})$$

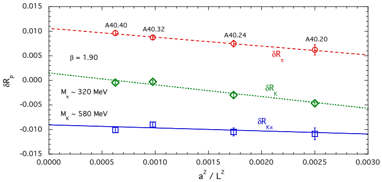
- **our result is**

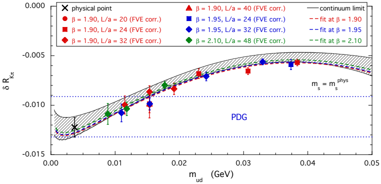
$$\boldsymbol{\delta R_{K\pi}}$$
$$= -0.0122(10)^{st}(2)^{tun}(8)^{\chi}(5)^{L}(4)^{a}(6)^{qQED}$$

$$= \mathbf{-0.0122(16)}$$

- this can (given the caveat concerning the definition of QCD) be compared with the result currently quoted by the PDG and obtained in **v.cirigliano and h.neufeld, PLB 700 (2011)**

$$\delta R_{K\pi} = -0.0112(21)$$





our result for $\Gamma[K^- \to \mu\bar{\nu}_\mu(\gamma)]$ and $\Gamma[\pi^- \to \mu\bar{\nu}_\mu(\gamma)]$

RM123+SOTON, PRL 120 (2018), arXiv:1904.08731

- by defining

$$\Gamma_P(E) = \Gamma_P^0 \left\{1 + \delta R_P(E)\right\} \ ,$$

- **our result are**

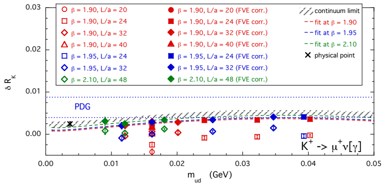
$$\boldsymbol{\delta R_K(E_K^{max}) = 0.0024(10)}$$

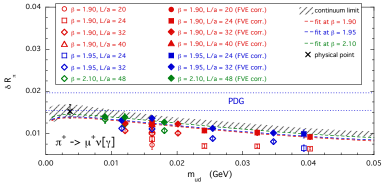
$$\boldsymbol{\delta R_\pi(E_\pi^{max}) = 0.0153(19)}$$

- this can (given the caveat concerning the definition of QCD) be compared with the result currently quoted by the PDG

$$\boldsymbol{\delta R_K(E_K^{max}) = 0.0064(24)}$$

$$\boldsymbol{\delta R_\pi(E_\pi^{max}) = 0.0176(21)}$$





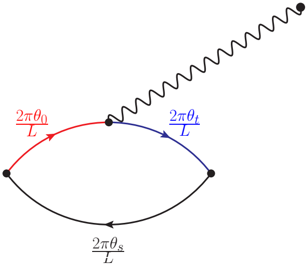
- I now move to the discussion of the **non-perturbative lattice calculation of the radiative leptonic decay rates** for the processes $P \mapsto \ell\bar{\nu}_\ell\gamma$

- as we have seen, in the region of small (soft) photon energies these are needed to properly define the measurable infrared–safe purely leptonic decay rates $P \mapsto \ell\bar{\nu}_\ell(\gamma)$

- in the region of experimentally detectable (hard) photon energies these represent important probes of the internal structure of mesons

- in the case of light pseuudoscalar mesons one can rely on chiral perturbation theory but the low–energy constants that enter these calculations are model dependent

- in the case of heavy–light mesons nothing is known from first-principles about these quantities



the RM123+SOTON collaboration:

g.martinelli, University of Rome La Sapienza
f.mazzetti, University of Rome La Sapienza
m.di carlo, University of Rome La Sapienza

g.m.de divitiis, University of Rome Tor Vergata
a.desiderio, University of Rome Tor Vergata
r.frezzotti, University of Rome Tor Vergata
m.garofalo, INFN of Rome Tor Vergata

d.giusti, University of Roma Tre
v.lubicz, University of Roma Tre
f.sanfilippo, INFN of Roma Tre
s.simula, INFN of Roma Tre

c.t.sachrajda, University of Southampton

- the **non-perturbative information** needed to compute the radiative decay-rates is encoded into the decay constant of the meson and into **two form-factors**

$$\epsilon^r_\mu(k) \int d^4y\, e^{ik\cdot y}\, \mathtt{T}\langle 0|j^\alpha_W(0) j^\mu_{em}(y)|P(p)\rangle =$$

