

Factorisation and Subtraction beyond NLO

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in collaboration with:

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based on:

Magnea et. al., arXiv:180609570, arXiv: 180905444

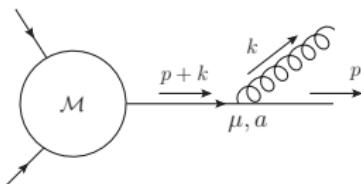
Motivations

- LHC data do not (yet?) manifest signals of new Physics.
- Small deviations from theoretical predictions can provide important tests for New Physics models.
- Theoretical predictions have to investigate higher perturbative orders for the comparison with experimental data.

@NNLO: several different schemes are available:

Antenna, Nested soft-collinear, ColorfulNNLO, N-jettiness, Unsubtraction, Geometric slicing, Q_\perp -subtraction, Projection to Born,

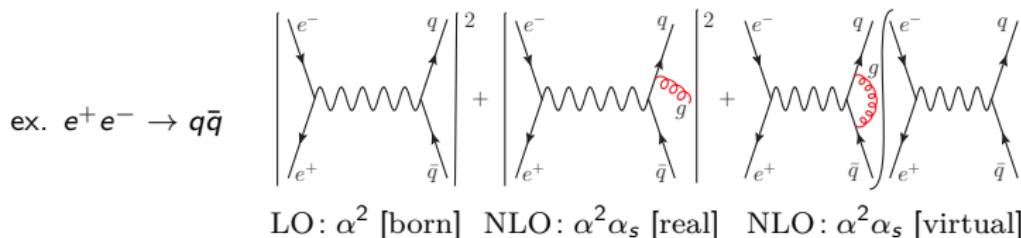
IR singularities cancellation



$$\begin{cases} m_p = 0 \\ \theta_{pk} = 0 \\ E_k = 0 \end{cases} \rightarrow \begin{array}{l} \text{Collinear} \\ \text{Soft} \end{array}$$

- IR divergences from **real radiation** phase space and from **virtual corrections**.
- **order-by-order cancellation** in perturbation theory for a generic IR-safe observable.

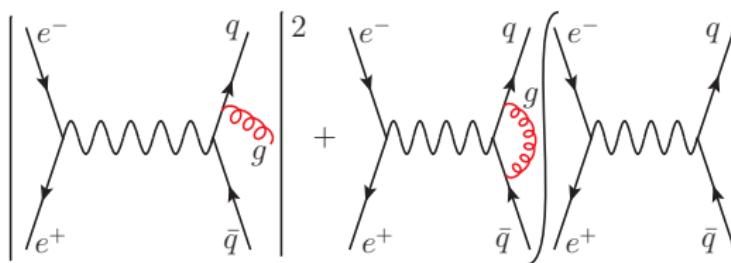
At NLO, the cancellation occurs between the following diagram sample



Real and Virtual contributions

hp: $d = 4 - 2\epsilon$, $\epsilon < 0$

Real contr. \rightarrow implicit poles in $\epsilon \rightarrow$ unresolved radiation PS.
Virtual contr. \rightarrow explicit poles in ϵ .

**Real contribution**

complicated phase space integration

**Subtraction****counterterm****Virtual contribution**

divide singularities according to their nature

**Factorisation****completeness**

The procedure is completely general and could be applied at any perturbation order.
(this talk: NLO + sketched NNLO)

Tools: Factorisation and Subtraction

Factorised virtual amplitude

A n -particle massless virtual amplitude factorises in regions according to
[9606312] [0908.3273]

Factorisation formula

$$\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^n \left[\frac{\mathcal{J}_i((p_i \cdot n_i)^2 / (n_i^2 \mu^2))}{\mathcal{J}_{i,\mathbb{E}}((\beta_i \cdot n_i)^2 / n_i^2)} \right] \mathcal{S}_n(\beta_i \cdot \beta_j) \mathcal{H}_n\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}\right)$$

where $p_i^\mu = Q\beta_i^\mu$, $\beta_i^2 = 0$, and $n_i^2 \neq 0$ auxiliary vector, μ renormalisation scale.

- universality

Functions' properties:

- gauge invariance
- all-orders definition in perturbation theory

Remarks: \mathcal{J}_E avoids soft-collinear double counting;
 $n_i^2 \neq 0$ avoids spurious collinear singularities (in practice $n_i^2 = 0$).

