Power-enhanced QED corrections to $B_q \to \mu\bar{\mu}$

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With M.Beneke and R.Szafron,

arXiv:1708.09157 and 1907.SOOON

GDR Workshop
"QED corrections to (semi-)leptonic *B* decays"
LPNHE, Paris
July 9, 2019

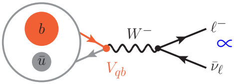
Outline

- Motivation for $B_q \to \ell\bar{\ell}$
- Introduction to EW and QED corrections
- SCET_1 and SCET_2 below μ_b
- Definition of soft matrix elements & Factorization
- Including ultra-soft radiation
- Phenomenological implications

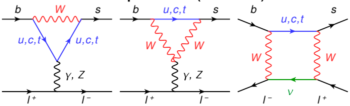
Motivation for $B_q \to \ell\bar{\ell}$

Motivation to study $B_q \to \ell\bar{\ell}$ and $B_u \to \ell\bar{\nu}_\ell$

- **Test CKM in SM** at tree- (CC) and loop-level (FCNC)



$\propto |V_{ub}|^2$



$\propto |V_{tb} V_{tq}^*|^2$

- **Helicity suppression of SM** $\Rightarrow$ sensitive to NP (pseudo-) scalar interactions

$$Br(B_q \to \ell\bar{\ell}) \propto |V_{tb} V_{tq}^*|^2 \; f_{B_q}^2 \times \left\{ \left| \frac{2m_\ell}{m_{B_q}} (C_{10} - C'_{10}) + (C_P - C'_P) \right|^2 + \beta_\ell^2 \, |C_S - C'_S|^2 \right\}$$

- **Hadronic uncertainty** $\Rightarrow$ from decay constant (at LO in QED)

$$\langle 0 | \bar{q} \gamma_\mu \gamma_5 b | \overline{B}_q(p) \rangle = i \, f_{B_q} p_\mu$$

$\Rightarrow$ from lattice in future $\delta f_{B_q} \lesssim 0.5\%$

$f_{B_u} = (189.4 \pm 1.4)$ MeV $\qquad f_{B_s} = (230.7 \pm 1.2)$ MeV $\qquad$ [FNAL/MILC 1712.09262]

$\Rightarrow$ theoretical control of $\delta Br \sim 1\%$ possible

!!! only other compareable precision in flavor: $Br(K^+ \to \pi^+ \nu\bar{\nu})$ (NA62), $Br(K_L \to \pi^0 \nu\bar{\nu})$ (KOTO), $\Delta M_{d,s}$ (lattice)

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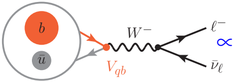

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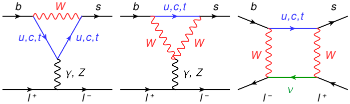

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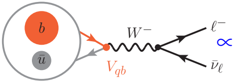

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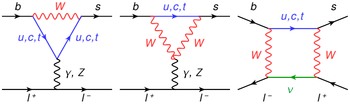

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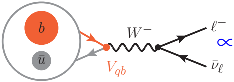

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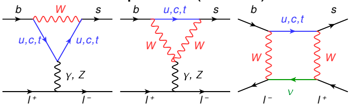

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Need to include also small effects, like QED etc.

Analysing NP in $B_s \to \mu\bar{\mu}$ via time-dependence

3 CP asymmetries

$$\frac{\Gamma\left[B_s(t) \to \mu_\lambda \bar{\mu}_\lambda\right] - \Gamma\left[\overline{B}_s(t) \to \mu_\lambda \bar{\mu}_\lambda\right]}{\Gamma\left[B_s(t) \to \mu_\lambda \bar{\mu}_\lambda\right] + \Gamma\left[\overline{B}_s(t) \to \mu_\lambda \bar{\mu}_\lambda\right]} = \frac{\mathcal{C}^\lambda \cos(\Delta M_s t) + \mathcal{S}^\lambda \sin(\Delta M_s t)}{\cosh(y_s t/\tau_{B_s}) + \mathcal{A}^\lambda_{\Delta\Gamma} \sinh(y_s t/\tau_{B_s})}$$

- $\mathcal{A}_{\Delta\Gamma}$ ... without flavor tagging
  $\mathcal{S}$ ... requires flavor-tagging
  $\mathcal{C}^\lambda$ ... requires helicity of leptons

$$\left|\mathcal{C}^\lambda\right|^2 + \left|\mathcal{S}^\lambda\right|^2 + \left|\mathcal{A}^\lambda_{\Delta\Gamma}\right|^2 = 1$$

- in **SM "clean" observables** (at LO QED): $\mathcal{A}_{\Delta\Gamma} = 1$ $\quad \mathcal{S} = 0$ $\quad \mathcal{C}^\lambda = 0$
  $\Rightarrow$ QED corrections are SM background to NP contributions

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Distinguishing NP

[Fleischer/Galarraga Espinosa/Jaarsma/Tetlalmatzi-Xolocotzi 1709.04735]

- even measurement of $\text{sgn}(\mathcal{C}^\lambda)$ can reduce degeneracy

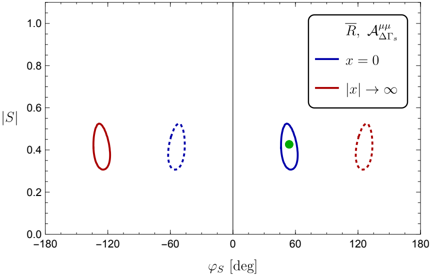
  Benchmark measurement

  $\mathcal{A}_{\Delta\Gamma} = +0.58 \pm 0.20$
  $\mathcal{S} = -0.80 \pm 0.20$

  $\to$ 4 solutions from Br and $\mathcal{A}_{\Delta\Gamma}$

  dashed: ruled out by $\mathcal{S}$

  blue: ruled out by $\text{sgn}(\mathcal{C}^\lambda)$



Experimental measurement $B_q \to \mu\bar{\mu}$

Exp	Run	[fb^{-1}]	Ref.
LHCb	1	3	[1307.5024]
LHCb	1+2	3 + 1.8	[1703.05747]
CMS	1	25	[1307.5025]
CMS+LHCb	1	25 + 3	[1411.4413]
ATLAS	1	25	[1604.04263]
ATLAS	1+2	25 + 36.2	[1812.03017]

single best measurement from LHCb

$$\overline{\mathrm{Br}}^{(0)}_{s\mu} \equiv \overline{Br}(B_s \to \mu\bar{\mu}) = (3.0^{+0.7}_{-0.6}) \times 10^{-9}$$

$$\overline{\mathrm{Br}}^{(0)}_{d\mu} \equiv \overline{Br}(B_d \to \mu\bar{\mu}) < 3.4 \times 10^{-10} \text{ @ } 95\%$$

$$\mathcal{A}_{\Delta\Gamma}(B_s \to \mu\bar{\mu}) = 8.24 \pm 10.72$$

LHCb → mass-eigenstate rate asymmetry

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Combination of LHCb, CMS and ATLAS

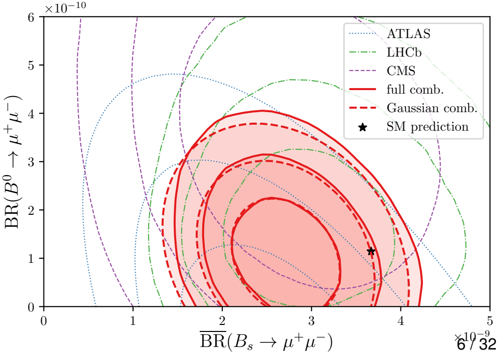
[Aebischer et al. 1903.10434]

Depending on treatment of $\overline{\text{Br}}^{(0)}_{d\mu}$

$$\overline{\text{Br}}^{(0)}_{s\mu} = \begin{cases} (2.67^{+0.45}_{-0.35}) \times 10^{-9} & \text{profiled} \\ (2.65^{+0.46}_{-0.33}) \times 10^{-9} & \text{SM-like} \end{cases}$$

Depending on treatment of $\overline{\text{Br}}^{(0)}_{s\mu}$

$$\overline{\text{Br}}^{(0)}_{d\mu} = \begin{cases} (1.00^{+0.86}_{-0.57}) \times 10^{-10} & \text{profiled} \\ (0.57^{+0.86}_{-0.36}) \times 10^{-10} & \text{SM-like} \end{cases}$$



Experimental prospects

of events for LHCb

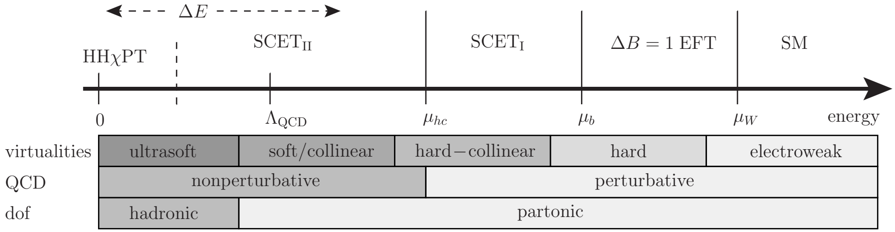
Events in channel	Run 1	Run 2	50/fb	300/fb
$B_s^0 \to \mu\bar{\mu}$	15	60	500	2 700
$B_s^0 \to \mu\bar{\mu}$ (3% tag-power)	—	—	—	80