$$\epsilon^r_\mu(k)\Bigg\{ - iF_V\, \frac{\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta}{m_P}$$

$$+ \left[F_A + \frac{m_P f_P}{p\cdot k}\right] \frac{(p\cdot k\, g^{\mu\alpha} - p^\mu k^\alpha)}{m_P}$$

$$+ \frac{m_P f_P}{p\cdot k}\, \frac{p^\mu p^\alpha}{m_P} \Bigg\}$$

- these can be expressed as functions of x_γ (and of m_P)

$$F_{A,V}(x_\gamma)\,, \qquad 0 \le x_\gamma = \frac{2p\cdot k}{m_P^2} \le 1$$

- the infrared divergent contribution (in red) is universal: it is proportional to the amplitude with no photons (f_P)

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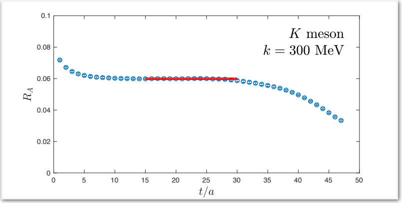

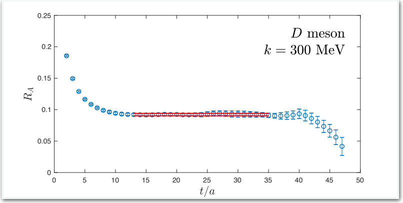

- by using the **RM123 method** we managed to get **excellent numerical signals** for the correlators from which $F_{A,V}$ can be extracted

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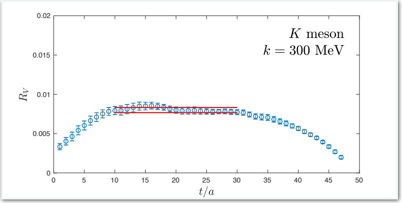
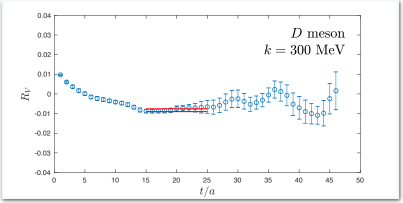
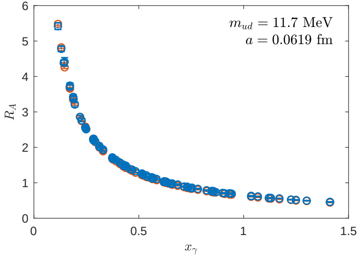


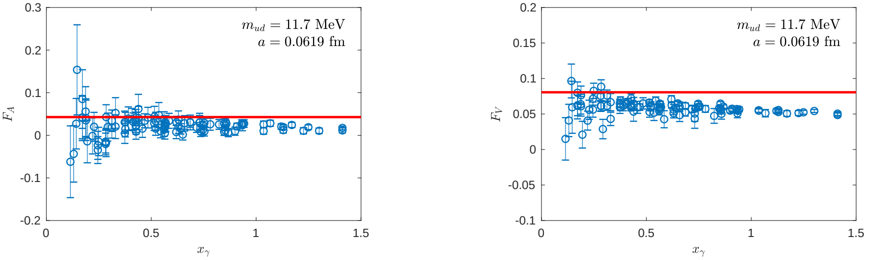




- in the case of **light mesons** (the plots correspond to the K) the structure-dependent form factors are very small

$$R_A = F_A + \frac{2 f_P}{m_P\, x_\gamma}\ , \qquad\qquad R_A^{pt} = \frac{2 f_P}{m_P\, x_\gamma}\ ,$$

- remarkably we are able to cover the full kinematical range $0 \leq x_\gamma \leq 1$



- in the case of **light mesons** (the plots correspond to the K) the structure-dependent form factors are very small and **in agreement with chiral perturbation theory**

$$F_A^\chi = \frac{8 m_P (L_9 + L_{10})}{f_P} \,, \qquad F_V^\chi = \frac{m_P}{4\pi^2 f_P} \,, \qquad L_9 + L_{10} \simeq 0.0017$$ **(arXiv:1405.6488)**

