Gravitational Waves from Dark Matter

Iason Baldes
In collaboration with Camilo Garcia-Cely

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20 May 2019

Two big discoveries in the past decade

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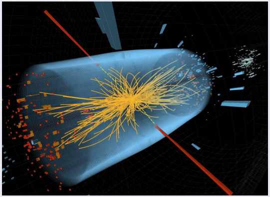

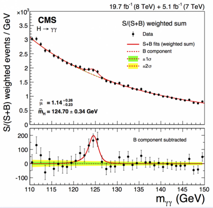

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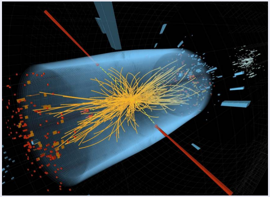
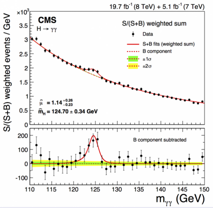
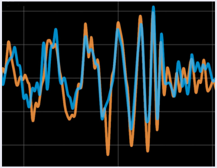




2016. Direct Detection of Gravitational Waves



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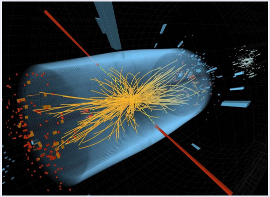

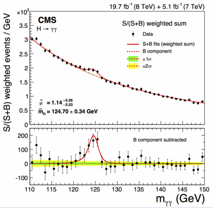

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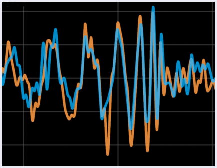

Let us merge the two ideas.

Gravitational Waves from an early Universe Phase Transition

Actually already done

by Witten '84, Hogan '86, ...

PHYSICAL REVIEW D VOLUME 30, NUMBER 2 15 JULY 1984

Cosmic separation of phases

Edward Witten*
Institute for Advanced Study, Princeton, New Jersey 08540
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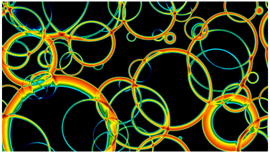
- Symmetry is typically restored at high T.
- Violent events (e.g. cosmological phase transitions) produce gravitational waves.

Gravitational Waves from an early Universe Phase Transition



From a simulation by Weir et. al.

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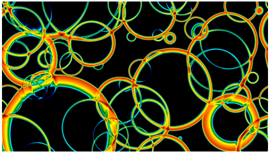

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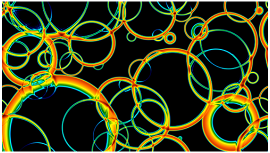

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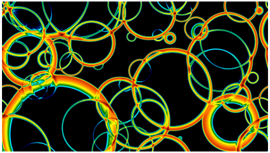

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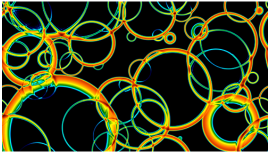

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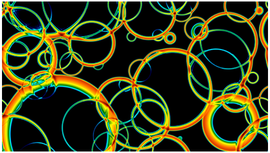

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The idea here is to explore a simple case study as to the feasibility of using GWs to detect SSB in a dark sector.

A simple DM model - Hambye 0811.0172

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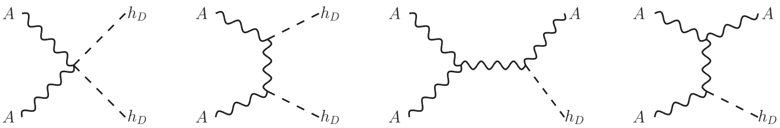

The Model: $SU(3)_C \times SU(2)_L \times U(1)_Y \times SU(2)_D$

$$\mathcal{L} \supset -\frac{1}{4} F_D \cdot F_D + (\mathcal{D} H_D)^\dagger (\mathcal{D} H_D) - \mu_2^2 \, H_D^\dagger H_D - \lambda_\eta \, (H_D^\dagger H_D)^2 - \lambda_{h\eta} \, H_D^\dagger H_D \, H^\dagger H$$

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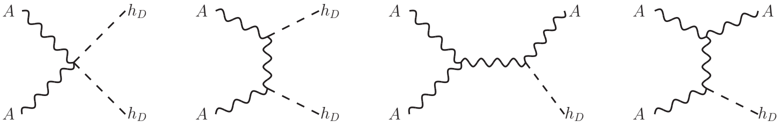

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Custodial SO(3) symmetry

Dark gauge bosons, A, are stable and form the DM!