Factorised virtual amplitude

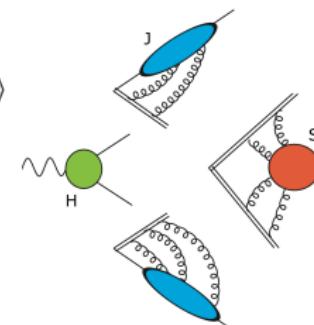
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where $p_i^\mu = Q\beta_i^\mu$, $\beta_i^2 = 0$, and $n_i^2 \neq 0$ auxiliary vector, μ renormalisation scale.

- Collinear function: $\bar{u}_s(p)\mathcal{J}_q = \langle p, s | \bar{\psi}(0) \Phi_n(0, \infty) | 0 \rangle$
- Soft-Collinear function: $\mathcal{J}_{\text{E}} = \langle 0 | \Phi_\beta(\infty, 0) \Phi_n(0, \infty) | 0 \rangle$
- Soft function: $\mathcal{S}_n(\beta_i \cdot \beta_j) = \langle 0 | \prod_{k=1}^n \Phi_{\beta_k}(\infty, 0) | 0 \rangle$
- Hard region: \mathcal{H}_n colour vector, finite reminder



Wilson line operator: $\Phi_v(\lambda_2, \lambda_1)$

Remarks: \mathcal{J}_{E} avoids soft-collinear double counting;
 $n_i^2 \neq 0$ avoids spurious collinear singularities (in practice $n_i^2 = 0$).

Subtraction pattern

Given a generic amplitude with n massless particle in the final state

$$\mathcal{A}_n(p_i) = \mathcal{A}_m^{(0)}(p_i) + \mathcal{A}_m^{(1)}(p_i) + \mathcal{A}_m^{(2)}(p_i) + \dots$$

An **IR-safe** observable X receives contribution at NLO according to

$$\frac{d\sigma^{\text{NLO}}}{dX} = \lim_{d \rightarrow 4} \left\{ \int d\Phi_n V_n \delta_n + \int d\Phi_{n+1} R_{n+1} \delta_{n+1} \right\}$$

$$\delta_i = \delta(X - X_i), \quad X_i \text{ the } i\text{-particle config.}, \quad V_n = 2\mathbf{Re}[\mathcal{A}_n^{(0)*} \mathcal{A}_n^{(1)}], \quad R_{n+1} = |\mathcal{A}_{n+1}^{(0)}|^2.$$

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Subtraction idea

make the real contribution finite before performing the PS integration by **adding and subtracting a counterterm**. [1806.09570]

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Subtraction idea

make the real contribution finite before performing the PS integration by **adding and subtracting a counterterm**. [1806.09570]

Counterterms and Subtraction at NLO

$$\begin{cases} \text{same singular limits as the real} \\ \text{analytically integrable in } d \text{ dim} \end{cases} : \frac{d\sigma_{ct}^{\text{NLO}}}{dX} = \int \Phi_{n+1} K_{n+1}, \quad I_n = \int d\Phi_{rad} K_{n+1}$$

$$\frac{d\sigma^{\text{NLO}}}{dX} = \underbrace{\int d\Phi_n \left[V_n + I_n \right] \delta_n}_{\text{finite in } d=4} + \underbrace{\int d\Phi_{n+1} \left[R_{n+1} \delta_{n+1} - K_{n+1} \delta_n \right]}_{\text{finite in } d=4}$$

Counterterms construction at NLO

Soft radiative contribution

We model **final state soft radiation** at amplitude and **cross-section level** as

$$S_{n,m}(k_1 \dots k_m; \beta_i) \equiv \langle k_1, \lambda_1 \dots k_m, \lambda_m | \prod_{i=1}^n \Phi_{\beta_i}(\infty, 0) | 0 \rangle = \sum_{p=0}^{\infty} S_{n,m}^{(p)}(k_1 \dots k_m; \beta_i)$$

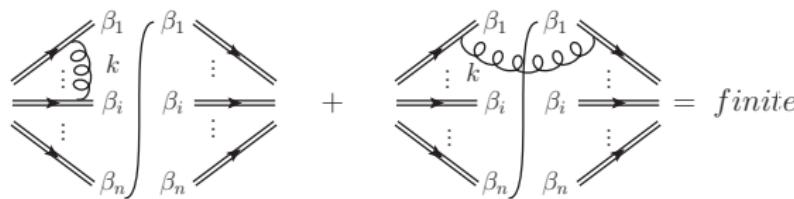
$$S_{n,m}(k_1 \dots k_m; \beta_i) \equiv \sum_{p=0}^{\infty} S_{n,m}^{(p)}(k_1 \dots k_m; \beta_i)$$

$$\equiv \sum_{\{\lambda_i\}} \langle 0 | \prod_i \Phi_{\beta_i}(0, \infty) | k_1, \lambda_1 \dots k_m, \lambda_m \rangle \langle k_1, \lambda_1 \dots k_m, \lambda_m | \prod_i \Phi_{\beta_i}(\infty, 0) | 0 \rangle$$