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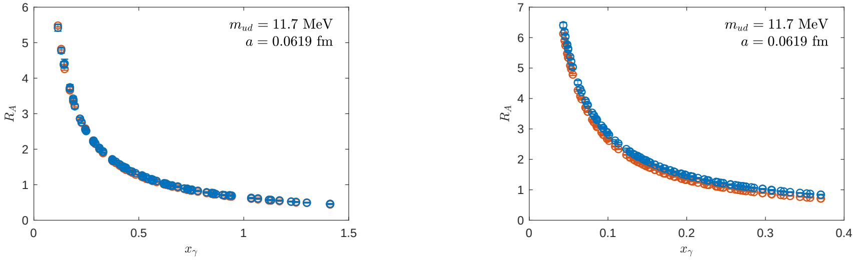


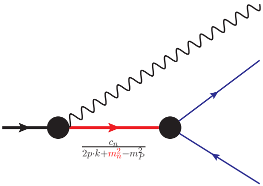
- in the case of **heavy mesons** (the plot on the left corresponds to the K while the one on the right to the D_s) we again obtain the correct infrared divergence

$$R_A = F_A + \frac{2f_P}{m_P\, x_\gamma}\ , \qquad R_A^{pt} = \frac{2f_P}{m_P\, x_\gamma}\ ,$$

- and we are able to cover the kinematical range $0 \le x_\gamma \le 0.4$

- **in the case of heavy mesons there is a strong enhancement of the structure-dependent form factors**

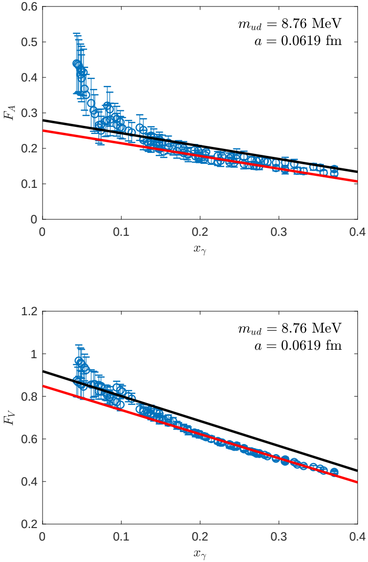
- this can be understood by using the argument of **d.becirevic, b.haas and e.kou, PLB 681 (2009)**



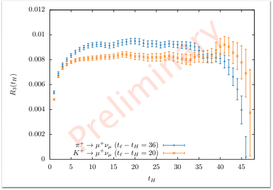
- in between the electromagnetic and the weak currents propagate internal states that give contributions to the form-factors proportional to

$$\frac{1}{x_\gamma + \frac{m_n^2 - m_P^2}{m_P^2}}$$

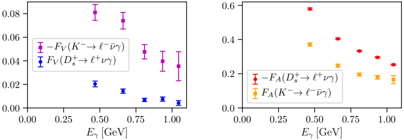
$$\frac{m_n^2 - m_P^2}{m_P^2} = \begin{cases} O(1)\ , & P = \{\pi, K\} \\ O(m_\pi/m_P)\ , & P = \{D, B\} \end{cases}$$

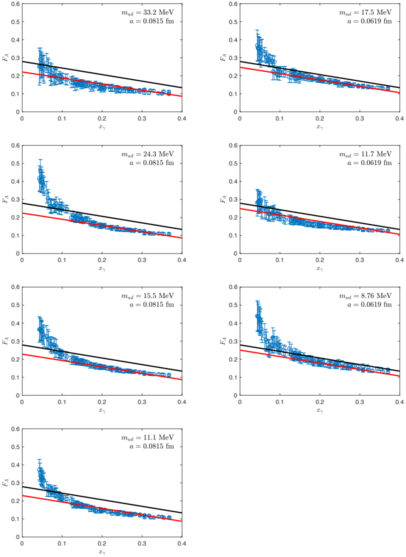


a.portelli at lattice19



c.kane, c.lehner, s.meinel, a.soni, arXiv:1907.00279





- **we are not alone in the world**:
- **the RBC/UKQCD collaboration** started a project to compute radiative corrections to $P \mapsto \ell\bar{\nu}_\ell(\gamma)$ and the real-photon decay $P \mapsto \ell\bar{\nu}_\ell\gamma$, physical results will be available soon
- we are finalizing the phenomenological analysis of our data on $P \mapsto \ell\bar{\nu}_\ell\gamma$, a paper on the subject will be available very soon!