[K. Petridis, @ Barchelona 2016; K. Alvarez Cartelle @ HL-LHC WS, CERN, Oct. 2017]

Exp	[fb^{-1}]	Prospects / Ref.	
		$B_s \to \mu\bar{\mu}$	
CMS	100	$\delta(Br) \sim 15\%$ error of SM	[CMS PAS FTR-13-022]
	3000	$\delta(Br) \sim 12\%$ error of SM	
LHCb	50	$\sigma(Br) \sim 0.15 \times 10^{-9}$ ($\approx 4\%$ of SM) (only stat. err)	[LHCb arXiv:1208.3355]
	300	$\sigma(Br) \sim 0.16 \times 10^{-9}$ ($\approx 4\%$ of SM) (for curr. syst. err = f_s/f_d (5.8%) and norm. mode (3%))	
		$\sigma(Br) \sim 0.13 \times 10^{-9}$ **($\lesssim 4\%$ of SM)** (for 3% syst. err)	
		$\delta(\tau_{\text{eff}}) \sim 2\%$, $\sigma(\mathcal{S}) \sim 0.2$	[A. Puig @ LHCb Upgrade WS, LAPP, Annecy, 03/2018]
ATLAS	Run 2	$\sigma(Br) \sim 0.83 \times 10^{-9}$ (stat + syst err)	
	1500	$\sigma(Br) \sim 0.47 \times 10^{-9}$ (stat + syst err)	[ATLAS ATL-PHYS-PUB-2019-005]
		$B_d \to \mu\bar{\mu}$	
CMS	100	$\delta(Br) \sim 66\%$ error of SM	[CMS PAS FTR-13-022]
	3000	$\delta(Br) \sim 18\%$ error of SM	
LHCb	300	**$\delta(R_{d/s}) \sim 10\%$** $R_{d/s} \equiv Br(B_d \to \mu\bar{\mu})/Br(B_s \to \mu\bar{\mu})$	
ATLAS	1500	$\sigma(Br) \sim 0.31 \times 10^{-10}$ (stat + syst err)	[ATLAS ATL-PHYS-PUB-2019-005]

Introduction to
EW and QED corrections

Scales in SM for $B_q \to \ell\bar{\ell}$



- **decoupling** of W, Z, H-boson and t-quark **@ EW** (electroweak) scale μ_W [see talk Mikolaj Misiak]

  $\Rightarrow$ @ NLO EW and NNLO QCD

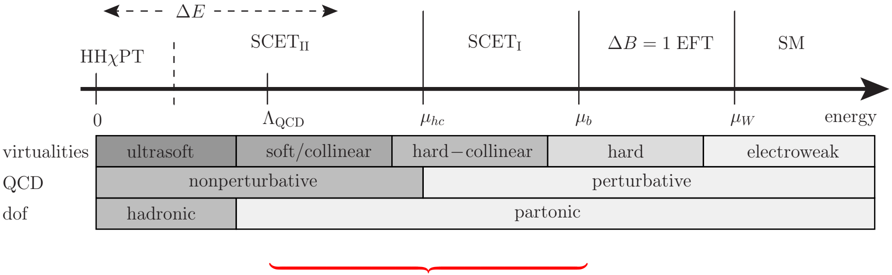
  $\to$ residual μ_W dependence of C_{10} from QCD < 0.1% and EW < 0.2%

  [Hermann/Misiak/Steinhauser 1311.1347, CB/Gorbahn/Stamou 1311.1348]

- **RG** evolution in $\Delta B = 1$ EFT **from $\mu_W \to \mu_b$**

  $\Rightarrow$ @ NLO QED and NNLO QCD

Scales in SM for $B_q \to \ell\bar{\ell}$



- **fixed-order** calculation of **NLO QED** corrections using expansion in $\lambda = \sqrt{m_\mu/m_b}$ [Beneke/CB/Szafron 1708.09157]

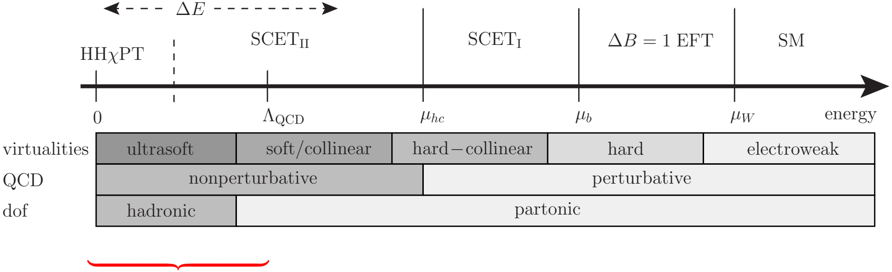
  $\Rightarrow$ leading contribution scales as $1/\lambda$ → power-enhancement

- explicite **2-step-matching on $\mathbf{SCET_1}$ and $\mathbf{SCET_2}$** for contribution of $O_{9,10}$, including resummation presented here [Beneke/CB/Szafron 1907.XXXXX]

  $\Rightarrow$ factorization into hadronic and leptonic parts

  $\Rightarrow$ definition of **process-dependent** hadronic matrix elements due to presence of QED

Scales in SM for $B_q \to \ell\bar{\ell}$



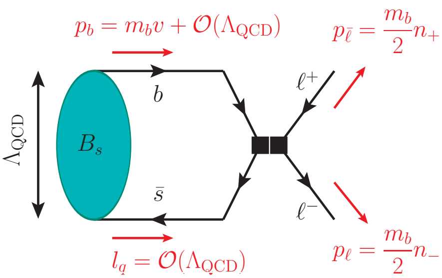
- initial and final state (FSR) radiation (including virtual crr's) of ultrasoft photons $E_{X_s} < \Delta E < \mu_c \sim \Lambda_{QCD}$

- **exponentiation of FSR** using "heavy-lepton" EFT here [Beneke/CB/Szafron 1907.XXXXX]

  $\Rightarrow$ recover soft-photon approximation from virtual crr's from scales μ_b down to ΔE [Buras/Girrbach/Guadagnoli/Isidori 1208.0934]

$B_q \to \ell\bar{\ell}$ Kinematics

External kinematics in B_s rest frame



- **soft** b- and s-quarks bound in B_s with

$$\lambda_s \equiv \sqrt{\frac{\Lambda_{\rm QCD}}{m_b}} \ll 1$$

$\Lambda_{\rm QCD} \approx (200 \ldots 400)$ MeV

- **(anti-) collinear** energetic leptons

$$p_\ell^2 = m_\ell^2 \quad \ll \quad E_\ell^2 \approx m_{B_s}^2/4$$

$$\lambda_\ell \equiv \sqrt{\frac{m_\ell}{m_b}} \ll 1$$

$m_e \approx 0.5$ MeV, $m_\mu \approx 105$ MeV

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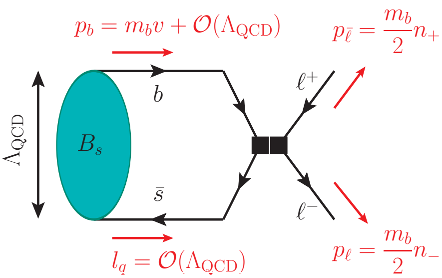

Can have simple case $\lambda_s \approx \lambda_\mu$ for $\ell = \mu$

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$m_e \approx 0.5$ MeV, $m_\mu \approx 105$ MeV

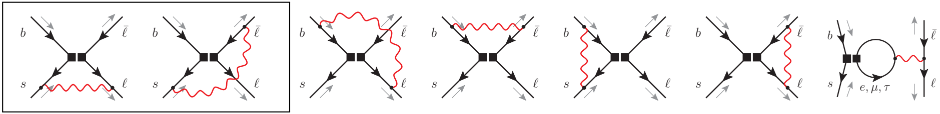
(compared to $\lambda_e \ll \lambda_s$ for $\ell = e$)

$\Rightarrow \Lambda \simeq \{\Lambda_{\rm QCD}, m_\mu\}$	kinematic invariants	virtuality	scale	EFT
	$p_b^2 \sim p_\ell \cdot p_{\bar{\ell}} \sim p_b \cdot p_{\ell,\bar{\ell}}$	$\sim m_b^2$	hard	QED
$\lambda \equiv \lambda_s \approx \lambda_\mu$	$p_b \cdot l_q \sim l_q \cdot p_{\ell,\bar{\ell}}$	$\sim m_b \Lambda$	**hard-collinear**	SCET_1
	$l_q^2 \sim p_\ell^2 \sim p_{\bar{\ell}}^2$	$\sim \Lambda^2$	soft/collinear	SCET_2

2-step-matching: full QED $\to$ SCET_1 $\to$ SCET_2 **Soft Collinear EFT = SCET**

Power-enhanced contribution in full QED

Example $O_{9,10}$: NLO QED corrections to $b\bar{s} \to \mu\bar{\mu}$ @ scale μ_b



- calculate diagrams **fully analytical** (package X [Patel 1612.00009])
- leading term in expansion in λ scales as $\sim \lambda^{-2} = m_b/\Lambda$ $\Leftarrow$ power-enhanced

  $\Rightarrow$ comes only from diag's 1) and 2)

  $\to$ result in [Beneke/CB/Szafron 1708.09152]