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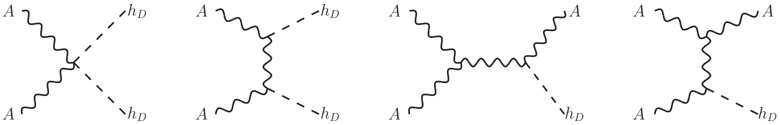

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Potential possibilities

1. Standard Potential with Mass terms - Hambye 0811.0172
2. Classically Scale Invariant
   - Hambye, Strumia 1306.2329, - Hambye, Strumia, Teresi 1805.01473

Standard Freezeout

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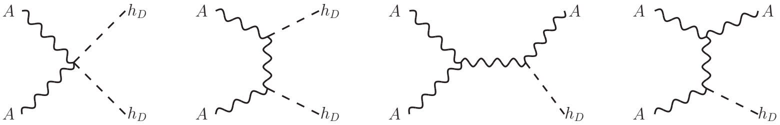

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$$g_D \approx 0.9 \times \sqrt{\frac{m_A}{1\ \mathrm{TeV}}}$$

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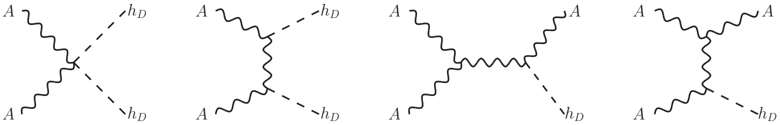

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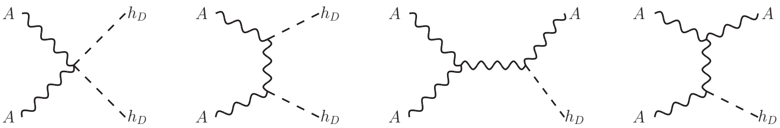

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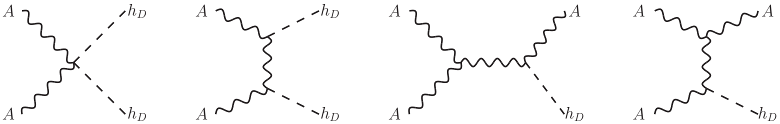

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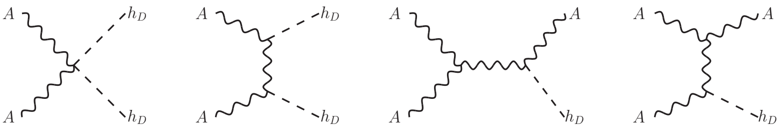

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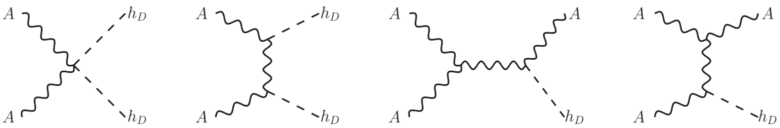

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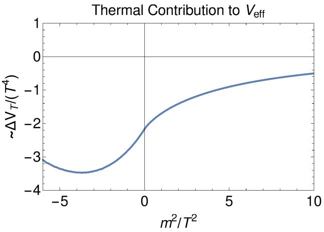
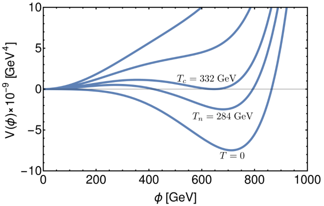
Close link between Ω_{DM} and SSB $\to$ Test using GWs!