- **QED radiative corrections are phenomenologically relevant for many observables** and have to be taken into account, possibly with the required non-perturbative accuracy

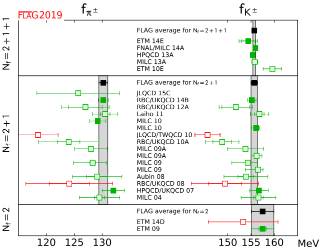
- in fact, if the precision is already **at the percent level is useless to improve** the accuracy of **lattice calculations without QED**

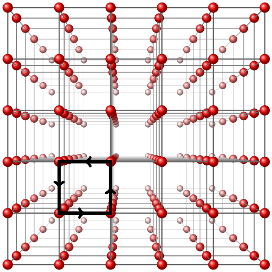
- including QED radiative corrections in a lattice simulation is a very **hard problem** because of

  - **soft divergences** and, more generally, **large finite volume effects**

  - non-perturbative corrections have to be extracted from **euclidean correlators**

  - it is highly non trivial to probe electrically charged states in a local finite-volume formulation of the theory





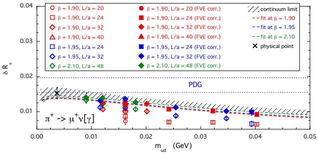
- the **RM123+SOTON collaboration** developed **a method to calculate QED radiative corrections to $\Gamma[P \to \ell\bar{\nu}(\gamma)]$**

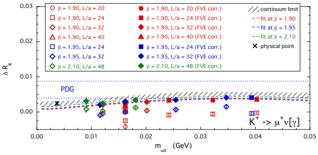
- with this method a $\log(L)$ **divergence is turned into a $1/L^2$ finite volume effect**

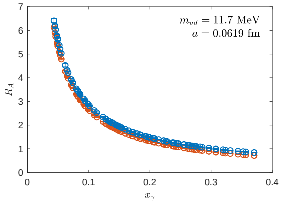
- the **RM123+SOTON collaboration** provided the **first non-perturbative results for the QED radiative corrections** to $\Gamma[K \to \mu\bar{\nu}_\mu(\gamma)]$ and $\Gamma[\pi \to \mu\bar{\nu}_\mu(\gamma)]$

- and is going to provide **soon phenomenologically relevant results for the radiative leptonic decays of π, K and $D_{(s)}$ mesons**

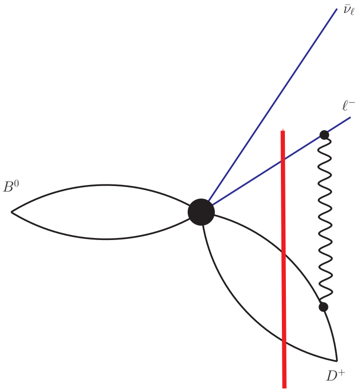
- **other collaborations started their work** on the subject and this will rapidly become a very active field of research







- the calculation of the **QED corrections to (radiative) leptonic decays in the case of B mesons doesn't present any conceptual issue**
- cutoff effects are the problem there but **strategies to cope with b-physics on the lattice exist and can be applied**
- **the problem is more challenging in the case of semileptonic decays** because, for generic kinematical configurations, the physical observable cannot be extracted from euclidean correlators by the leading exponential contributions
- nevertheless, **the RM123+SOTON method can be extended to the case of semileptonic decays**, we have already analyzed the problem in great detail
- the infrared divergence is again proportional to the leading order decay rate (obvious) and **the $O(1/L)$ corrections are again universal** although, as expected from Low's theorem, **their evaluation requires the knowledge of the derivatives of the form-factors $f_\pm(s_D)$ with respect to $s_D = (p_B - p_D)^2$**



backup material

- notice that $\Gamma_V(L)$ and $\Gamma_V^{pt}(L)$ are ultraviolet divergent in the Fermi theory
- the divergence can be reabsorbed into a renormalization of G_F, both in the full theory and in the point-like effective theory
- we have analyzed the renormalization of the four-fermion weak operator on the lattice in details and calculated non-perturbatively the renormalization constants in the RI-MOM scheme
- we have then matched the non-perturbative results to the so-called W-regularization at $O(\alpha)$ (**a.sirlin, NPB 196 (1982); e.braaten and c.s.li PRD 42 (1990)**)

$$\frac{1}{k^2} \mapsto \frac{1}{k^2} - \frac{1}{k^2 + m_W^2} \; , \qquad H_W = \frac{G_F V_{CKM}}{\sqrt{2}} \left\{ 1 + \frac{\alpha}{\pi} \log \frac{m_Z}{m_W} \right\} O_1^{\text{W-reg}} \; ,$$