Completeness relation

$$\sum_{m=0}^{\infty} \int d\Phi_m S_{n,m}(k_1 \dots k_m; \beta_i) = \langle 0 | \prod_i \Phi_{\beta_i}(0, \infty) \prod_i \Phi_{\beta_i}(\infty, 0) | 0 \rangle$$

$$\rightarrow S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1 S_{n,1}^{(0)}(k, \beta_i) = \text{finite}$$



Collinear radiative contribution

Similarly, **final-state collinear radiation** at **amplitude** and **cross-section level** read

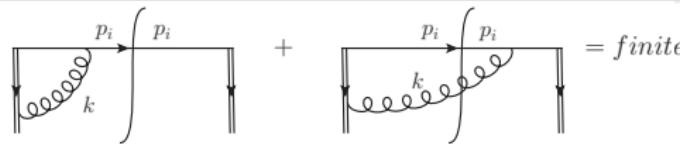
$$\bar{u}_s(p) \mathcal{J}_{q,m}(k_1 \dots k_m; p, n) \equiv \langle p, s; k_1, \lambda_1; \dots; k_m, \lambda_m | \bar{\psi}(0) \Phi_n(0, \infty) | 0 \rangle$$

$$\begin{aligned} J_{q,m}(k_1, \dots, k_m; p, n) &\equiv \sum_{p=o}^{\infty} J_{q,m}^{(p)}(k_1, \dots, k_m; p, n) \\ &\equiv \int d^d x e^{il \cdot x} \sum_{\{\lambda_i\}} \langle 0 | \Phi_n(\infty, x) \psi(x) | p, s; k_j, \lambda_j \rangle \langle p, s; k_j, \lambda_j | \bar{\psi}(0) \Phi_n(0, \infty) | 0 \rangle \end{aligned}$$

where $\mathcal{J}_{q,0}$ is the virtual function, with $\mathcal{J}_{q,0}^{(0)} = 1$, and $l^\mu = p_i^\mu + \sum_i^m k_i^\mu$ the total momentum flowing in the final state.

Completeness relation

$$\begin{aligned} \sum_{m=0}^{\infty} \int d\Phi_{m+1} J_{q,m}(k; l, p, n) &= \text{Disc} \left[\int d^d x e^{il \cdot x} \langle 0 | \Phi_n(\infty, x) \psi(x) \bar{\psi}(0) \Phi_n(0, \infty) | 0 \rangle \right] \\ &\rightarrow J_{i,0}^{(1)}(l, p, n) + \int d\Phi_1 J_{i,1}^{(0)}(k; l, p, n) = \text{finite} \end{aligned}$$



Counterterms at NLO

Factorisation → virtual contribution decomposed in functions

$$V_n = \mathcal{H}_n^{(0)\dagger}(p_i) S_{n,0}^{(1)} \mathcal{H}_n^{(0)}(p_i) + \sum_{i=1}^n [J_{i,0}^{(1)}(p_i) - J_{i,\text{E},0}^{(1)}(\beta_i)] |\mathcal{A}_n^{(0)}(p_i)|^2.$$

Completeness → virtual functions linked to real functions

$$J_{i,0}^{(1)}(l, p, n) + \int d\Phi_1 J_{i,1}^{(0)}(k; l, p, n) = \text{finite},$$

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1 S_{n,1}^{(0)}(k, \beta_i) = \text{finite},$$

