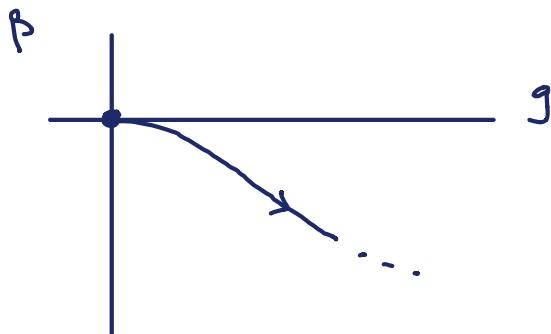


II. Freedom on the lattice

1. Consider QCD w. n_f flavor, masses

$$\beta(g) = -[b_0 g^3 + b_1 g^5 + \dots]$$

$$b_0 = \frac{1}{(4\pi)^2} \left(11 - \frac{2}{3} n_f \right), \quad b_1 = \frac{1}{(4\pi)^4} \left(102 - \frac{38}{3} n_f \right).$$



$\beta(g) < 0$ for $g \approx 0$ \rightarrow asymptotic freedom at short distances

g is a marginal deformation away from the free theory.

$$(*) -\alpha \frac{d}{da} g = -b_0 g^3 \quad @ 1\text{-loop}$$

$$\frac{dg}{g^3} = b_0 \frac{da}{a} \Rightarrow -\frac{1}{2} \left(\frac{1}{g^2} - \frac{1}{g_0^2} \right) = b_0 \log \frac{a}{a_0}$$

$$\Rightarrow \frac{1}{g^2} = \frac{1}{g_0^2} + b_0 \log \frac{a_0^2}{a^2}$$

$$\frac{1}{g_0^2} = b_0 \log \left(\frac{1}{a^2 a_0^2} \right) \Rightarrow \frac{1}{g^2(a)} = b_0 \log \left(\frac{1}{a^2 a_0^2} \right).$$

$$g(a)^2 = \frac{1}{b_0 \log\left(\frac{a_2}{a_0^2 g_2}\right)} \quad \wedge \text{ integration constant}$$

$g(a)$ is determined by \wedge i.e. by one renormalization cond

NB. A dimensionful parameter emerges when integrating the RG evolution \rightarrow trace anomaly.

(o) beyond pert. th.

$$\frac{da}{a} = - \frac{dg}{\beta(g)} \Rightarrow \log\left(\frac{a_2}{a_0}\right) = \int_{g_2}^{g_1} \frac{dx}{\beta(x)}$$

$a_2 \rightarrow 0$, log divergence on LHS

\hookrightarrow RHS is divergent because $\frac{1}{\beta(x)}$ is not integrable at $x \approx$

$$\frac{1}{\beta(x)} = - \frac{1}{b_0 x^3} \frac{1}{1 + \frac{b_1}{b_0} x^2 + O(x^4)} \approx - \frac{1}{b_0 x^3} \left[1 - \frac{b_1}{b_0} x^2 + O(x^4) \right]$$

$$\begin{aligned} \int_{g_2}^{g_1} \frac{dx}{\beta(x)} &= \left[\frac{1}{2b_0 g_1^2} - \frac{1}{2b_0 g_2^2} + \frac{b_1}{b_0^2} \log g_1 - \frac{b_1}{b_0^2} \log g_2 \right] + \\ &+ \int_{g_2}^{g_1} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right]. \end{aligned}$$

$$\frac{\Lambda}{a_0} \left(b_0 g_1^2 \right)^{-b_1/2b_0^2} \exp \left[-\frac{1}{2b_0 g_1^2} \right] \exp \left\{ - \int_0^{g_1} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\} =$$

$$= \frac{\Lambda}{a_0} \left(b_0 g_2^2 \right)^{-b_1/2b_0^2} \exp \left[-\frac{1}{2b_0 g_2^2} \right] \exp \left\{ - \int_0^{g_2} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\} .$$

$$(a1) = \left(b_0 g^2 \right)^{-b_1/2b_0^2} \exp \left[-\frac{1}{2b_0 g^2} \right] \exp \left\{ - \int_0^g dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\}.$$

Given Λ , the relation b/w g & a is fixed uniquely

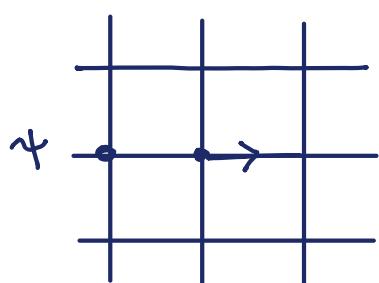
e.g. m_p is all we need to find Λ in QCD.

\hookrightarrow or any other dimensionful quantity.

Exercise: show that the dependence of Λ on the scheme can be determined exactly @ 1-loop in pert. theory.

2. Lattice formulation

$$x^M = a n^{\mu}$$



$$\partial_\mu \psi \rightarrow \nabla_\mu \psi = \frac{1}{a} [\psi(x+a\hat{\mu}) - \psi(x)] .$$

local transf.: $\psi(x) \mapsto g(x) \psi(x)$

Introduce a parallel transporter along a link: $U(x, \mu) = \text{P exp} i \int_x^{x+a\mu} dz_\mu A^\mu$

$$U(x, \mu) \mapsto g(x) U(x, \mu) g(x+a\mu)^+$$

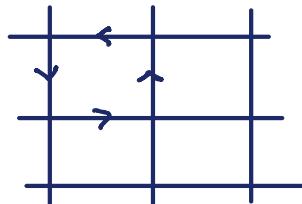
Define a covariant derivative

$$\nabla_\mu \psi(x) = \frac{1}{a} \left[U(x, \mu) \psi(x + a\hat{\mu}) - \psi(x) \right].$$

$$\nabla_\mu \psi(x) \mapsto g(x) \nabla_\mu \psi(x)$$

and hence $\bar{\psi}(x) \nabla_\mu \psi(x)$ is invariant.