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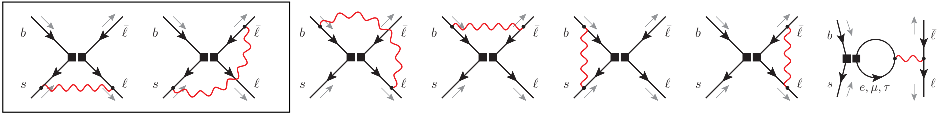

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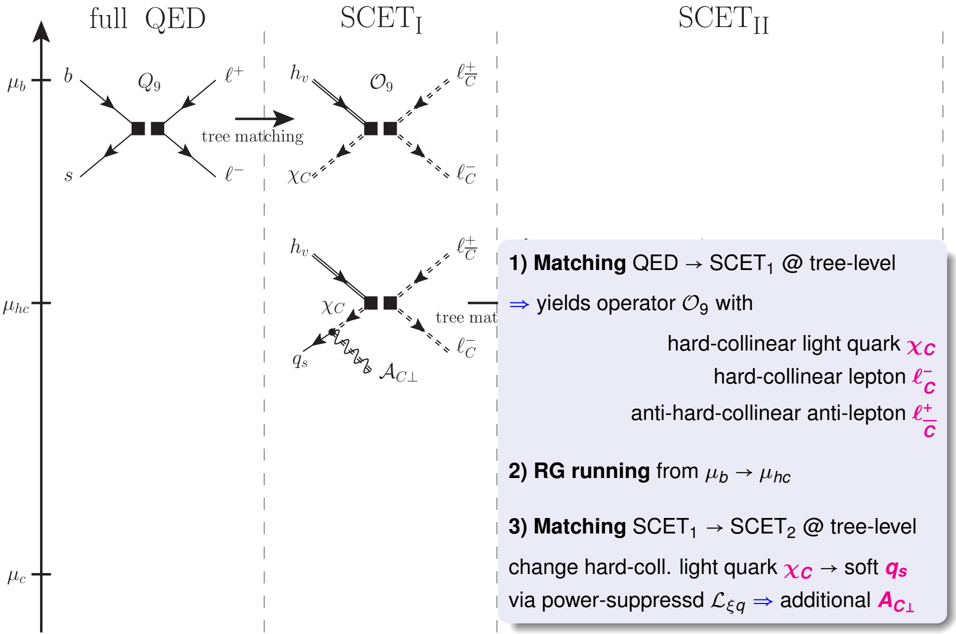
Analyse using **Method of Regions (MoR)** [Beneke/Smirnov hep-ph/9711391]

- can reproduce with MoR the full-analytic + λ-expansion result
  $\Rightarrow$ provides guidance towards construction of SCET
- only diag's 1) and 2) contain hard-collinear region
  $\Rightarrow$ **power-enhancement arises from $\text{SCET}_1 \to \text{SCET}_2$** [see talk Robert Szafron]

$\Rightarrow$ Also done for O_7
$\to$ additional need for analytic regulator when doing MoR

SCET_1 and SCET_2 below μ_b

SCET roadmap to power-enhanced contr'n from $\mathcal{O}_{9,10}$



1) Matching QED → SCET$_1$ @ tree-level

⇒ yields operator $\mathcal{O}_9$ with

hard-collinear light quark χ_C

hard-collinear lepton ℓ^-_C

anti-hard-collinear anti-lepton $\ell^+_{\bar{C}}$

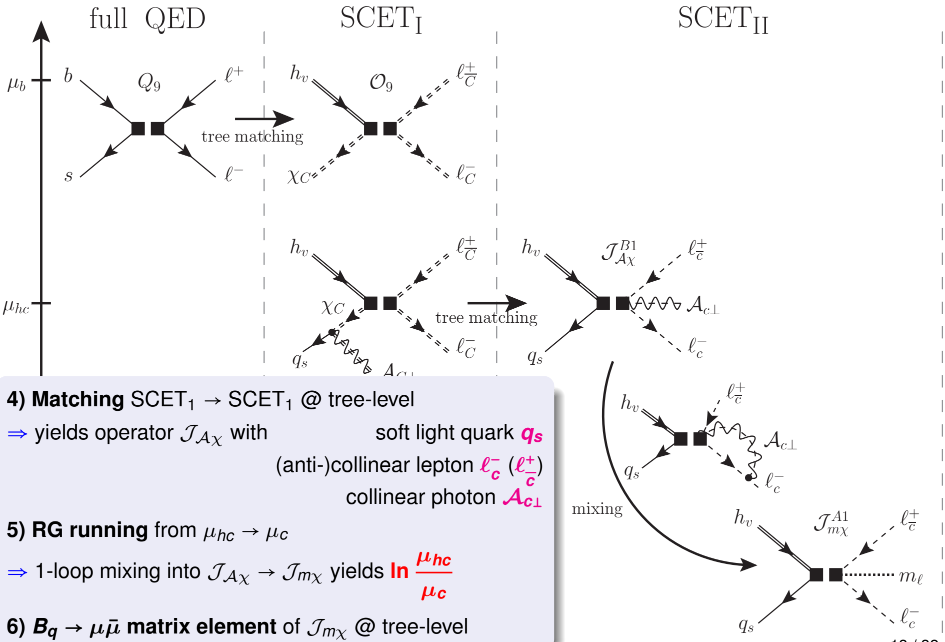
2) RG running from $\mu_b \to \mu_{hc}$

3) Matching SCET$_1$ → SCET$_2$ @ tree-level

change hard-coll. light quark χ_C → soft q_s

via power-suppressd $\mathcal{L}_{\xi q}$ ⇒ additional $A_{C\perp}$

SCET roadmap to power-enhanced contr'n from $O_{9,10}$



4) **Matching** SCET$_1$ → SCET$_1$ @ tree-level

⇒ yields operator $\mathcal{J}_{\mathcal{A}\chi}$ with soft light quark q_s

(anti-)collinear lepton ℓ_c^- ($\ell_{\bar{c}}^+$)

collinear photon $\mathcal{A}_{c\perp}$

5) **RG running** from $\mu_{hc} \to \mu_c$

⇒ 1-loop mixing into $\mathcal{J}_{\mathcal{A}\chi} \to \mathcal{J}_{m\chi}$ yields $\ln \frac{\mu_{hc}}{\mu_c}$

6) **$B_q \to \mu\bar{\mu}$ matrix element** of $\mathcal{J}_{m\chi}$ @ tree-level

SCET$_1$: Operators & Matching

SCET$_1$ Operators with hard-collinear and anti-collinear light quark

$$\widetilde{\mathcal{O}}_{9(10)}(s,t) = g^{\perp}_{\mu\nu}(i\varepsilon^{\perp}_{\mu\nu})\left[\overline{\chi}_C(sn_+)\,\gamma^{\mu}_{\perp} P_L\, h_v(0)\right]\left[\overline{\ell}_C(tn_+)\,\gamma^{\nu}_{\perp}\,\ell_{\overline{C}}(0)\right]$$

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$\boxed{\chi_C = W^{\dagger}_C \xi_C}$ contain Wilson lines with (anti-)collinear photons — analogous for $\overline{\ell}_C$ and $\ell_{\overline{C}}$

$$W_{\xi C(\overline{C})}(x) \equiv \mathrm{P}\exp\left[ie\,Q_{\xi}\int_{-\infty}^{0} ds\, n_{\pm}A_{C(\overline{C})}(x+sn_{\pm})\right] \qquad Q_{\xi} = \text{ electric charge of } \xi_C$$

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Fourier trafo from position space (s,t) → momentum fractions (u, $\overline{u} \equiv 1-u$)

$$\mathcal{O}_i(u) = n_+p_C \int \frac{dr}{2\pi}\, e^{-iu\,r(n_+p_C)}\,\widetilde{\mathcal{O}}_i(0,r)$$

- $n_+p_C = n_+(p_{\chi} + p_{\ell})$ total hc-momentum
- $u = \dfrac{n_+p_{\ell}}{n_+p_C}$ and $\overline{u} = \dfrac{n_+p_{\chi}}{n_+p_C}$

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- $u = \dfrac{n_+p_\ell}{n_+p_C}$ and $\bar{u} = \dfrac{n_+p_\chi}{n_+p_C}$

Matching condition:

$$\mathcal{N}\sum_{k=9,10} C_k(\mu_b)\, O_k = \sum_i \int_0^1 du\, H_i(u,\mu_b)\, \mathcal{O}_i(u) \qquad \text{tree-level @ } \mu_b$$

$\Rightarrow$ "hard functions": $H_i(u,\mu_b)$ = SCET$_1$ Wilson coefficients

$$H_9(u,\mu_b) = \mathcal{N}\, C_9^{\text{eff}}(u,\mu_b) \qquad H_{\overline{9}} = H_9$$

$$H_{10}(u,\mu_b) = \mathcal{N}\, C_{10}(\mu_b) \qquad H_{\overline{10}} = H_{10} \qquad \mathcal{N} \equiv \frac{4G_F}{\sqrt{2}} V_{tb}V^*_{ts}\frac{\alpha_{\text{em}}(\mu_b)}{4\pi}$$

SCET$_1$: Renormalization group (RG) running

RG equation of hard functions

$$\frac{dH_i(u,\mu)}{d\ln\mu} = \int du' \left[\Gamma^1_{\rm cusp}\underbrace{\left(\ln\frac{m_{B_q}}{\mu} - \frac{i\pi}{2}\right)}_{\downarrow}\delta(u-u') + \Gamma_i(u',u)\right] H_i(u',\mu)$$