⇒ Starting from the virtual structure we can **identify real emission counterterms** through the completeness relations, such that

$$V_n + \int \Phi_{\text{rad}} K_{n+1} \rightarrow \text{finite}$$

Counterterm list at NLO

$$K_{n+1}^s = \mathcal{H}_n^{(0)\dagger}(p_i) S_{n,1}^{(0)} \mathcal{H}_n^{(0)}(p_i)$$

$$K_{n+1}^c = \sum_{i=1}^n J_{i,1}^{(0)}(k_i; l, p_i, n_i) |\mathcal{A}_n^{(0)}(p_1 \dots p_{i-1}, l, p_{i+1} \dots p_n)|^2$$

$$K_{n+1}^{sc} = \sum_{i=1}^n J_{i,\text{E},1}^{(0)}(k_i; l, p_i, n_i) |\mathcal{A}_n^{(0)}(p_1 \dots p_{i-1}, l, p_{i+1} \dots p_n)|^2$$

Counterterms at NNLO

NNLO Subtraction pattern

@NNLO:

- more configurations contribute

$$\frac{d\sigma^{\text{NNLO}}}{dX} = \int d\Phi_n VV_n \delta_n(X) + \int d\Phi_{n+1} RV_{n+1} \delta_{n+1}(X) + \int d\Phi_{n+2} RR_{n+2} \delta_{n+2}(X)$$

- more counterterms to add and subtract

$$\int d\Phi_{n+2} K^{(1)} \delta_{n+1} : \quad K^{(1)} \rightarrow \text{same 1-unr. singularities as RR}$$

$$\int d\Phi_{n+2} (K^{(12)} + K^{(2)}) \delta_n : \quad K^{(12)} + K^{(2)} \rightarrow \text{same 2-unr. singularities as RR.}$$

[1-unr.(2-unr.), pure 2-unr.]

$$\int d\Phi_{n+1} K^{(RV)} \delta_n : \quad K^{(RV)} \rightarrow \text{same 1-unr. singularities as RV}$$

and integrate in the radiative phase space

$$I^{(i)} = \int d\Phi_{\text{rad},i} K^{(i)}, \quad I^{(12)} = \int d\Phi_{\text{rad},1} K^{(12)}, \quad I^{(RV)} = \int d\Phi_{\text{rad}} K^{(RV)},$$

Subtraction pattern at NNLO

$$\frac{d\sigma^{\text{NNLO}}}{dX} = \int d\Phi_n \left[\underbrace{RV_n + I^{(2)} + I^{(RV)}}_{\text{finite in } d=4 \text{ and in } \Phi_n} \delta_n \right. \\ \left. + \int d\Phi_{n+1} \left[\underbrace{(RV_{n+1} + I^{(1)})}_{\text{finite in } d=4, \text{ singular in } \Phi_{n+1}} \delta_{n+1} - \underbrace{(K^{(RV)} - I^{(12)})}_{\text{finite in } d=4, \text{ singular in } \Phi_{n+1}} \delta_n \right] \right. \\ \left. + \int d\Phi_{n+2} \left[\underbrace{RR_{n+2} \delta_{n+2} - K^{(1)} \delta_{n+1}}_{\text{finite in } d=4 \text{ and in } \Phi_{n+2}} - (K^{(12)} + K^{(2)}) \delta_n \right] \right]$$

NNLO counterterms for the purely soft contribution

Factorisation @NNLO → pure soft contribution

$$\begin{aligned} \left. VV_n \right|_{\text{soft}} &\equiv (VV)_n^{(2s)} + (VV)_n^{(1s)} \\ &= \mathcal{H}_n^{(0)\dagger} S_{n,0}^{(2)} \mathcal{H}_n^{(0)} + (\mathcal{H}_n^{(0)\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(1)} + \text{h.c.}) \end{aligned}$$

Completeness → virtual functions linked to real functions

$$S_{n,0}^{(2)}(\beta_i) + \int d\Phi_1 S_{n,1}^{(1)}(k, \beta_i) + \int d\Phi_2 S_{n,2}^{(0)}(k_1, k_2, \beta_i) = \text{finite}$$

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1 S_{n,1}^{(0)}(k, \beta_i) = \text{finite}$$

⇒ from the virtual structure we can identify **real emission soft counterterms** according to their kinematics

Purely Soft counterterms at NNLO

$$K_{n+2}^{(1s)} = \mathcal{H}_{n+1}^{(0)\dagger} S_{n+1,1}^{(0)} \mathcal{H}_{n+1}^{(0)}$$

$$K_{n+2}^{(2s)} = \mathcal{H}_n^{(0)\dagger} S_{n,2}^{(0)} \mathcal{H}_n^{(0)}$$

$$K_{n+1}^{(\text{RV}, s)} = \mathcal{H}_n^{(0)\dagger} S_{n,1}^{(1)} \mathcal{H}_n^{(0)} + (\mathcal{H}_n^{(0)\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(1)} + \text{h.c.})$$