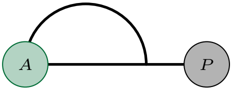
$$O_1^{\text{W-reg}} = \sum_{i=1}^{5} Z_{1i} O_i^{latt}(a)$$

- indeed, this is the scheme conventionally used to extract G_F from the muon decay

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} \left[1 - \frac{8 m_e^2}{m_\mu^2} \right] \left[1 + \frac{\alpha}{2\pi} \left(\frac{25}{4} - \pi^2 \right) \right]$$

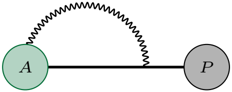
- power-law finite volume effects arise when internal states can go on-shell, e.g.

$$\boldsymbol{k} = \frac{2\pi \boldsymbol{n} + \boldsymbol{\theta}}{L} \;,$$



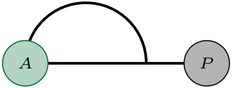
$$\Delta\mathcal{O}(p, L) = \mathcal{O}(p, L) - \mathcal{O}(p, \infty)$$

$$= \left(\frac{1}{L^3} \sum_{\boldsymbol{k}} - \int \frac{d^3k}{(2\pi)^3} \right) \int \frac{dk^0}{2\pi} f_{\mathcal{O}}(p, k)$$



- power-law finite volume effects arise when internal states can go on-shell, e.g.

$$\boldsymbol{k} = \frac{2\pi \boldsymbol{n} + \boldsymbol{\theta}}{L}\ , \qquad \alpha > 0\ ,$$



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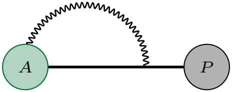

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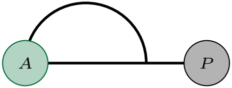

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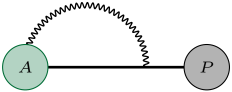

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$$= \frac{g_{\mathcal{O}}(p)\xi(p, \boldsymbol{\theta})}{L^{3-\alpha}} + O\left(\frac{1}{L^{4-\alpha}} \right) \,,$$

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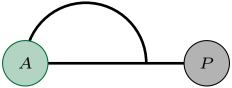
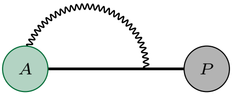
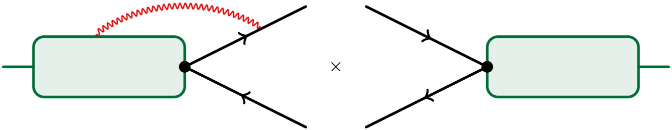




$$\xi(p, \boldsymbol{\theta}) = \left\{ \sum_{\boldsymbol{n}} - \int \frac{d^3n}{(2\pi)^3} \right\} \frac{1}{(2\pi\boldsymbol{n}\cdot\boldsymbol{p} + \boldsymbol{\theta}\cdot\boldsymbol{p})^\alpha}$$



$\sim \frac{1}{2p\cdot k + k^2}$

- **the key point of our method is the universality of infrared divergences**

- to see how this works, let's consider the contribution to the decay rate coming from the diagrams shown in the figure

$$\Gamma_V^{P\ell} = \int \frac{d^4k}{(2\pi)^4}\, H^{\alpha\mu}(k,p)\, \frac{1}{k^2}\, \frac{\mathcal{L}_{\alpha\mu}(k)}{2p_\ell \cdot k + k^2}$$

- infrared divergences (and power-law finite volume effects) come from the singularity at $k^2 = 0$ of the integrand

- the tensor $\mathcal{L}_{\alpha\mu}$ is a regular function, it contains the numerator of the lepton propagator and the appropriate normalization factors

$$\mathcal{L}_{\alpha\mu}(k) \equiv \mathcal{L}_{\alpha\mu}(k, p_\nu, p_\ell) = O(1)$$

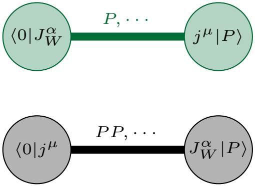
- the hadronic tensor is a QCD quantity

$$H^{\alpha\mu}(k,p) = i\int d^4x\, e^{ik\cdot x}\, T\langle 0|\, J_W^\alpha(0)\, j^\mu(x)\, |P\rangle$$

- it satisfies the WIs coming from QED gauge invariance, e.g.

$$k_\mu\, H^{\alpha\mu}(k,p) = -f_P\, p^\alpha \ ,$$

- and, given the kinematics of the process, it is singular *only* at the single-meson pole