Finite temperature effective potential





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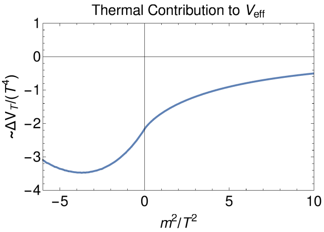

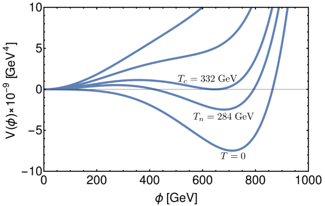

$$V_{\text{eff}} = V_{\text{tree}}(\phi) + V_1^0(\phi) + V_1^T(\phi, T) + V_{\text{Daisy}}(\phi, T)$$

Thermal Contribution

$$\frac{2\pi^2}{T^4} V_1^T(\phi, T) = \int_0^\infty y^2 \text{Log}\left(1 - e^{-\sqrt{y^2 + m_i^2(\phi)/T^2}}\right) \text{d}y$$

$$\approx -\frac{\pi^4}{45} + \frac{\pi^2 m^2}{12 T^2} - \frac{\pi m^3}{6 T^3} - \frac{m^4}{32 T^4} \text{Ln}\left(\frac{m^2}{220 T^2}\right)$$

Calculation of the GW spectrum

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Euclidean Action

$$S_3 = 4\pi \int r^2 \left(\frac{1}{2} \left(\frac{d\phi_i}{dr} \right)^2 + \Delta V(\phi, \eta, T) \right) dr$$

Nucleation when $\Gamma/V \sim T^4 e^{-S_3/T} \sim H^4$.

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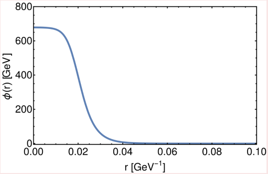

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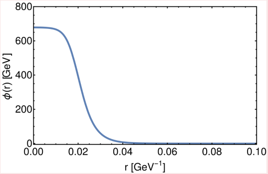

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Find the latent heat and timescale of the PT

$$\alpha = \frac{1}{\rho_{\text{rad}}} \left(1 - T \frac{\partial}{\partial T} \right) \left(V[\phi_0, \eta_0] - V[\phi_n, \eta_n] \right) \Big|_{T_n}$$

$$\beta = -\frac{d}{dt} \left(\frac{S_3}{T} \right) = H\, T_n \frac{d}{dT} \left(\frac{S_3}{T} \right) \Big|_{T_n}$$

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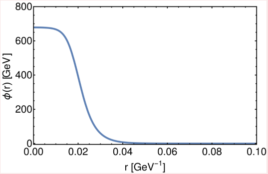

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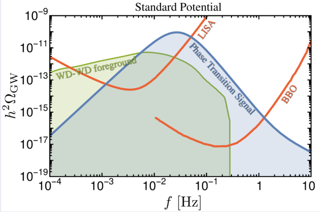
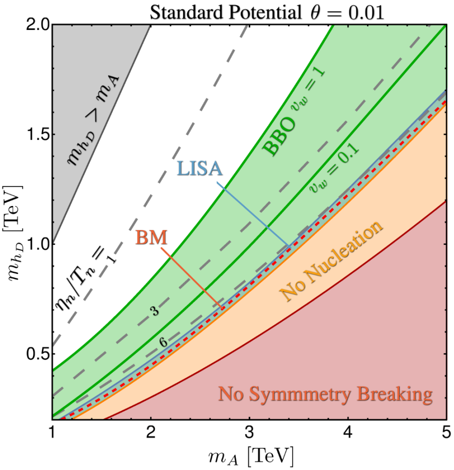
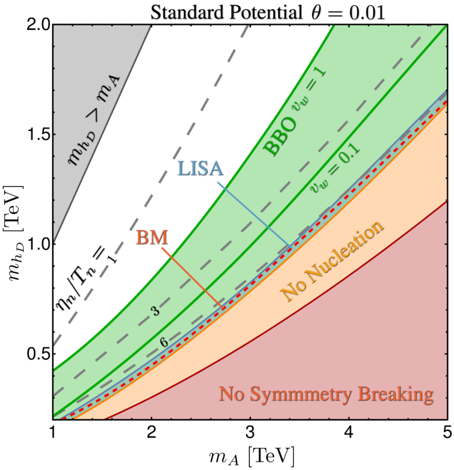


Results



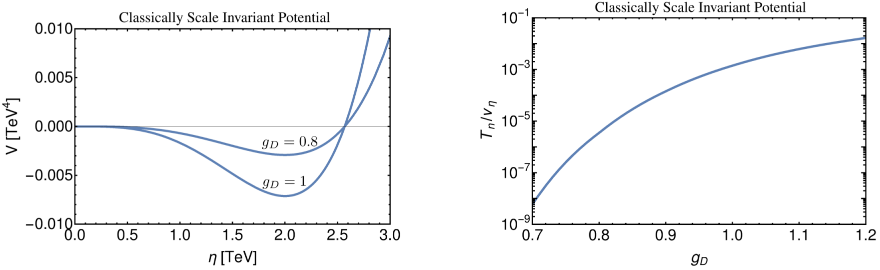
Standard Potential $\theta = 0.01$
$m_{h_D} > m_A$
$\eta_n/T_n = 1$
3
6
BM
LISA
BBO
$v_w = 1$
$v_w = 0.1$
No Nucleation
No Symmmetry Breaking
m_{h_D} [TeV]
m_A [TeV]
2.0
1.5
1.0
0.5
1
2
3
4
5