NNLO counterterms for the purely collinear contribution: single leg

Factorisation @NNLO → pure hard-collinear contribution, 2-particle, one hard leg

$$VV_n \Big|_{2hc,i} = \left[J_{i,0}^{(2)} - J_{i,\epsilon,0}^{(2)} - J_{i,\epsilon,0}^{(1)} (J_{i,0}^{(2)} - J_{i,\epsilon,0}^{(1)}) \right] |\mathcal{A}_n^{(0)}|^2$$

Completeness → virtual functions linked to real functions

$$J_{i,0}^{(2)} + \int d\Phi_1 J_{i,1}^{(1)} + \int d\Phi_2 J_{i,2}^{(0)} = \text{finite}$$

$$\left[J_{i,\epsilon,0}^{(1)} + \int d\Phi_1 J_{i,\epsilon,1}^{(0)} \right] \left[J_{i,0}^{(1)} - J_{i,\epsilon,0}^{(1)} + \int d\Phi'_1 (J_{i,1}^{(0)} - J_{i,\epsilon,1}^{(0)}) \right] = \text{finite}$$

⇒ from the virtual structure we can identify some **real emission hard-collinear counterterms**

Purely hard-coll counterterms at NNLO for 2 particles from the same leg

$$K_{n+2,i}^{(2hc)} = \left[J_{i,2}^{(0)} - J_{i,\epsilon,2}^{(0)} - J_{i,\epsilon,1}^{(0)} (J_{i,1}^{(0)} - J_{i,\epsilon,1}^{(0)}) \right] |\mathcal{A}_n^{(0)}|^2$$

$$K_{n+2,i}^{(1hc)} = \left[J_{i,1}^{(0)} - J_{i,\epsilon,1}^{(0)} \right] |\mathcal{A}_{n+1}^{(0)}|^2$$

$$K_{n+1,i}^{(RV,hc)} = \left[J_{i,1}^{(1)} - J_{i,\epsilon,1}^{(1)} - J_{i,0}^{(1)} J_{i,\epsilon,1}^{(0)} - J_{i,1}^{(0)} J_{i,\epsilon,0}^{(1)} + 2 J_{i,\epsilon,0}^{(1)} J_{i,\epsilon,1}^{(0)} \right] |\mathcal{A}_n^{(0)}|^2$$

Complete list of counterterms

$$K_{n+2}^{(2s)} = \mathcal{H}_n^{(0)\dagger} S_{n,2}^{(0)} \mathcal{H}_n^{(0)}$$

$$K_{n+2}^{(1,s)} = \mathcal{H}_{n+1}^{(0)\dagger} S_{n+1,1}^{(0)} \mathcal{H}_{n+1}^{(0)}$$

$$K_{n+1}^{(\text{RV}, s)} = \mathcal{H}_n^{(0)\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(0)\dagger} S_{n,1}^{(1)} \mathcal{H}_n^{(0)}$$

$$K_{n+2,i}^{(2hc)} = [J_{i,2}^{(0)} - J_{\mathbb{E},i,2}^{(0)} - J_{\mathbb{E},i,1}^{(0)}(J_{i,1}^{(0)} - J_{\mathbb{E},i,1}^{(0)})] |\mathcal{A}_n^{(0)}|^2$$

$$K_{n+2,i}^{(1,hc)} = (J_{i,1}^{(0)} - J_{\mathbb{E},i,1}^{(0)}) |\mathcal{A}_{n+1}^{(0)}|^2$$

$$K_{n+1,i}^{(\text{RV}, hc)} = [J_{i,1}^{(1)} - J_{\mathbb{E},i,1}^{(1)} - J_{i,0}^{(1)} J_{\mathbb{E},i,1}^{(0)} - J_{\mathbb{E},i,0}^{(1)} J_{i,1}^{(0)} + 2 J_{\mathbb{E},i,0}^{(1)} J_{\mathbb{E},i,1}^{(0)}] |\mathcal{A}_n^{(0)}|^2$$

$$K_{n+2,ij}^{(2hc)} = (J_{i,1}^{(0)} - J_{\mathbb{E},i,1}^{(0)}) (J_{j,1}^{(0)} - J_{\mathbb{E},j,1}^{(0)}) |\mathcal{A}_n^{(0)}|^2$$