- has **cusp anomalous part** $\propto \ln\frac{-(s_{\ell\bar\ell}+i0^+)}{\mu^2} = \ln\frac{m^2_{B_q}}{\mu^2} - i\pi \qquad s_{\ell\bar\ell} \equiv (n_+p_\ell)(n_-p_{\bar\ell}) = m^2_{B_q}$

$$\Gamma^1_{\rm cusp} = \Gamma_c + \Gamma_s \qquad \Gamma_c = \frac{\alpha_{\rm em}}{\pi}2Q_\ell^2 \qquad \text{only} \propto Q_\ell^2$$

$$\Gamma_s = \frac{\alpha_s}{\pi}C_F + \frac{\alpha_{\rm em}}{\pi}Q_q(2Q_\ell + Q_q) \qquad \propto Q_q$$

$\Rightarrow$ resummation of Sudakov-type "double"-log's

!!! **dilepton-inv. mass $s_{\ell\bar\ell}$** in log's $\Rightarrow$ B-meson mass m_{B_q} (NOT b-quark mass)

- **Solution** for leading double log-resummation $\Rightarrow$ two Sudakov factors

$$H_i(u,\mu_{hc}) = \underbrace{e^{S_{\ell,1}}}_{\Downarrow} \times e^{S_{q,1}} \times H_i(u,\mu_b) \qquad S_{\ell,1} = -\frac{\Gamma_c}{2}\ln^2\frac{\mu_b}{\mu_{hc}} \qquad S_{q,1} \propto \Gamma_s \ln^2\frac{\mu_b}{\mu_{hc}}$$

$\Rightarrow$ $e^{S_{\ell,1}}$ becomes part of "universal final-state radiation" à la Yennie-Frautschi-Suura

[Yennie/Frautschi/Suura Annals Phys. 13 (1961) 379]

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SCET$_2$: Matching & Operators

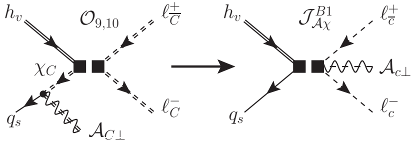
SCET$_1$ operators have (anti-)hard-collinear light quark

⇒ need to transform to soft light quark q_s via **power-suppr. SCET$_1$** interactions

$$\mathcal{L}^{(1)}_{\xi q} = \bar{q}_s(x_-)\, W^\dagger_{\xi C}(x) i\not{D}_\perp\, \xi_C(x) + \text{h.c.}$$

[Beneke/Chapovsky/Diehl/Feldmann hep-ph/0206152]

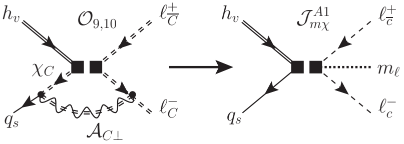
Tree-level



$$\widetilde{\mathcal{J}}_{\mathcal{A}\chi}(v,t) = \left[\bar{q}_s(vn_-)\, Y(vn_-,0)\, \frac{\not{n}_-}{2} P_L h_v(0)\right]$$
$$\times \left[Y^\dagger_+\, Y_-\right](0) \left[\bar{\ell}_c(0) (2\mathcal{A}\!\!\!/_{c\perp}(tn_+) P_R) \ell_{\bar{c}}(0)\right]$$

and $\widetilde{\mathcal{J}}_{\mathcal{A}\overline{\chi}}(v)$ from $\mathcal{O}_{\overline{9},\overline{10}}$

One-loop ⇐ in SCET$_1$ need power-suppr. $\mathcal{L}^{(1)}_{\xi m} = m_\ell \bar{\ell}_C \left[i\not{D}_{C\perp},\, (in_+ D_C)^{-1}\right] \frac{\not{n}_+}{2} \ell_C$



$$\widetilde{\mathcal{J}}_{m\chi}(v) = \left[\bar{q}_s(vn_-)\, Y(vn_-,0)\, \frac{\not{n}_-}{2} P_L h_v(0)\right]$$
$$\times \left[Y^\dagger_+\, Y_-\right](0) \left[\bar{\ell}_c(0) (4\, m_\ell P_R)\, \ell_{\bar{c}}(0)\right]$$

and $\widetilde{\mathcal{J}}_{m\overline{\chi}}(v)$ from $\mathcal{O}_{\overline{9},\overline{10}}$

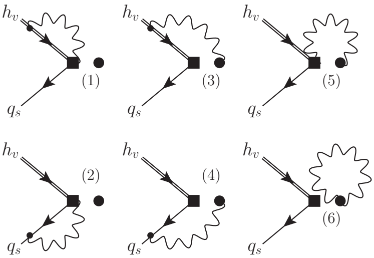
Decoupling trafo in SCET$_1$ generates "soft" Wilson lines

[Bauer/Pirjol/Stewart hep-ph/0109045]

$$Y_\pm(x) = \exp\left[ie\, Q_\ell \int_{-\infty}^0 ds\, n_\mp A_s(x + sn_\mp)\right] \qquad Y(x,y) = \exp\left[ie\, Q_q \int_y^x dz_\mu\, A^\mu_s(z)\right] \times \text{QCD}$$

SCET$_2$: Renormalization group (RG) running

Decoupling in SCET$_1$ $\Rightarrow$ in SCET$_2$ the soft & collinear sectors do not interact at leading power
$\Rightarrow$ soft and (anti-)collinear parts of the operator have separate RG's !!! but IR regularisation



soft photons

IR dependence of diag 6 not cancelled

$$\left[\bar{q}_s(vn_-)\,Y(vn_-,0)\,\frac{\not{n}_-}{2}\,P_L h_v(0)\right]\left[Y_+^\dagger\, Y_-\right](0)$$

$$\Downarrow$$

all-order subraction prescription **IR-finite**

$$\frac{\left[\bar{q}_s(vn_-)\,Y(vn_-,0)\,\frac{\not{n}_-}{2}\,P_L h_v(0)\right]\left[Y_+^\dagger\, Y_-\right](0)}{\langle 0|\left[Y_+^\dagger\, Y_-\right](0)|0\rangle}$$

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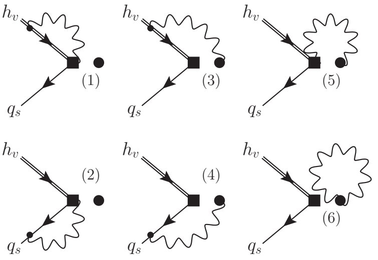

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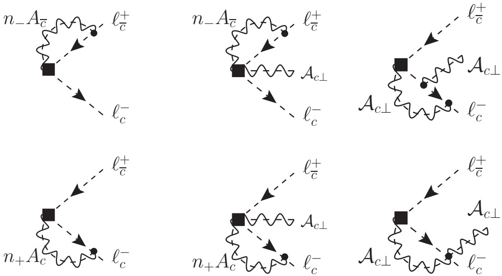

(anti-)collinear photons

$\mathcal{J}_{m\chi}$ $\mathcal{J}_{\mathcal{A}\chi}$

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$$\Big[\bar{q}_s(vn_-)\,Y(vn_-,0)\,\frac{\not{n}_-}{2}\,P_L h_v(0)\Big]\Big[Y_+^\dagger\,Y_-\Big](0)$$

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$\Rightarrow$ IR dep of diag 6 cancels with $n_- A_{\bar{c}}$ and $n_+ A_c$ diags

$\Rightarrow$ include in renormalization of (anti-)collinear sector

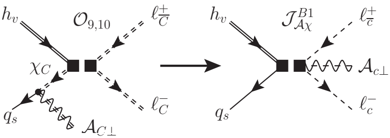
$$R_+ R_- \equiv \langle 0|\big[Y_+^\dagger\,Y_-\big](0)|0\rangle$$

$\Rightarrow$ similar to TMDPDF [Echevarria/Idilbi/Scimemi 1211.1947]

SCET_2: Matching

$\text{SCET}_1 \to \text{SCET}_2$ threshold corrections = **"jet functions"**

$$\mathcal{O}_i(u) = \int_0^\infty d\omega \left[J_m(u;\omega)\, \mathcal{J}_{m\chi}(\omega) + \int_0^1 dw\, J_A(u;\omega,w)\, \mathcal{J}_{\mathcal{A}\chi}(\omega,w) \right],$$



tree-level @ μ_{hc}: $(\omega \sim n_- l_q)$

$$J_A^{(0)}(u;\omega,w,\mu_{hc}) = J_{\overline{A}}^{(0)}(u;\omega,w,\mu_{hc}) = -\frac{Q_q}{\omega}\delta(u-w)$$

calculated also $J_m^{(1)}$

Schematically $\text{SCET}_1 \to \text{SCET}_2$

$H_9 \otimes_u \mathcal{O}_9 + H_{10} \otimes_u \mathcal{O}_{10}$	$\to$	$(H_9 + H_{10}) \otimes_u J_A \otimes_{\omega,w} \mathcal{J}_{\mathcal{A}\chi}$	collinear
$H_{\overline{9}} \otimes_u \mathcal{O}_{\overline{9}} + H_{\overline{10}} \otimes_u \mathcal{O}_{\overline{10}}$	$\to$	$(H_{\overline{9}} - H_{\overline{10}}) \otimes_u J_{\overline{A}} \otimes_{\omega,w} \mathcal{J}_{\mathcal{A}\overline{\chi}}$	anti-collinear