Product of link variables along a closed loop is also invariant.



$$P_{\mu\nu}(x) = \text{tr} [U(x, \mu) U(x + a\mu, \nu) \dots]$$

$$\begin{aligned} S[U, \psi, \bar{\psi}; g, a] = & \sum_x \left\{ \beta \sum_{\mu < \nu} \left[1 - \frac{1}{2N} (P_{\mu\nu}(x) + \text{c.c.}) \right] + \right. \\ & \left. + \bar{\psi}(x) \gamma_\mu \nabla^\mu \psi(x) \right\} \end{aligned}$$

$\beta = \frac{2N}{g^2}$ is the bare gauge coupling.

Fix β , compute $(aM) = \hat{M}$, check that $aM \lesssim 1$.

↳ set the scale by requiring

$$\boxed{\frac{\hat{M}}{a} = M_{\text{phys}}}.$$

3. IR fixed point

(Banks & Zaks)

5.

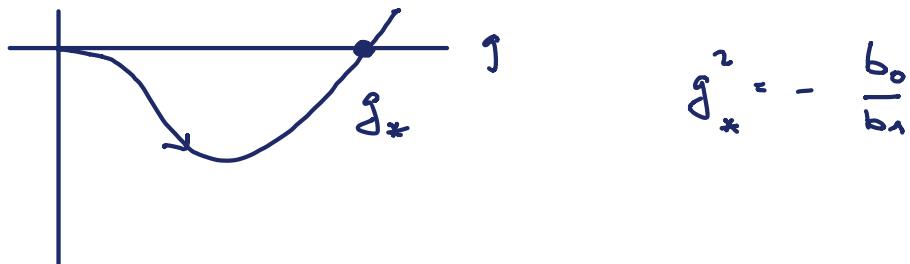
Consider fermions in the fund. representation of $SU(3)$

$$\beta(g) = -b_0 g^3 - b_1 g^5 + \dots$$

$$b_0 = \frac{1}{(4\pi)^2} \left[11 - \frac{2}{3} n_f \right] \quad b_0 > 0, \quad n_f < \frac{33}{2}$$

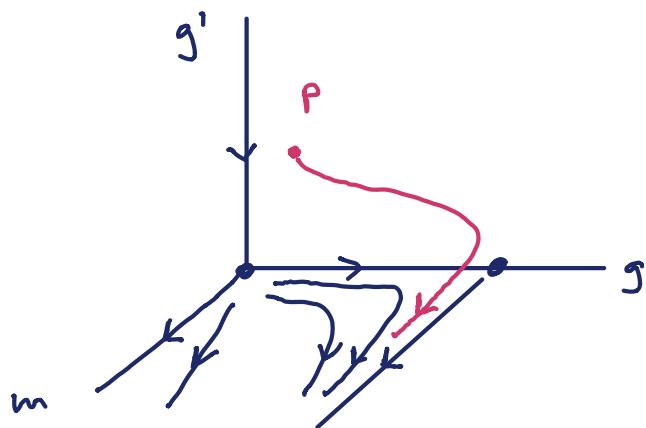
$$b_1 = \frac{1}{(4\pi)^4} \left[102 - \frac{38}{3} n_f \right], \quad b_1 < 0, \quad \frac{306}{38} < n_f < \frac{33}{2}$$

Range in n_f when $b_0 > 0, b_1 < 0$



if $\beta_*^2 \ll 1$ trust pert. calculation

↳ can we find non-pert. evidence of IR fixed pt.?



Scaling relations - example

Hadronic spectrum:

$$c_H(t; g, \hat{m}, a) = \int d^3x \langle H(t, \vec{x}) H(\omega)^+ \rangle_{g, \hat{m}, a}$$

$H(x)$ interpolating field, i.e. field w. the quantum # of the state H .

$$c_H(t; g, \hat{m}, a) \sim e^{-M_H t}$$

RG transf. $a' = b a$

$$c_H(t; g, \hat{m}, a) = b^{-2\gamma_H} c_H(t; g', \hat{m}', a')$$

$$\begin{cases} \delta g' = b^{y_g} \delta g, & y_g < 0 \text{ at IR fixed pt.} \\ \hat{m}' = b^{y_m} \hat{m}, & y_m > 0 \end{cases}$$

Dimensional analysis

$$c_H(t; \hat{m}', a') = b^{-2d_H} c_H(t b'; \hat{m}', a')$$

choose $b = \hat{m}^{-1/y_m}$

$$c_H(t; g, \hat{m}, a) = k_H c_H(t^{\hat{m}^{-1}/y_m}; a)$$

7.

for a given value of a , the time dependence of $c_H(t; g, \hat{m}, a)$ is through the scaling variable $t^{\hat{m}^{-1}/y_m}$.

$$\Rightarrow (a M_H) \propto \hat{m}^{1/y_m}.$$