$$K_{n+1,ij}^{(\text{RV}, hc)} = [(J_{i,0}^{(1)} - J_{\mathbb{E},i,0}^{(1)}) (J_{j,1}^{(0)} - J_{\mathbb{E},j,1}^{(0)}) + (i \leftrightarrow j)] |\mathcal{A}_n^{(0)}|^2$$

$$K_{n+2,i}^{(1hc, 1s)} = (J_{i,1}^{(0)} - J_{\mathbb{E},i,1}^{(0)}) \mathcal{H}_n^{(0)\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(0)}$$

$$K_{n+1,i}^{(\text{RV}, 1hc, 1s)} = (J_{i,0}^{(1)} - J_{\mathbb{E},i,0}^{(1)}) \mathcal{H}_n^{(0)\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(0)} + (J_{i,1}^{(0)} - J_{\mathbb{E},i,1}^{(0)}) \mathcal{H}_n^{(0)\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(0)}$$

$$K_{n+1,i}^{(\text{RV}, 1hc)} = (J_{i,1}^{(0)} - J_{\mathbb{E},i,1}^{(0)}) (\mathcal{H}_n^{(0)\dagger} S_{n,0}^{(0)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\dagger} S_{n,0}^{(0)} \mathcal{H}_n^{(0)})$$

Outlook

Outlook

Some work is done:

- Important information has been derived from the factorisation approach.
- General all-orders definition of soft and/or collinear counterterms has been proposed.
- Existing results at NLO are reproduced and systematised.
- New strategy of tracing the real emission counterterms starting from virtual poles.

a lot of work remains to be done:

- detailed analytic subtraction algorithm is under way
- higher orders investigation

Backup

Practical implementation of the Subtraction method 1/2

Ingredients:

- partition of the phase space Φ_{n+1} with sector functions \mathcal{W}_{ij} ($\sigma_{ij} = \frac{1}{e_i \omega_{ij}}$)
 - minimum amount of singularities
 - sum to unity

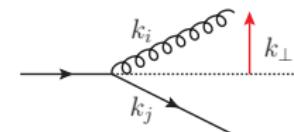
$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k,l \neq k} \sigma_{kl}} \rightarrow \sum_{i,j \neq i} \mathcal{W}_{ij} = 1, \quad \mathbf{s}_i \sum_{j \neq i} \mathcal{W}_{ij} = 1, \quad \mathbf{c}_{ij} \sum_{a,b \in \text{perm}(ij)} \mathcal{W}_{ab} = 1$$

- momentum mapping: $\{k_1, \dots, k_{n+1}\} \rightarrow \{\bar{k}_1, \dots, \bar{k}_n\}$
 - phase space factorisation $d\Phi_{n+1} = d\bar{\Phi}_n d\bar{\Phi}_{\text{rad}}$
 - n on-shell particles conserving momentum.
- singular structure of \mathbf{R} in sector ij [9908523]

$$\mathbf{s}_i R(\{k\}) = -\mathcal{N} \sum_{c,d} \underbrace{\delta_{f_{ig}} \frac{s_{cd}}{s_{ic} s_{id}}}_{\text{eikonal kernel}} B_{cd}(\{k\}_f)$$

$$\mathbf{C}_{ij} R(\{k\}) = \mathcal{N} \frac{1}{s_{ij}} \underbrace{P_{ij}^{\mu\nu}(s_{ir}, s_{jr})}_{\text{AP splitting kernel}} B_{\mu\nu}(\{k\}_f, k)$$

$$\mathbf{s}_i \mathbf{C}_{ij} R(\{k\}) = 2 \mathcal{N} C_{f_j} \delta_{f_{ig}} \frac{s_{jr}}{s_{ij} s_{id}} B(\{k\}_f)$$



$\bar{\mathbf{s}}_i R = \text{leading in } R (k_i^\mu \rightarrow 0)$

$\bar{\mathbf{C}}_{ij} R = \text{leading in } R (k_\perp^\mu \rightarrow 0)$

Practical implementation of the Subtraction method 2/2

The local counterterm in the barred kinematic

$$\bar{K}_{ij} \equiv (\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij} - \bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij}) R \mathcal{W}_{ij}$$