- since $H_{10} = H_{\overline{10}}$ $\Rightarrow$ cancel in sum of (coll + anti-coll) $\Rightarrow$ no C_{10} contribution

SCET$_2$: Matching

SCET$_1$ → SCET$_2$ threshold corrections = **"jet functions"**

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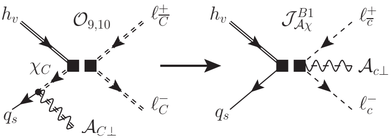

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Schematically SCET$_1$ → SCET$_2$

$H_9 \otimes_u \mathcal{O}_9 + H_{10} \otimes_u \mathcal{O}_{10}$	→	$(H_9 + H_{10}) \otimes_u J_A \otimes_{\omega,w} \mathcal{J}_{\mathcal{A}\chi}$	collinear
$H_{\bar{9}} \otimes_u \mathcal{O}_{\bar{9}} + H_{\overline{10}} \otimes_u \mathcal{O}_{\overline{10}}$	→	$(H_{\bar{9}} - H_{\overline{10}}) \otimes_u J_{\bar{A}} \otimes_{\omega,w} \mathcal{J}_{\mathcal{A}\bar{\chi}}$	anti-collinear

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Remark $B_u \to \mu\bar{\nu}_\mu$

- $B_u \to \mu\bar{\nu}_\mu$ has NO anti-collinear because $\bar{\nu}_\mu$ electrically neutral
- single operator $\propto C[\bar{u}\gamma^\mu P_L b][\bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell]$ $\quad \Rightarrow C_9 = C$ and $C_{10} = -C$

  $\Rightarrow H_9 + H_{10} = 0$

  $\Rightarrow$ no power-enhancement

Definition of matrix elements and Factorization

Process-depedent soft matrix elements

Define process-dependent $\Leftarrow$ because of additional soft Wilson lines

$$\frac{\langle 0|\bar{q}_s(vn_-)\,Y(vn_-,0)\,\not{n}_-\gamma_5\,h_v(0)\,Y_+^\dagger(0)\,Y_-(0)|\overline{B}_q(p)\rangle}{R_+\,R_-} \equiv -i\mathscr{F}_{B_q}\,m_{B_q}\int_0^\infty d\omega\,e^{-i\omega v}\,\Phi_+(\omega)$$

$$\frac{\langle 0|\bar{q}_s(0)\gamma^\mu\gamma_5\,h_v(0)\,Y_+^\dagger(0)\,Y_-(0)|\overline{B}_q(p)\rangle}{R_+\,R_-} \equiv +i\mathscr{F}_{B_q}\,m_{B_q}\,v^\mu$$

- $\Phi_+(\omega)$ . . . generalized ***B*-meson LCDA**
- $\mathscr{F}_{B_q}$ . . . generalized ***B*-meson decay constant** in HQET (static)

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- $\Phi_+(\omega)$ . . . generalized ***B*-meson LCDA**
- $\mathscr{F}_{B_q}$ . . . generalized ***B*-meson decay constant** in HQET (static)

Process-dependence in presence of QED

- B^0 decays into electrically neutral final states: $B_q \to \gamma\gamma$ or $B_q \to \nu_i\bar{\nu}_j$

$$\langle 0|\bar{q}_s(vn_-)Y(vn_-,0)\not{n}_-\gamma_5\, h_v(0)|\bar{B}_q(p)\rangle \equiv -i\mathscr{F}^0_{B_q} m_{B_q} \int_0^\infty d\omega\, e^{-i\omega v}\,\Phi^0_+(\omega)$$

- B^+ decay: $B_u \to \ell\bar{\nu}_\ell$

$$\frac{\langle 0|\bar{q}'_s(vn_-)\widetilde{Y}(vn_-,0)\not{n}_-\gamma_5\, h_v(0)\,Y_+(0)^\dagger|\bar{B}_u(p)\rangle}{\langle 0|Y_v(0)\,Y_+^\dagger(0)|0\rangle} \equiv -i\mathscr{F}^\pm_{B_u} m_{B_u} \int_0^\infty d\omega\, e^{-i\omega v}\Phi^\pm_+(\omega)$$

$\Rightarrow$ the three LCDA's Φ_+, Φ^0_+ and $\Phi^\pm_+$ have different anomalous dimensions

Soft matrix elements: expansion and RG

Expansion in $\alpha_{\rm em}(\mu_s)$

$$\mathscr{F}_{B_q} = \sum_{n=0}^{\infty} \left(\frac{\alpha_{\rm em}}{4\pi}\right)^n F_{B_q}^{(n)}$$

$$\mathscr{F}_{B_q} \Phi_+(\omega) = \sum_{n=0}^{\infty} \left(\frac{\alpha_{\rm em}}{4\pi}\right)^n F_{B_q}^{(n)}\, \phi_+^{(n)}(\omega)$$

$\Rightarrow$ need here only $n = 0$

- $\phi_+(\omega, \mu_s) \equiv \phi_+^{(0)}(\omega, \mu_s)$
  usual B-meson LCDA in absence of QED
- $F_{B_q}(\mu_s) \equiv F_{B_q}^{(0)}(\mu_s)$
  usual HQET B-meson decay constant
  $\Rightarrow$ related to full-QCD f_{B_q} via threshold corr's @ μ_b

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  usual HQET B-meson decay constant
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RG equation in SCET$_2$ with $\Gamma^s(\omega, \omega')$ from soft photons $\Leftarrow$ after subtraction prescription

$$\frac{d}{d\ln\mu}\left[\mathscr{F}_{B_q}(\mu)\, \Phi_+(\omega, \mu)\right] = -\mathscr{F}_{B_q} \int_0^\infty d\omega'\, \Gamma^s(\omega, \omega')\, \Phi_+(\omega')$$

!!! $\phi_+^{(0)}$ includes QED, but not ϕ_+ $\Rightarrow$ for $\mu \neq \mu_s$ run differently

$$\left[F_{B_q}^{(0)}\, \phi_+^{(0)}(\omega)\right](\mu) = U^{QED}(\omega, \omega'; \mu, \mu_s) \otimes_{\omega'} \left[F_{B_q}\, \phi_+(\omega')\right](\mu)$$

where $U^{QED}(\mu, \mu_s)$ fulfils RGE

$$\frac{d}{d\ln\mu} U^{QED}(\mu, \mu_s) = \left[\Gamma^s - \Gamma^s\big|_{\alpha_{em}\to 0}\right] U^{QED}(\mu, \mu_s) \qquad U^{QED}(\mu_s, \mu_s) = \delta(\omega - \omega')$$

Collinear matrix elements

Matrix elements of collinear parts of operators: anti-collinear analogously $+ \to -$, $c \to \bar{c}$, ...

$$\mathcal{J}_{m\chi}: \qquad \big\langle \ell(p_\ell) \big| R_+ \bar{\ell}_c(0) \big| 0 \big\rangle = Z_\ell \, \bar{u}_c(p_\ell),$$

$$\mathcal{J}_{\mathcal{A}\chi}: \qquad \int \frac{dt}{2\pi} \, e^{-it\overline{w}\, n_+ p_\ell} \big\langle \ell(p_\ell) \big| R_+ \bar{\ell}_c(0) \mathcal{A}^\mu_{c\perp}(tn_+) \big| 0 \big\rangle = Z_\ell \, M_A(w) \, m_\ell \left[\bar{u}_c(p_\ell) \gamma^\mu_\perp \right]$$

- contain factors $R_\pm$ from subtraction prescription
- contain only leptons and photons $\Rightarrow$ can be calculated perturbatively in QED
- here considered to order @ $\mu_c \sim \mu_s$

$$Z_\ell = Z_{\bar{\ell}} = 1 + \mathcal{O}(\alpha_{\rm em}) \qquad M_A^{(1)}(w) = M_{\bar{A}}^{(1)}(w) = -\frac{\alpha_{\rm em}}{4\pi} Q_\ell \, \overline{w} \left(\ln \frac{\mu_c^2}{m_\ell^2} - \ln \overline{w}^2 \right)$$

Collinear matrix elements

Matrix elements of collinear parts of operators: anti-collinear analogously $+ \to -$, $c \to \bar{c}$, ...

$$\mathcal{J}_{m\chi}: \quad \langle \ell(p_\ell) | R_+ \bar{\ell}_c(0) | 0 \rangle = Z_\ell \, \bar{u}_c(p_\ell),$$

$$\mathcal{J}_{\mathcal{A}\chi}: \quad \int \frac{dt}{2\pi} e^{-it\overline{w}\, n_+ p_\ell} \langle \ell(p_\ell) | R_+ \bar{\ell}_c(0) \mathcal{A}^\mu_{c\perp}(tn_+) | 0 \rangle = Z_\ell \, M_A(w) \, m_\ell \left[\bar{u}_c(p_\ell) \gamma^\mu_\perp \right]$$