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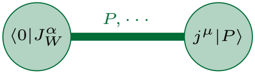

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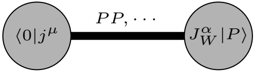

- and, given the kinematics of the process, it is singular *only* at the single-meson pole

- the singularity can be isolated by considering the *point-like* tensor, built in such a way to satisfy the same WIs of the full theory

$$H_{pt}^{\alpha\mu}(k,p) = f_P \left\{ \delta^{\alpha\mu} - \frac{(p+k)^{\alpha}\,(2p+k)^{\mu}}{2p\cdot k + k^2} \right\} ,$$

$$H_{SD}^{\alpha\mu}(k,p) = H^{\alpha\mu}(k,p) - H_{pt}^{\alpha\mu}(k,p)\ , \qquad k_\mu\, H_{pt}^{\alpha\mu}(k,p) = -f_P\, p^{\alpha}\ , \qquad k_\mu\, H_{SD}^{\alpha\mu}(k,p) = 0$$

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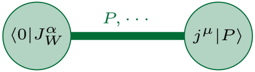
$$H^{\alpha\mu}(k,p) = i \int d^4x \, e^{ik\cdot x} \, T \langle 0 | \, J_W^{\alpha}(0) \, j^{\mu}(x) \, | P \rangle$$



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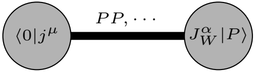

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- the structure dependent contributions are regular and, since there is no constant two-index tensor orthogonal to k,

$$H_{SD}^{\alpha\mu}(k,p) = \left(p\cdot k \, \delta^{\alpha\mu} - k^{\alpha} p^{\mu} \right) F_A + \epsilon^{\alpha\mu\rho\sigma} p_{\rho} k_{\sigma} F_V + \cdots \; = O(k)$$

- at $O(e^2)$ with massive charged particles, singularities arise only at

$$k^2 = (\pm i|\boldsymbol{k}|)^2 + \boldsymbol{k}^2 = 0$$

- the blobs on the right are QCD vertexes, e.g.

$$\Delta(p+k)\Gamma^\mu(p,k)\Delta(p) =$$

$$iN(p) \int d^4x d^4y e^{-ipy-ikx} T\langle 0|P(y)j^\mu(x)P^\dagger(0)|0\rangle \ ,$$

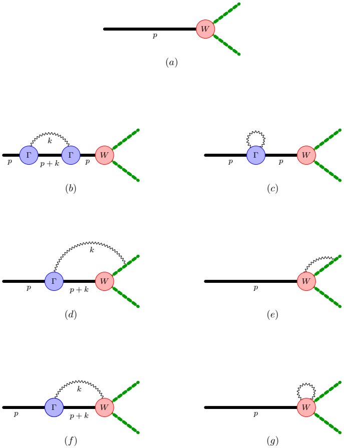
$$\Delta(p) = N(p) \int d^4y e^{-ipy} T\langle 0|P(y)P^\dagger(0)|0\rangle \ ,$$

$$N^{-1}(p) = |\langle P(\boldsymbol{p})|P^\dagger(0)|0\rangle|^2 \ ,$$

- gauge WIs constrain the first two terms in the expansion, e.g.

$$k_\mu \Gamma^\mu(p,k) = \Delta^{-1}(p+k) - \Delta^{-1}(p) \ ,$$

$$\Gamma^\mu(p,k) = 2p^\mu + k^\mu + O(k^2)$$



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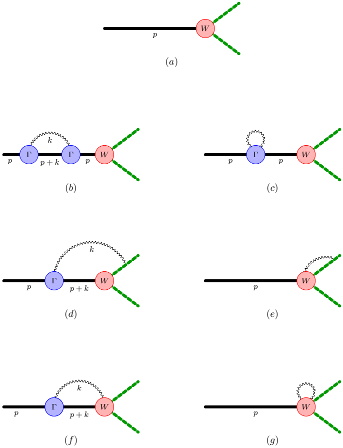


the first two terms in $1/L$ are universal!!

$$\left\{\Gamma_V - \Gamma_V^{pt}\right\}(L) = \Gamma_V^{SD}(\infty) + O\left(\frac{1}{L^2}\right)$$