LISA can test only limited parameter space of standard, polynomial type, potentials. BBO can do somewhat better. But we are really after a scenario which generically returns a lot of supercooling.

Classically Scale Invariant Potential

- Hambye, Strumia 1306.2329



Potential at $T = 0$

$$V_1^0(\eta) \simeq \frac{9 g_D^4 \eta^4}{512\pi^2}\left(\mathrm{Ln}\left[\frac{\eta}{v_\eta}\right] - \frac{1}{4}\right)$$

The thermal contribution of the gauge bosons is added to this.
Universe generically becomes vacuum dominated before PT.
For $T_n < \Lambda_{\mathrm{QCD}}$ need to add effects of QCD

- Iso, Serpico, Shimada 1704.04955

DM relic density

DM relic density

DM and PT possibilities

- **Regime (i): standard freeze-out.**
  (ia). $T_n > \Lambda_{\mathrm{QCD}}$.
  (ib). $T_n < \Lambda_{\mathrm{QCD}}$. (QCD effects break the scale invariance)

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Super-cool DM - Hambye, Strumia, Teresi 1805.01473

$$Y_{\rm DM}|_{\rm super-cool} = Y_{\rm DM}^{\rm eq} \frac{T_{\rm RH}}{T_{\rm infl}} \left(\frac{T_{\rm end}}{T_{\rm infl}} \right)^3$$

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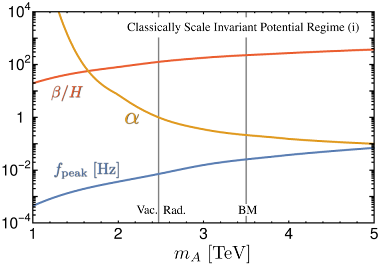
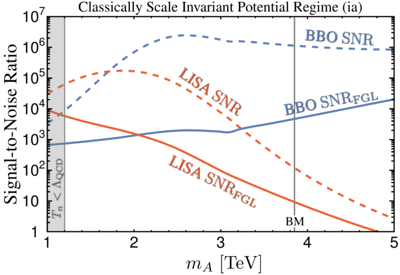
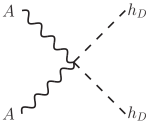
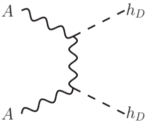
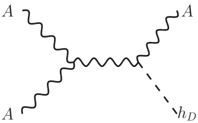
Regime (ia) and (iia) are ameable for testing using GWs!

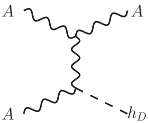












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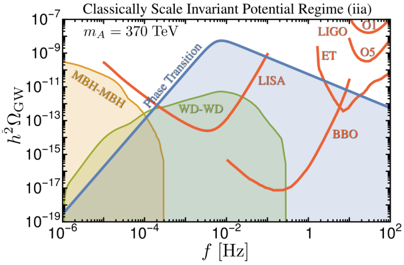
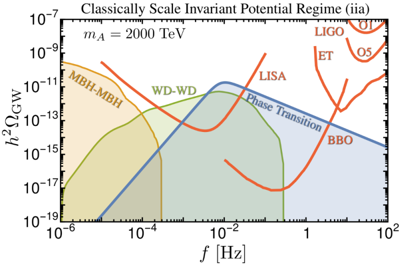
GW signal Regime (iia) - Super-cool DM





Super-cool DM

$$Y_{\rm DM}|_{\rm super-cool} = Y_{\rm DM}^{\rm eq} \frac{T_{\rm RH}}{T_{\rm infl}} \left(\frac{T_{\rm end}}{T_{\rm infl}} \right)^3$$

Here $g_D \simeq 1$ and $m_A \gtrsim 370$ TeV.