- contain factors $R_\pm$ from subtraction prescription
- contain only leptons and photons $\Rightarrow$ can be calculated perturbatively in QED
- here considered to order @ $\mu_c \sim \mu_s$

$$Z_\ell = Z_{\bar{\ell}} = 1 + \mathcal{O}(\alpha_{\text{em}}) \qquad M_A^{(1)}(w) = M_{\bar{A}}^{(1)}(w) = -\frac{\alpha_{\text{em}}}{4\pi} Q_\ell \, \overline{w} \left(\ln \frac{\mu_c^2}{m_\ell^2} - \ln \overline{w}^2 \right)$$

Matrix elements of SCET$_2$ operators factorize into collinear $\times$ soft @ scale $\mu \sim \mu_s \sim \mu_c \sim \Lambda$

$$\langle \bar{\ell}(p_{\bar{\ell}}) \, \ell(p_\ell) | \mathcal{J}_{m\chi}(\omega) | \bar{B}_q(p) \rangle = T_+ \times m_{B_q} \mathscr{F}_{B_q} \Phi_+(\omega)$$

$$\langle \bar{\ell}(p_{\bar{\ell}}) \, \ell(p_\ell) | \mathcal{J}_{\mathcal{A}\chi}(\omega, w) | \bar{B}_q(p) \rangle = T_+ \, M_A(w) \times m_{B_q} \mathscr{F}_{B_q} \Phi_+(\omega)$$

$$T_+(\mu) \equiv m_\ell \, Z_\ell(\mu) Z_{\bar{\ell}}(\mu) \times \left[\bar{u}_c(p_\ell) \, P_R \, v_{\bar{c}}(p_{\bar{\ell}}) \right]$$

$\Rightarrow$ collinear and soft parts obey their own RG equations

SCET_2 RG evolution

Will use SCET_2 RG to run matrix elements – instead of Wilson coefficient – from $\mu_c \to \mu_{hc}$

SCET_2 RG evolution

Will use SCET_2 RG to run matrix elements – instead of Wilson coefficient – from $\mu_c \to \mu_{hc}$

RG of collinear parts → operator mixing 2×2 RG equation

$$\frac{d}{d\ln\mu}\begin{pmatrix} T_+ \\ T_+ M_A \end{pmatrix} = -\begin{pmatrix} \Gamma^c_{m\chi,\, m\chi} & 0 \\ \Gamma^c_{\mathcal{A}\chi,\, m\chi} & \Gamma^c_{\mathcal{A}\chi,\, \mathcal{A}\chi} \end{pmatrix} \otimes_{w'} \begin{pmatrix} T_+ \\ T_+ M_A \end{pmatrix}$$

Solution in double-log approximation

$$T_+(\mu_{hc}) = e^{S_{\ell,2}}\, T_+(\mu_c)$$

$$\big[T_+ M_A\big](\mu_{hc}) = e^{S_{\ell,2}}\, T_+(\mu_c)\left(M_A(\mu_c) - \underbrace{\frac{\alpha_{\text{em}}}{4\pi} Q_\ell\, \overline{w}\, \ln\frac{\mu_{hc}^2}{\mu_c^2}}_{\text{found in fixed-order}}\right)$$

⇒ same cusp part Γ_c as in SCET_1

$$S_{\ell,2} \equiv -\frac{\Gamma_c}{2}\left[\ln^2\frac{m_{B_q}}{\mu_c} - \ln^2\frac{m_{B_q}}{\mu_{hc}}\right]$$

⇒ combine $e^{S_{\ell,2}}$ with SCET_1 $e^{S_{\ell,1}}$

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Will use SCET$_2$ RG to run matrix elements – instead of Wilson coefficient – from $\mu_c \to \mu_{hc}$

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⇒ combine $e^{S_{\ell,2}}$ with SCET$_1$ $e^{S_{\ell,1}}$

RG of soft parts in double-log approximation

$$\left[\mathscr{F}_{B_q}\, \Phi_+(\omega)\right](\mu_{hc}) \approx \left[F^{(0)}_{B_q}\, \phi^{(0)}_+(\omega)\right](\mu_{hc})$$
$$= e^{S_{q,2}(\mu_{hc},\, \mu_c;\, \omega)}\left[F_{B_q}\, \phi_+(\omega)\right](\mu_{hc})$$

⇒ QED cusp part: $-\Gamma_s^{QED}$ of SCET$_1$

$$S_{q,2} = \frac{\Gamma_s^{QED}}{2}\left[\ln^2\frac{\omega}{\mu_c} - \ln^2\frac{\omega}{\mu_{hc}}\right]$$

⇒ **require input of F_{B_q} and ϕ_+ @ μ_{hc}**

!!! only QED part

→ avoids QCD running below μ_{hc}

Factorization of power-enhanced amplitude

General factorization of power-enhanced amplitude from $O_{9,10}$

$$
\begin{aligned}
i\mathcal{A}_9 = T_+ \Big[& (H_9 + H_{10}) \otimes_u (J_m + J_A \otimes_w M_A) \\
& + (H_{\overline{9}} - H_{\overline{10}}) \otimes_u \big(J_{\overline{m}} + J_{\overline{A}} \otimes_w M_{\overline{A}}\big) \Big] \otimes_\omega m_{B_q} \mathscr{F}_{B_q} \Phi_+ ,
\end{aligned}
$$

$\Rightarrow$ because of strict symmetries of QED **between coll. and anti-coll. sectors** after decoupling of hard virtualities, relations

- $H_9^{(0)} = H_{\overline{9}}^{(0)}$ $\quad$ $H_{10}^{(0)} = H_{\overline{10}}^{(0)}$
- $J_A^{(0)} = J_{\overline{A}}^{(0)}$ $\quad$ $J_m^{(0,1)} = J_{\overline{m}}^{(0,1)}$ $\quad$ $M_A^{(1)} = M_{\overline{A}}^{(1)}$

found at leading order will hold also at higher orders

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$$i\mathcal{A}_9 = T_+ \Big[(H_9 + H_{10}) \otimes_u (J_m + J_A \otimes_w M_A) + (H_{\overline{9}} - H_{\overline{10}}) \otimes_u \big(J_{\overline{m}} + J_{\overline{A}} \otimes_w M_{\overline{A}}\big) \Big] \otimes_\omega m_{B_q} \mathscr{F}_{B_q} \Phi_+ ,$$

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Factorization formula of power-enhanced amplitude from $O_{9,10}$ in SCET$_2$

$$i\mathcal{A}_9 = 2H_9(u, \mu_{hc}) \otimes_u T_+(\mu)\Big[J_m(u;\omega) + J_A(u;\omega, w)\otimes_w M_A(w)\Big](\mu) \otimes_\omega m_{B_q}\Big[\mathscr{F}_{B_q}\Phi_+(\omega)\Big](\mu)$$

Combination with leading O_{10} amplitude

SCET$_1$ matching O_{10} on operator

$$\widetilde{\mathcal{O}}_m = m_\ell \left[\overline{q}_s(0)\, P_R\, h_v(0)\right]\left[\overline{\ell}_C(0)\,\gamma_5\,\ell_{\overline{C}}(0)\right]$$

$\Rightarrow$ same cusp anomalous dimension

$$H_m(\mu_b) = \mathcal{N}\,\frac{2\,C_{10}(\mu_b)}{m_b}$$

SCET$_2$ matching trivial

$$\Rightarrow \widetilde{\mathcal{J}}_m^{A1} = m_\ell\Big[\overline{q}_s(0)P_R h_v(0)\Big]\Big[Y_+^\dagger Y_-\Big](0)\Big[\overline{\ell}_c(0)\gamma_5\,\ell_{\overline{c}}(0)\Big]$$

$$i\,\mathcal{A}_{10} = \frac{1}{2} m_{B_q} m_\ell f_{B_q}\, e^{S_\ell(\mu_b,\,\mu_c)}\, H_m(\mu_b)[\overline{u}_c(p_\ell)\gamma_5 v_{\overline{c}}(p_{\overline{\ell}})]$$

$\Rightarrow$ SCET$_1$ and SCET$_2$ Sudakov exponents add up: $S_\ell(\mu_b,\mu_c) \equiv S_{\ell,1}(\mu_b,\mu_{hc}) + S_{\ell,2}(\mu_{hc},\mu_c)$

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Final ampl. $\boxed{\mathcal{A}_{10} \sim 1\cdot\lambda^{12} \;\Leftrightarrow\; \mathcal{A}_9 \sim \frac{\alpha_{em}}{\pi}\cdot\lambda^{10}\cdot\ln\frac{\mu_{hc}}{\mu_c}}$ $\quad \lambda^2 \sim \frac{\Lambda}{m_b} \sim \frac{1}{20} \;\Leftrightarrow\; \frac{\alpha_{em}}{\pi} \sim \frac{1}{420}$