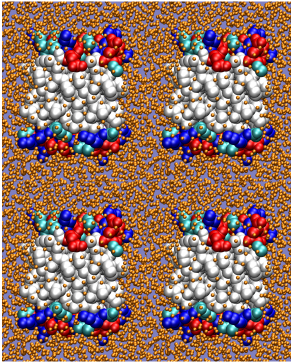
- *it is impossible to have a net electric charge in a periodic box!*

- classically, this is a consequence of Gauss's law

$$S = \int_{L^3} d^4x \left\{ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}_f \left(\gamma_\mu D_\mu^f + m_f \right) \psi_f \right\}$$

$$\partial_k \underbrace{F_{0k}(x)}_{E_k(x)} - \underbrace{ieq_f \bar{\psi}_f \gamma_0 \psi_f(x)}_{e\rho(x)} = 0$$

$$Q = \int_{L^3} d^3x\, \rho(x) = \frac{1}{e} \int_{L^3} d^3x\, \partial_k E_k(x) = 0$$



- this is well known in the field of classical simulations, charged molecules are usually studied by *adding* an appropriate number of "counter-ions" to have a neutral system

- notice: if the volume is not large enough these counter-ions *affect* the low-energy dynamics of the system

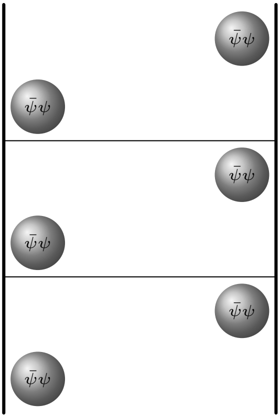
- *it is impossible to have a net electric charge in a periodic box!*

- at the quantum level, Gauss's law becomes the generator of local gauge transformations

$$\int_{\text{pbc in space}} \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \mathcal{D}A_\mu\, e^{-S}$$

$$= \langle \Phi |\, e^{-TH} \underbrace{\int_{\text{pbc in space}} \mathcal{D}\alpha\, e^{i\int_{L^3} d^3x\, \alpha(\boldsymbol{x})\, [\partial_k E_k - e\rho](\boldsymbol{x})}}_{P_g} |\Psi\rangle \;,$$

$$[H, \partial_k E_k(\boldsymbol{x}) - e\rho(\boldsymbol{x})] = 0$$



- a physical state is invariant under local gauge transformations and *necessarily* neutral

$$P_g^2 = P_g \;, \qquad |\Psi\rangle_{\text{phys}} = P_g\, |\Psi\rangle \;, \qquad Q\, |\Psi\rangle_{\text{phys}} \;=\; \frac{1}{e}\left\{\int_{L^3} d^3x\, \partial_k E_k(\boldsymbol{x})\right\} |\Psi\rangle_{\text{phys}} \;=\; 0$$

- *it is impossible to have a net electric charge in a periodic box!*

- one may think to overcome this problem by gauge fixing and to extract, say, the electron mass from the correlator

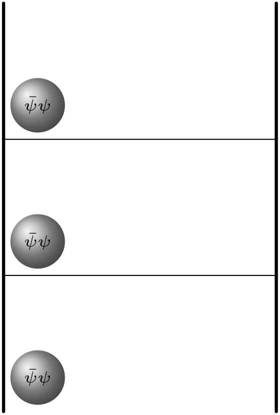
$$\langle \psi(x)\, \bar{\psi}(0) \rangle$$

- notice that after gauge fixing the theory is still invariant under *global* gauge transformations (electric charge is conserved)

- moreover, *large* gauge transformations survive gauge fixing ($n \in \mathbb{Z}^4$)

$$\psi(x) \mapsto e^{2\pi i \sum_\mu \frac{x_\mu n_\mu}{L_\mu}} \psi(x) \ ,$$

$$A_\mu(x) \mapsto A_\mu(x) + \frac{2\pi n_\mu}{L_\mu}$$



- as a consequence the correlator vanishes unless the two operators are in the same point,

$$\psi(x)\, \bar{\psi}(0) \mapsto e^{2\pi i \sum_\mu \frac{x_\mu n_\mu}{L_\mu}} \psi(x)\, \bar{\psi}(0) \ , \qquad \langle \psi(x)\, \bar{\psi}(0) \rangle = 0 \ , \qquad x \neq 0$$

- in order to study charged particles in a periodic box it has been suggested long ago (**duncan et al. 96**) to *quench* (a set of) the zero momentum modes of the gauge field, for example

$$\langle \mathcal{O} \rangle = \int_{\text{pbc in space}} \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \mathcal{D}A_\mu \prod_\mu \delta\left\{ \int_{TL^3} d^4x\, A_\mu(x) \right\} e^{-S}\, \mathcal{O}$$