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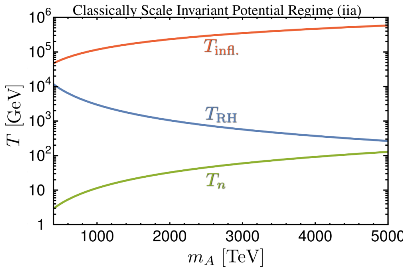
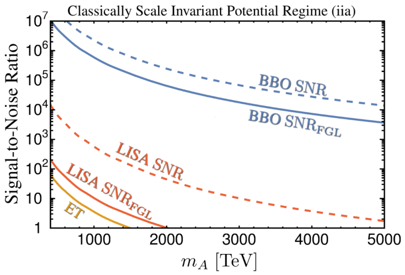
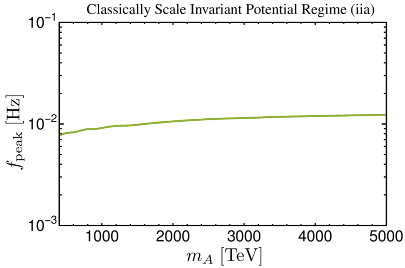




We correct for the period of matter domination after the PT.

$$f_{\rm peak} \to \left(\frac{T_{\rm RH}}{T_{\rm infl}}\right)^{1/3} f_{\rm peak} \qquad \Omega_{\rm GW} \to \left(\frac{T_{\rm RH}}{T_{\rm infl}}\right)^{4/3} \Omega_{\rm GW}$$

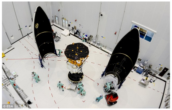
Peak Frequency Regime (iia) - Super-cool DM

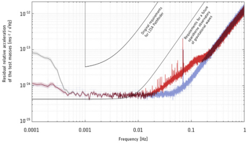


Key prediction of the model

We find the peak frequency here is $\sim 10^{-2}$ Hz almost independent of m_A.

Summary





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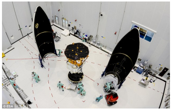

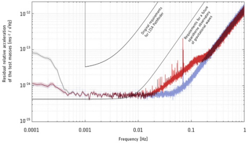

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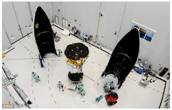


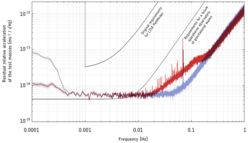

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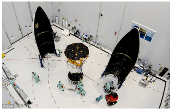

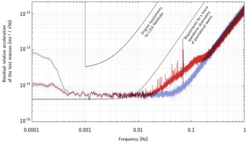

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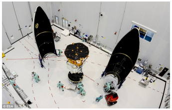

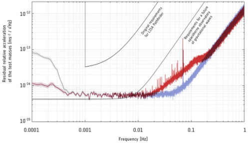

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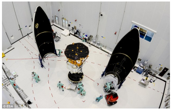

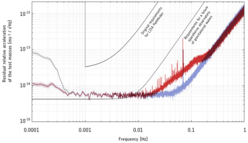

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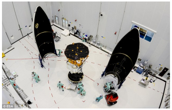

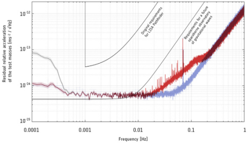

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- Phase transitions: another pheno avenue to explore in your favourite models.
- Much work still needed → exciting times ahead.

Backup

The terms of the one-loop effective potential

Effective Potential

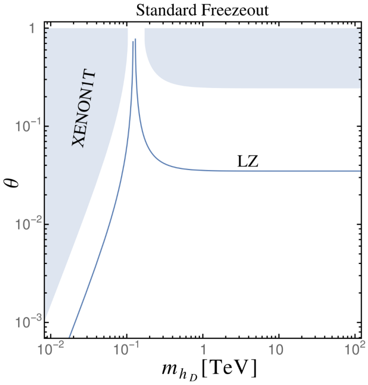
$$V_{\text{eff}} = V_{\text{tree}}(\phi) + V_1^0(\phi) + V_1^T(\phi, T) + V_{\text{Daisy}}(\phi, T)$$

$$V_1^0(\phi) = \sum_i \frac{g_i(-1)^F}{64\pi^2}\left\{ m_i^4(\phi)\left(\text{Log}\left[\frac{m_i^2(\phi)}{m_i^2(v)}\right] - \frac{3}{2}\right) + 2m_i^2(\phi)m_i^2(v)\right\}$$

$$V_1^T(\phi, T) = \sum_i \frac{g_i(-1)^F T^4}{2\pi^2} \times \int_0^\infty y^2 \text{Log}\left(1 - (-1)^F e^{-\sqrt{y^2 + m_i^2(\phi)/T^2}}\right) \mathrm{d}y$$

$$V_{\text{Daisy}}^{\phi}(\phi, T) = \frac{T}{12\pi}\left\{ m_\phi^3(\phi) - \left[m_\phi^2(\phi) + \Pi_\phi(\phi, T)\right]^{3/2}\right\}$$