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$$\begin{aligned} i(\mathcal{A}_{10}+\mathcal{A}_9) &= m_\ell f_{B_q}\, e^{S_\ell}\,\mathcal{N} C_{10}(\mu_b)\,[\bar{u}_c\gamma_5 v_{\bar{c}}] \\ &+ \frac{\alpha_{em}(\mu_c)}{4\pi} Q_\ell Q_q\, m_\ell\, m_{B_q} f_{B_q}\, e^{S_\ell}\,[\bar{u}_c(1+\gamma_5)v_{\bar{c}}] \\ &\times e^{S_{q,1}(\mu_b,\,\mu_{hc})} \int_0^1 du\,(1-u)\,\mathcal{N} C_9^{\rm eff}(u,\mu_b) \\ &\times \int_0^\infty \frac{d\omega}{\omega}\, e^{S_{q,2}(\mu_{hc},\,\mu_c;\,\omega)}\,\phi_+(\omega;\mu_{hc})\left[\ln\frac{\omega m_b}{m_\ell^2} + \ln\frac{u}{1-u}\right] \end{aligned}$$

fixed-order result for
$e^{S_\ell} = e^{S_{q,1}} = e^{S_{q,2}} = 1$

virtual FSR
$\Rightarrow$ part of exponentiation

SCET$_1$ resummation

SCET$_2$ resummation

Including ultra-soft radiation

Ultra-soft radiation

Matrix element (ME) including ultra-soft photons: u-soft state X_s with photons

$$\langle \ell\bar{\ell} X_s | H_k \otimes J_k \otimes \mathcal{J}_k | \overline{B}_s \rangle = \mathcal{A}_k(\mu_c) \, \langle X_s | S^\dagger_{v_\ell}(0) S_{v_{\bar{\ell}}}(0) | 0 \rangle \qquad k = 9, 10$$

- **HQET-like theory for "heavy" leptons** → lepton velocity $p_\ell = E_\ell v_\ell$

  after decoupl trafo with u-soft Wilson lines $\boxed{\ell \to S_{v_\ell} \ell^{(0)}}$ = static "heavy"-lepton
  $\Rightarrow$ factorization into $\mathcal{A}_{9,10}$ and u-soft ME

- u-soft photons do not couple to electrically B_q-meson $\Rightarrow$ their u-soft Wilson lines cancel

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- u-soft photons do not couple to electrically B_q-meson ⇒ their u-soft Wilson lines cancel

Decay rate incl. u-soft radiation

[$S^{(1)}$ for example in Manteuffel/Schabinger/Zhu 1408.5134]

$$\Gamma[B_q \to \mu\bar{\mu}](\Delta E) = \Gamma^{(0)} \times \underbrace{\left| e^{S_\ell} \right|^2}_{\text{virt. FSR}} \times \underbrace{\sum_{X_s} \left| \langle X_s | S^\dagger_{v_\ell}(0) S_{v_{\bar{\ell}}}(0) | 0 \rangle \right|^2 \underbrace{\theta(\Delta E - E_{X_s})}_{\text{measurement function}}}_{\approx \exp\left[\frac{\alpha_{em}}{4\pi} Q_\ell^2 S^{(1)}(v_\ell, v_{\bar{\ell}}, \Delta E)\right]}$$

and non-radiative rate including SCET_1 + SCET_2 resummation

$$\Gamma^{(0)} \equiv \frac{m_{B_q}}{8\pi} \beta_\mu \left(\left| A_{10} + A_9 + A_7 \right|^2 + \beta_\mu^2 \left| A_9 + A_7 \right|^2 \right) \qquad i\mathcal{A}_i \equiv e^{S_\ell} A_i \left[\bar{u}_c(p_\ell) \Gamma_i v_{\bar{c}}(p_{\bar{\ell}}) \right]$$

$$\Gamma[B_q \to \mu\bar{\mu}](\Delta E) = \Gamma^{(0)} \times \left(\frac{2\Delta E}{m_{B_q}} \right)^{-\frac{2\alpha_{em}}{\pi}\left(1 + \ln \frac{m_\mu^2}{m_{B_q}^2}\right)}$$

virtual FSR & u-soft parts combine to usual soft-photon exponentiation

[Buras/Girrbach/Guadagnoli/Isidori 1208.0934]

Phenomenological implications

Resummation for power-enhanced amplitude

SCET$_1$ compare here **fixed-order result ↔ resummed result**

- $e^{S_{q,1}}$ enters as global factor

  $$iA_9 \propto H_9(\mu_{hc}) = e^{S_{q,1}} H_9(\mu_b)$$

- main effect from QCD-cusp

  ← strong running of α_s

- μ_{hc} dep. cancels with $\phi_+(\mu_{hc})$

μ_{hc} GeV	$e^{S_{q,1}}$ QCD+QED	$e^{S_{q,1}}$ only QED	$\frac{\lambda_B(\mu_0)}{\lambda_B(\mu_{hc})} e^{S_{q,1}}$ QCD+QED
0.8	0.734	0.996	0.771
1.1	0.843	0.998	0.827
1.6	0.922	0.999	0.845

⇒ Resummation decreases by ~ 20% fixed order result

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SCET$_2$

- enters as global factor as $\quad i\mathcal{A}_9 \propto \int_0^\infty d\omega\, e^{S_{q,2}(\mu_{hc},\mu_c;\omega)}\, \phi_+(\omega;\mu_{hc})\, J_i(u;\omega,w;\mu_{hc})$

- inverse + logarithmic moments of ϕ_+

$$\frac{1}{\lambda_B(\mu)} \equiv \int_0^\infty \frac{d\omega}{\omega} \phi_{B+}(\omega,\mu) \qquad \frac{\sigma_n(\mu)}{\lambda_B(\mu)} \equiv \int_0^\infty \frac{d\omega}{\omega} \ln^n \frac{\mu_0}{\omega} \phi_{B+}(\omega,\mu)$$

- neglect evolution of logarithmic moments in RG $\quad \mu_{hc} = \mu_0 = 1\,\text{GeV},\ \Lambda = 0.25\,\text{GeV}$

$$i\mathcal{A}_9 \propto \frac{e^{S_{q,2}(\mu_{hc},\mu_c;\mu_0)}}{\lambda_B(\mu_{hc})} \left[\ln \frac{m_b \mu_0}{m_\ell^2} - \sigma_1(\mu_{hc})\right] \qquad e^{S_{q,2}} = \begin{cases} 1.00177 & 1/\alpha_{\rm em}(\mu_0) = 134.28 \\ 1.00176 & 1/\alpha_{\rm em}(\Lambda) = 135.45 \end{cases}$$

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Resummation of QED logs in SCET$_1$ & SCET$_2$ numerically negligible

$B_q \to \mu\bar{\mu}$ non-radiative branching ratio

Non-radiative rate
(time-integrated)

$$\overline{\mathrm{Br}}^{(0)}_{q\mu} \equiv \frac{\Gamma^{(0)}[B_q \to \mu\bar{\mu}]}{\Gamma_q^{\mathrm{tot}}} \quad \text{with} \quad \Gamma_q^{\mathrm{tot}} = \begin{cases} \Gamma_s^H & \text{for } q = s \\ \Gamma_d^{\mathrm{tot}} & \text{for } q = d \end{cases}$$

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Semi-numeric result "LO × (1 + NLO / LO)"

$$\overline{\text{Br}}^{(0)}_{s\mu} = 3.677 \cdot 10^{-9} \times \Big(\overbrace{1}^{\mathcal{A}_{10}} + \frac{\text{GeV}}{10^3 \cdot \lambda_B} \Big[\overbrace{S_9\,(-6.5 + 1.3\sigma_1)}^{\mathcal{A}_9} + \overbrace{S_7\,(4.7 - 1.5\sigma_1 + 0.2\sigma_2)}^{\mathcal{A}_7} \Big] \Big)$$

$$= 3.677 \cdot 10^{-9} \times (1 - 0.0166\, S_9 + 0.0105\, S_7)$$

$$= 3.660 \cdot 10^{-9}$$

destructive interference between $\mathcal{A}_9$ and $\mathcal{A}_7$

$$\overline{\text{Br}}^{(0)}_{d\mu} = 1.031 \cdot 10^{-10} \times \Big(1 + \frac{\text{GeV}}{10^3 \cdot \lambda_B} \Big[S_9\,(-6.0 + 1.2\sigma_1) + S_7\,(4.7 - 1.5\sigma_1 + 0.2\sigma_2) \Big] \Big)$$

$$= 1.031 \cdot 10^{-10} \times (1 - 0.0155\, S_9 + 0.0103\, S_7)$$

$$= 1.027 \cdot 10^{-10}$$

$V_{ub} V_{us}^* \ll V_{ub} V_{ud}^*$ contribution

- $S_9 \equiv \exp[S_{q,\text{I}}(\mu_b, \mu_{hc})] \simeq 0.8$ is SCET$_1$ Sudakov factor, dominated by QCD
- $S_7 \simeq S_9$ for what concerns numerically leading QCD effect in SCET$_1$
- f_{B_q} from $N_f = 2 + 1 + 1$ [FLAG 1902.08191]

$B_q \to \mu\bar{\mu}$ non-radiative branching ratio

Non-radiative rate
(time-integrated)

$$\overline{\text{Br}}_{q\mu}^{(0)} \equiv \frac{\Gamma^{(0)}[B_q \to \mu\bar{\mu}]}{\Gamma_q^{\text{tot}}} \quad \text{with} \quad \Gamma_q^{\text{tot}} = \begin{cases} \Gamma_s^H & \text{for } q = s \\ \Gamma_d^{\text{tot}} & \text{for } q = d \end{cases}$$

Error budget

$N_f = 2+1$ [FLAG 1902.08191]

$$\overline{\text{Br}}_{s\mu}^{(0)} = \begin{pmatrix} 3.599 \\ 3.660 \end{pmatrix} \left[1 + \begin{pmatrix} 0.032 \\ 0.011 \end{pmatrix}_{f_{B_s}} + 0.031|_{\text{CKM}} + 0.011|_{m_t} + 0.006|_{\text{pmr}} + 0.012|_{\text{non-pmr}} \, {}^{+0.003}_{-0.005}|_{\text{LCDA}}\right] \cdot 10^{-9}$$