$$\begin{aligned}\rightarrow \quad \bar{K} &= \sum_{i,j \neq i} \bar{K}_{ij} = \sum_i (\bar{\mathbf{S}}_i R) \left[\overbrace{\bar{\mathbf{S}}_i \sum_{j \neq i} \mathcal{W}_{ij}}^{=1} \right] + \sum_{i,j > i} (\bar{\mathbf{C}}_{ij} R) \left[\overbrace{\bar{\mathbf{C}}_{ij} (\mathcal{W}_{ij} + \mathcal{W}_{ji})}^{=1} \right] \\ &\quad - \sum_{i,j \neq i} (\bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij} R) \left[\underbrace{\mathbf{S}_i \bar{\mathbf{C}}_{ij} \mathcal{W}_{ij}}_{=1} \right] \\ &= \sum_i \bar{\mathbf{S}}_i R + \sum_{i,j > i} \bar{\mathbf{C}}_{ij} (1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j) R\end{aligned}$$

The integrated counterterm $I_n(\{\bar{k}\}) = \int d\Phi_{rad} \bar{K}_{n+1}$:

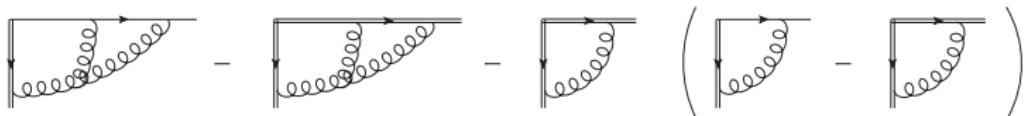
- exact in ϵ
- same pole as the virtual contribution
- finite parts checked

Collinear structure at amplitude-level @NNLO

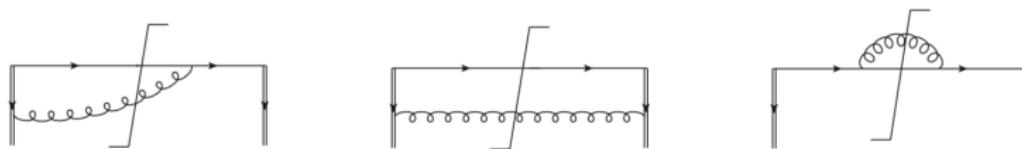
The collinear region arises from the quotient between the collinear and the eikonal functions

$$\begin{aligned}
 \frac{\prod_i^n \mathcal{J}_i(p_i)}{\prod_i^n \mathcal{J}_{\epsilon,i}(\beta_i)} = 1 &+ \sum_i \left[\mathcal{J}_i^{(1)}(p_i, n) - \mathcal{J}_{\epsilon,i}^{(1)}(\beta_i, n) \right] \\
 &+ \sum_i \left[\mathcal{J}_i^{(2)}(p_i, n) - \mathcal{J}_{\epsilon,i}^{(2)}(\beta_i, n) \right] \\
 &+ \sum_{i,j>i} \left[\mathcal{J}_i^{(1)}(p_i, n) - \mathcal{J}_{\epsilon,i}^{(1)}(\beta_i, n) \right] \left[\mathcal{J}_j^{(1)}(p_j, n) - \mathcal{J}_{\epsilon,j}^{(1)}(\beta_j, n) \right] \\
 &- \sum_i \mathcal{J}_{\epsilon,i}^{(1)}(\beta_i, n) \left[\mathcal{J}_i^{(1)}(p_i, n) - \mathcal{J}_{\epsilon,i}^{(1)}(\beta_i, n) \right].
 \end{aligned}$$

Pure hard-collinear content:



Collinear splitting at NLO



Single-radiative jet function at the lowest perturbative order in the coupling constant

$$\sum_s J_{q,1}(k; l, p, n) = \frac{4\pi\alpha_s C_F}{(l^2)^2} (2\pi)^d \delta^d(l - p - k) \left[-l \gamma_\mu \not{\phi} \gamma^\mu l + \frac{l^2}{k \cdot n} (l \not{\phi} \not{\phi} + \not{\phi} \not{\phi} l) \right]$$

⇒ **Sudakov parametrisation** for momenta p^μ and k^μ

$$p^\mu = z l^\mu + \mathcal{O}(l_\perp), \quad k^\mu = (1-z) l^\mu + \mathcal{O}(l_\perp), \quad n^2 = 0.$$

⇒ **Leading behaviour** in the $l_\perp \rightarrow 0$ limit

$$\sum_s J_{q,1}(k; l, p, n) = \frac{8\pi\alpha_s C_F}{l^2} (2\pi)^d \delta^d(l - p - k) \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right],$$

The leading order unpolarised DGLAP splitting function $P_{q \rightarrow qg}$.