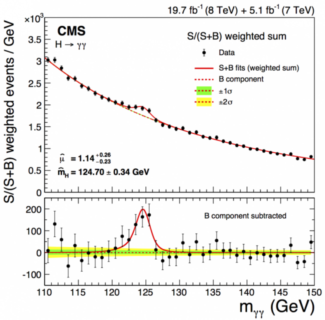
Direct Detection - Limit on Mixing

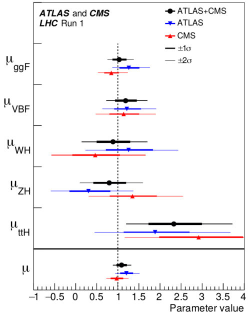


$$\sigma_{\rm SI} = \frac{g_D^4\, f^2\, m_N^4\, v_\eta^2}{64\pi\,(m_N + m_A)^2\, v_\phi^2}\left(\frac{1}{m_h^2} - \frac{1}{m_{h_D}^2}\right)^2 \sin^2 2\theta$$

For $m_A \gtrsim \mathcal{O}(100)$ GeV, need $\theta \lesssim 0.2$.

LHC constraints - Limit on Mixing

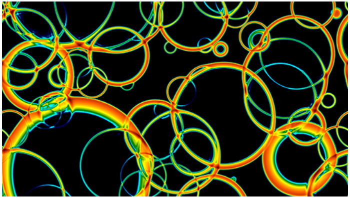




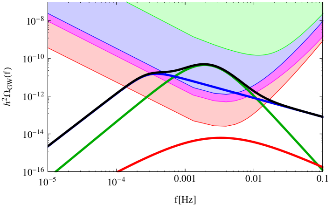
$\mu = 1.09 \pm 0.11$	LHC Run 1	$7 + 8$ TeV	1606.02266
$\mu = 1.10 \pm 0.06$	LHC Run 2	13 TeV	1810.02521

$$\theta \lesssim \mathcal{O}(0.1)$$

Predicted GW spectra



From a simulation by Weir et. al.



LISA working group 1512.06239

$$h^2\Omega_{\mathrm{GW}}(f) \equiv h^2 \frac{f}{\rho_c} \frac{d\rho_{\mathrm{GW}}}{df}$$

Three contributions

1. Scalar field contribution
2. Sound waves in the plasma
3. Magnetohydrodynamic Turbulence.

Predicted GW spectra

The spectra depend on the macroscopic properties

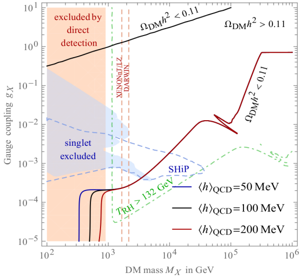
- Latent heat α
- Timescale of the transition β^{-1}
- The Hubble scale (or almost equivalently T_n)
- The wall velocity v_w

These are all calculable from microphysics (although v_w is technically challenging).
We can calculate these quantities and then match onto results from simulations/semi-analytic studies.

If enough of a plasma is present - Bodeker, Moore 1703.08215

- Runaway wall is prevented by $P_{\rm LO} \sim T^2 \Delta M^2$ or $P_{\rm NLO} \sim \gamma g^2 T^3 \Delta M$
- Scalar field contribution is suppressed.