- by using this procedure one is also *quenching* large gauge transformations that are no longer a symmetry and charged particles can propagate

- the *assumption* is that the induced modifications on the infrared dynamics of the theory should disappear once the infinite volume limit is taken

- the point to note is that the resulting finite volume theory, although it may admit an hamiltonian description, is *non-local*
  m.hayakawa, s.uno Prog.Theor.Phys. 120 (2008)

$$\text{QED}_\text{L}: \quad \prod_{\mu,t} \delta\left\{ \int_{L^3} d^3x\, A_\mu(t,\boldsymbol{x}) \right\} \quad \mapsto \quad \int_{\text{pbc in space}} \mathcal{D}\alpha_\mu(t)\, e^{-\int_{L^3} d^4x\, \alpha_\mu(t)\, A_\mu(t,\boldsymbol{x})}$$

b.lucini, a.patella, a.ramos, n.t, JHEP 1602(2016)

- consider $C^\star$ boundary conditions (first suggested by **wise and polley 91**)

$$\psi_f(x + L\boldsymbol{k}) = C^{-1}\bar{\psi}_f^T(x)$$

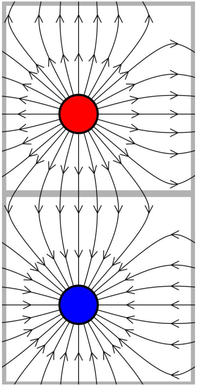
$$\bar{\psi}_f(x + L\boldsymbol{k}) = -\psi_f^T(x)C$$

$$A_\mu(x + L\boldsymbol{k}) = -A_\mu(x) \ , \qquad U_\mu(x + L\boldsymbol{k}) = U_\mu^*(x) \ ,$$

- the gauge field is anti-periodic ($|\boldsymbol{p}|_{min} = \pi/L$): *no zero modes by construction!*

- this means no large gauge transformations and

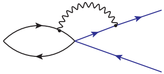
$$Q = \int_{L^3} d^3x \, \rho(x) \ = \ \frac{1}{e}\int_{L^3} d^3x \, \partial_k E_k(x) \neq 0$$



- a fully gauge invariant formulation is possible: for example the electrostatic potential is unique with anti-periodic boundary conditions

$$\partial_k\partial_k\Phi(\boldsymbol{x}) = \delta^3(\boldsymbol{x}) \ , \qquad \Phi(\boldsymbol{x} + L\boldsymbol{k}) = -\Phi(\boldsymbol{x})$$

- at $O(\alpha)$ the systematics associated with the quenching of the zero modes can be understood; this is what we did in the applications described so far; for example,



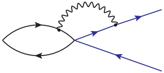
$$= \frac{1}{L^3}\sum_{\boldsymbol{k}} \int^{1/a} \frac{dk^0}{2\pi}\,\frac{1-\delta_{\boldsymbol{k},\mathbf{0}}}{k^2}\, H^{\mu\nu}(k)\, L_{\mu\nu}(k) \ ,$$

$$H^{\mu\nu}(k) = \int d^4x\, e^{ikx}\, T\langle 0|j^\mu(x)\, J^\nu_W(0)|P(p)\rangle \ ,$$

$$L^{\mu\nu}(k) = \bar{v}_{\nu_\ell}\gamma^\nu \frac{1}{i(\not{p}_\ell + \not{k}) + m_\ell}\gamma^\mu u_\ell$$

- the ultraviolet behaviour of this object can be understood by taking

$$j^\mu(x)\, J^\nu_W(0) \sim \frac{O^{\mu\nu}(0)}{x^3} \ , \qquad H^{\mu\nu}(k) \sim \frac{1}{k} \ ,$$



$$\sim \frac{1}{L^3}\sum_{\boldsymbol{k}} \int^{1/a} \frac{dk^0}{2\pi}\,\frac{1-\delta_{\boldsymbol{k},\mathbf{0}}}{k^4}$$

- in the local theory the diagram has a logarithmic divergence (absent with a propagating W) that renormalizes G_F
- the effect of the zero-modes subtraction is a term

$$\frac{1}{L^3}\int^{1/a} \frac{dk^0}{(k^0)^4} \sim \frac{a^3}{L^3}$$

i.e. no new ultraviolet divergences but tricky interplay between cutoff and finite volume effects!