$N_f = 2+1+1$

$$\overline{\text{Br}}_{d\mu}^{(0)} = \begin{pmatrix} 1.049 \\ 1.027 \end{pmatrix} \left[1 + \begin{pmatrix} 0.045 \\ 0.014 \end{pmatrix}_{f_{B_d}} + 0.046|_{\text{CKM}} + 0.011|_{m_t} + 0.003|_{\text{pmr}} + 0.012|_{\text{non-pmr}} \, {}^{+0.003}_{-0.005}|_{\text{LCDA}}\right] \cdot 10^{-10}$$

- **main parametric** long-distance (f_{B_q}) and short-distance (CKM and m_t) (use $|V_{cb}|_{\text{incl}}$)
- **non-QED:** parametric (Γ_q, α_s) and non-parametric (μ_W, μ_b and higher order)
- ***B*-meson LCDA:** λ_B and $\sigma_{1,2}$ entering power-enhanced QED crr'n

Radiative branching ratio for experiments

Radiative branching ratio measured in experiment

$$\overline{\text{Br}}_{q\mu}(\Delta E) \equiv \overline{\text{Br}}^{(0)}_{q\mu} \times \Omega(\Delta E; \alpha_{\text{em}})$$

- use here **measurement function** $\theta(\Delta E - E_{X_s})$ in sum over final states X_s → include arbitrary number ultra-soft photons
- $\Delta E = \sqrt{m^2_{B_q} - q^2}$ is window in $q^2 = (p_+ + p_-)^2$ around end-point of $B_q \to \ell\bar{\ell} + n\gamma$ spectrum
- E_{X_s} . . . total energy of X_s
- radiative factor

$$\Omega(\Delta E; \alpha_{\text{em}}) \equiv \left(\frac{2\Delta E}{m_{B_q}}\right)^{-\frac{2\,\alpha_{\text{em}}}{\pi}\left(1+\ln\frac{m^2_\mu}{m^2_{B_q}}\right)}$$

!!! our EFT approach predicts m_{B_q}

⇒ contrary to exponentiation approach, where fixed by hand to minimize higher order crr's [Isidori 0709.2439]

Radiative branching ratio for experiments

Radiative branching ratio measured in experiment

$$\overline{\text{Br}}_{q\mu}(\Delta E) \equiv \overline{\text{Br}}^{(0)}_{q\mu} \times \Omega(\Delta E; \alpha_{\text{em}})$$

- use here **measurement function** $\theta(\Delta E - E_{X_s})$ in sum over final states X_s → include arbitrary number ultra-soft photons
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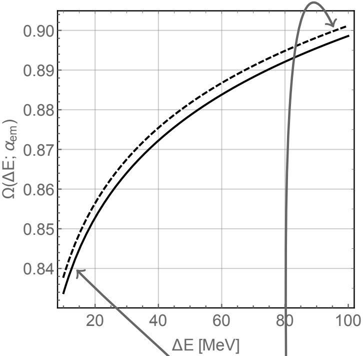


$$\Omega(\Delta E; \alpha_{\text{em}}) \equiv \left(\frac{2\Delta E}{m_{B_q}}\right)^{-\frac{2\,\alpha_{\text{em}}}{\pi}\left(1+\ln\frac{m_\mu^2}{m_{B_q}^2}\right)}$$

$$\alpha_{\text{em}}^{-1} = \{134.28,\ 138.0\}$$

$$\Omega(60\,\text{MeV}) = \begin{cases} \{0.8838,\ 0.8867\} & B_s \\ \{0.8848,\ 0.8877\} & B_d \end{cases}$$

!!! our EFT approach predicts m_{B_q}

⇒ contrary to exponentiation approach, where fixed by hand to minimize higher order crr's [Isidori 0709.2439]

⇒ 0.3% uncertainty from two choices of α_{em}

Summary

Presented **power-enhanced NLO QED corrections for $B_{s,d} \to \mu\bar{\mu}$** in SM

- for $O_{9,10}$ operators in 2-step matching **QED(μ_b) → SCET$_1$(μ_{hc}) → SCET$_2$(μ_c)**

  ⇒ factorization of soft hadronic from (anti-)collinear leptonic part

- clarify **process-dependence** of soft hadronic matrix elements in presence of QED

  ⇒ important for lattice or other nonperturbative methods

- resummation of **leading double-log's**

  ⇒ most important from $\mu_b \to \mu_{hc}$ for QCD part → decrease of 20% of amplitude

- resum **ultra-soft photon radiation** assuming window in $q^2 = (p_\ell + p_{\bar{\ell}})^2$

  ⇒ recover soft-photon approximation combining virtual corrections
  from SCET$_1$ + SCET$_2$ + "heavy-lepton" EFT (ultra-soft photons)

$Br(B_s \to \mu\bar{\mu})$ one of few flavor observables with ≈ 2% long-distance theory control

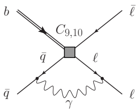
- no power-enhanced effect in $B_u \to \mu\bar{\nu}_\mu$
- important step for better understanding of QED factorization theorems

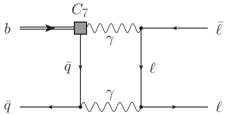
Next steps: clarify technical issues in SCET for O_7

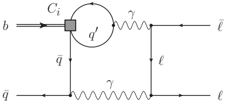
Backup Slides

Power-enhanced contribution at fixed order

Leading QED corrections **in $\boldsymbol{\lambda}$-expansion** to $b\bar{s} \to \mu\bar{\mu}$ $C_{10} \approx -4$, $C_9 \approx +4$, $C_7 \approx -0.3$



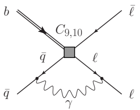


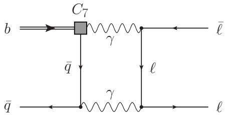


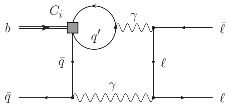
$\mathcal{O}_{9(10)} \propto [\bar{s}\gamma_\mu P_L b][\bar{\ell}\gamma^\mu(\gamma_5)\ell]$ $\mathcal{O}_7 \propto m_b[\bar{s}\sigma_{\mu\nu}P_R b]F^{\mu\nu}$ $\mathcal{O}_i \propto [\bar{s}\Gamma_i P_L b]\sum_q[\bar{q}'\Gamma_i q']$

Power-enhanced contribution at fixed order

Leading QED corrections **in λ-expansion** to $b\bar{s} \to \mu\bar{\mu}$ $\quad$ $C_{10} \approx -4$, $C_9 \approx +4$, $C_7 \approx -0.3$







$\mathcal{O}_{9(10)} \propto [\bar{s}\gamma_\mu P_L b][\bar{\ell}\gamma^\mu(\gamma_5)\ell]$ $\quad$ $\mathcal{O}_7 \propto m_b[\bar{s}\sigma_{\mu\nu}P_R b]F^{\mu\nu}$ $\quad$ $\mathcal{O}_i \propto [\bar{s}\Gamma_i P_L b]\sum_q[\bar{q}'\Gamma_i q']$

$$\frac{i\mathcal{A}}{\mathcal{N}} = \overbrace{m_\ell f_{B_q} C_{10}\,[\bar{\ell}\,\gamma_5\,\ell]}^{\text{LO}} + \frac{\alpha_{\text{em}}}{4\pi} Q_\ell Q_q\, m_\ell f_{B_q} \overbrace{\frac{m_B}{\lambda_B}}^{\text{power-enh.}} [\bar{\ell}\,(1+\gamma_5)\,\ell] \times \Bigg\{$$

$$\int_0^1 du\,\bar{u}\,C_9^{\text{eff}}(u m_b^2)\left[L + \ln\frac{u}{\bar{u}} - \sigma_1\right] - Q_\ell C_7^{\text{eff}}\left[\underbrace{L^2}_{\text{large (Log)}^2} - 2L(\sigma_1+1) + 2\sigma_1 + \sigma_2 + \frac{2\pi^2}{3}\right]\Bigg\} + \ldots$$

- power enhancement: $\quad m_B \approx 5$ GeV $\leftrightarrow$ $\lambda_B \approx (0.27 \pm 0.08)$ GeV $\quad \Rightarrow \quad m_B/\lambda_B \approx 18$
- $L \equiv \ln\frac{m_b\mu_0}{m_\mu^2}$ with $\mu_0 = 1$ GeV — log (hard-collinear)^2/(collinear)^2 $\quad \Rightarrow L \approx \ln 500 \approx 6$
- only limited knowledge of B-meson DA: $\sigma_1 \approx (1.5 \pm 1.0)$, $\sigma_2 \approx (3 \pm 2)$ $\quad \Rightarrow$ large uncert.

$$\frac{1}{\lambda_B(\mu)} \equiv \int_0^\infty \frac{d\omega}{\omega}\,\phi_{B+}(\omega,\mu) \qquad \frac{\sigma_n(\mu)}{\lambda_B(\mu)} \equiv \int_0^\infty \frac{d\omega}{\omega}\,\ln^n\frac{\mu_0}{\omega}\,\phi_{B+}(\omega,\mu)$$