Super-cool DM relic density

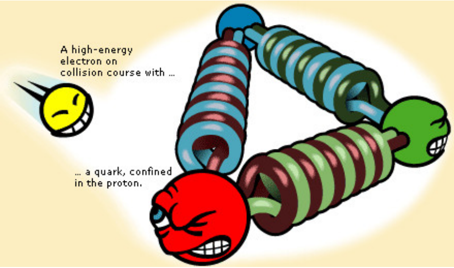


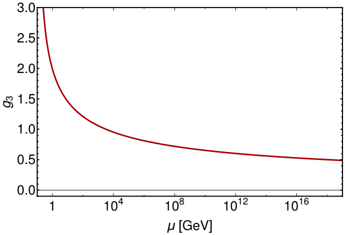
Super-cool DM - Hambye, Strumia, Teresi 1805.01473

$$Y_{\rm DM}|_{\rm super-cool} = Y_{\rm DM}^{\rm eq} \frac{T_{\rm RH}}{T_{\rm infl}} \left(\frac{T_{\rm n}}{T_{\rm infl}}\right)^3$$

$$Y_{\rm DM}|_{\rm sub-thermal} = M_{\rm Pl} M_{\rm DM} \langle \sigma_{\rm ann} v_{\rm rel}\rangle \sqrt{\frac{\pi g_*}{45}} \int_{z_{\rm RH}}^{\infty} \frac{dz}{z^2} Y_{\rm eq}^2$$

Taking into account QCD





If $T_n \lesssim \Lambda_{\rm QCD}$, QCD confinement must be taken into account.

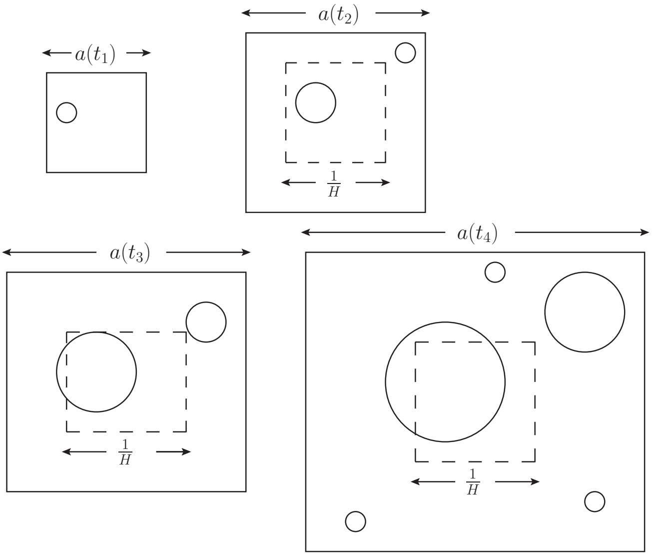
- When QCD confines a mass scale enters the potential.
- EW Symmetry is broken by the quark condensate.
- The Higgs gets a VEV $\langle h \rangle \sim \Lambda_{\rm QCD}$ induced by $y_t h \langle \overline{t_L} t_R \rangle$.
  - Witten '81
- This gives a mass term $V_{\rm eff} \supset -\lambda_{h\eta} \Lambda_{QCD}^2 \eta^2$.
- The thermal barrier disappears at $T \sim m_h \Lambda_{QCD} / m_A$.
  - Iso, Serpico, Shimada 1704.04955

Why is the signal suppressed for $T_n < \Lambda_{\mathrm{QCD}}$?

- With massless quarks QCD PT is first order at $T \sim \Lambda_{QCD}$: GW signal
  - Helmboldt, Kubo, van der Woude 1904.07891
- However inflation continues until $T \sim m_h \Lambda_{QCD}/m_A$
  $\rightarrow$ suppresses signal.
- $SU(2)_D$ PT is also first order.
- But due to mass term $V_{\mathrm{eff}} \supset -\lambda_{h\eta}\Lambda_{QCD}^2\eta^2$ signal is weak.

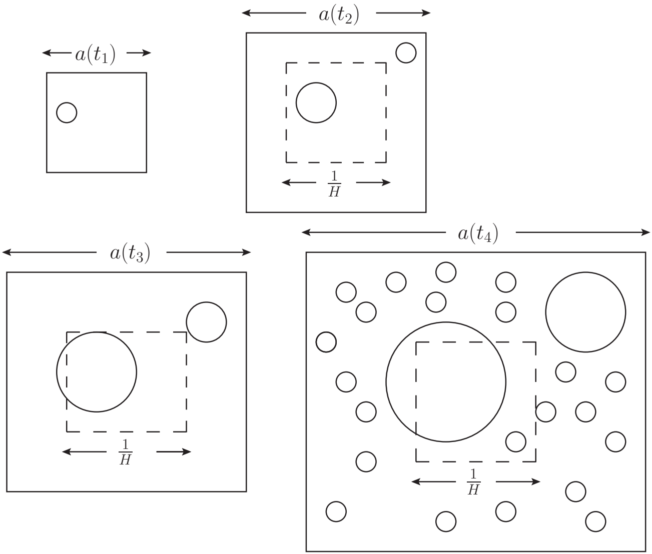
So we focus on $T_n > \Lambda_{\mathrm{QCD}}$ instead.

Completion of the Phase Transtion



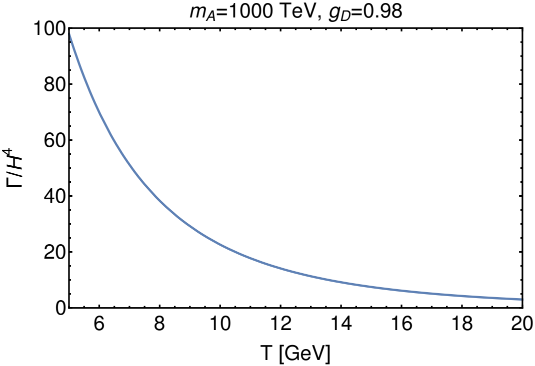
If nucleation rate is low, we can form bubbles which never meet.

Completion of the Phase Transtion



If nucleation grows enough, sufficient bubbles to meet will nucleate.

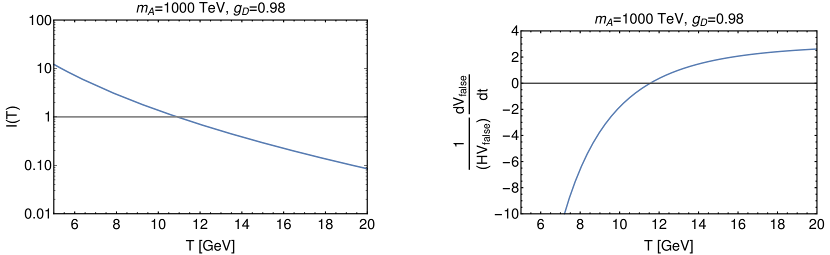
Completion of the Phase Transtion



In the classically scale invariant potential we have a slow transition but an exponentially growing nucleation rate.

Completion of the Phase Transtion

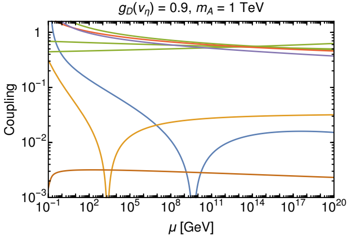
We can explicitly check the volume of false vacuum decreases and the bubbles will percolate.



$$P(T) \equiv e^{-I(T)} \lesssim 1/e \implies I(T) = \frac{4\pi}{3} \int_{t_c}^{t} dt' \Gamma(t') a(t')^3 r(t,t')^3 \gtrsim 1$$

$$\frac{1}{H\mathcal{V}_{\text{false}}} \frac{d\mathcal{V}_{\text{false}}}{dt} = 3 + T \frac{dI}{dT} \lesssim -1.$$

Radiative Symmetry Breaking



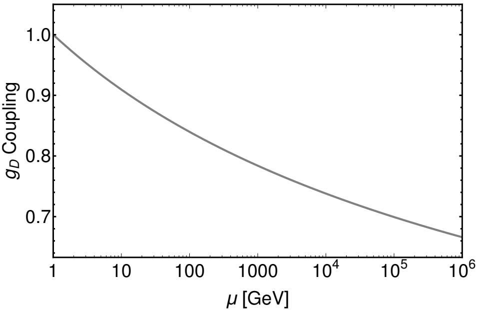
We start with a classically scale invariant theory

- The dark gauge coupling drives the exotic quartic negative in the IR

$$\beta_{\lambda_\eta} = \frac{1}{(4\pi)^2}\left(\frac{9}{8}g_D^4 - 9g_D^2\lambda_\eta + 2\lambda_{h\eta}^2 + 24\lambda_\eta^2\right)$$

- This signals radiative symmetry breaking - Coleman, E. Weinberg '73
- The potential is approximated in the flat direction in field space - Gildener, S. Weinberg '76

Dark Running



$$\frac{dg_D}{d\, ln(\mu)} = \frac{g_D^3}{(4\pi)^2}\left(-\frac{22}{3} + \frac{1}{6}\right)$$