

# LATTICE FIELD THEORY

## I. Wilsonian approach

Introduce a UV cutoff  $a^{-1}$ , and consider the most general action that is allowed by the symmetries of the system.

$$S[\phi; g, a] = \int_a^D d^D x \left[ \frac{1}{2} (\partial \phi)^2 + \sum_k a^{d_k - D} g_k O_k \right]$$

$[O_k] = d_k$ ,  $g_k$  are dimensionless

There are several ways of varying the scales in the theory.

### 1. Dilatations "stretching space".

$$x \mapsto x' = b x, \quad \phi(x) \mapsto \phi'(x') = b^{-d_\phi} \phi(x)$$

$$\text{when } d_\phi = \frac{D-2}{2}$$

Let us consider an infinitesimal transf.  $b = e^t \approx 1+t$ , then

$$\phi'(x+tx) = \phi'(x) + t x \cdot \partial \phi'(x) = \phi'(x) + t x \cdot \partial \phi(x)$$

$$= (1 - d_\phi t) \phi(x)$$

$$\delta \phi(x) = \frac{\phi'(x) - \phi(x)}{t} = -[d_\dagger + x \cdot \partial] \phi(x)$$

$$\begin{aligned} \delta(m^2 \phi^2) &= m^2 2 (\delta \phi) \phi = -2 m^2 [d_\dagger + x \cdot \partial] \phi \phi \\ &= -m^2 [2 d_\dagger + x \cdot \partial] \phi^2 = -m^2 [D-2-D] \phi^2 \\ &= 2 m^2 \phi^2 \end{aligned}$$

$$\begin{aligned} \delta \phi^4 &= -4 [d_\dagger + x \cdot \partial] \phi \phi^3 = -[4 d_\dagger + x \cdot \partial] \phi^4 \\ &= -[2D-4-D] \phi^4 = -[D-4] \phi^4. \end{aligned}$$

More generally:  $O_k(x) = \phi(x)^k$

$$\begin{aligned} \delta \left( \int d^D x \mu^{D-d_k} g_k \phi(x)^k \right) &= \int d^D x \mu^{D-d_k} g_k k \delta \phi(x) \phi(x)^{k-1} \\ &= - \int d^D x \mu^{D-d_k} g_k k [d_\dagger + x \cdot \partial] \phi(x) \phi(x)^{k-1} \\ &= - \int d^D x \mu^{D-d_k} g_k [d_k + x \cdot \partial] \phi(x)^k \\ &= \int d^D x (\Delta - d_k) \mu^{D-d_k} g_k \phi(x)^k \end{aligned}$$

$$\delta S = \int d^D x t(x) [-\partial_\mu j^\mu(x) - \Delta(x)]$$

$$-\Delta(x) = \sum_k (D - d_k) \mu^{D-d_k} g_k O_k(x)$$

3.

Ward id.

$$\int \mathcal{D}\phi e^{-S[\phi]} \mathcal{F}(\phi) = \int \mathcal{D}\phi' e^{-S[\phi']} \mathcal{F}(\phi')$$

$$\left\{ \begin{array}{l} S[\phi'] = S[\phi] + \delta S, \quad \mathcal{D}\phi' = \mathcal{D}\phi \left[ 1 + \int d^D x t(x) X(x) \right] \\ \mathcal{F}(\phi') = \mathcal{F}(\phi) + \delta \mathcal{F} \end{array} \right.$$

$$\langle (-\delta S) \mathcal{F} \rangle + \langle (\delta A) \mathcal{F} \rangle + \langle \delta \mathcal{F} \rangle = 0$$

$$\int d^D x t(x) \langle \left( \partial_\mu j^\mu(x) + \Delta(x) + X(x) \right) \mathcal{F} + \frac{\delta \mathcal{F}}{\delta \phi(x)} \delta \phi(x) \rangle = 0$$

Hence

$$\langle \partial_\mu j^\mu(x) \mathcal{F} \rangle + \langle \left( \Delta(x) + X(x) \right) \mathcal{F} \rangle + \left\langle \frac{\delta \mathcal{F}}{\delta \phi(x)} \delta \phi(x) \right\rangle = 0$$

Familiar example:  $\mathcal{F} = \phi(x_1) \dots \phi(x_n)$

$$\langle \partial_\mu j^\mu(x) \phi(x_1) \dots \phi(x_n) \rangle + \langle \left( \Delta(x) + X(x) \right) \phi(x_1) \dots \phi(x_n) \rangle -$$

$$- \sum_i \delta(x-x_i) \left[ \Delta_\phi + x \cdot \partial \right] \langle \phi(x_1) \dots \phi(x) \dots \phi(x_n) \rangle = 0$$

Integrated ("no-momentum") Ward id.

4.

$$\int d^D x \langle (\Delta(x) + X(x)) \phi(x_1) \dots \phi(x_n) \rangle = \sum_i [d_{\phi_i} + x_i \cdot \partial_i] \langle \phi(x_1) \dots \phi(x_n) \rangle$$

$$\text{i.e. } \delta \langle \phi(x_1) \dots \phi(x_n) \rangle = \langle \tilde{T}_\mu^\mu(x) \phi(x_1) \dots \phi(x_n) \rangle$$

$$\tilde{T}_\mu^\mu(x) = \int d^D x' (\Delta(x') + X(x')) .$$

## 2. RG transformations

$$F(E, g, a) = \langle F(\phi; E) \rangle_{g, a} = \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi; g, a]} F(\phi; E)$$

$$aE \ll 1$$

Change the cutoff:  $a \mapsto a' = b a$

then the couplings to new values  $g'$  i.e.

$$F(E; g, a) = F(E; g', a') + O(aE).$$

↳ enough conditions to fix all the couplings?

couplings depend on the cutoff:  $g_k(a)$

## Flow in the space of couplings

$$\beta_k(g) = -a \frac{d}{da} g_k(a), \quad \gamma_\phi(g) = + \frac{1}{2} a \frac{d}{da} \log Z_\phi(a)$$

Invariance of low-energy physics

$$Z_\phi^{-n/2} \langle \phi(x_1) \dots \phi(x_n) \rangle_{g, a} = Z'_\phi^{-n/2} \langle \phi(x_1) \dots \phi(x_n) \rangle_{g', a'}$$

$$-a \frac{d}{da} \left[ Z_\phi^{-n/2} \langle \phi(x_1) \dots \phi(x_n) \rangle_{g, a} \right] = 0$$

$$\left[ -a \frac{d}{da} + \sum_k \beta_k(g) \frac{\partial}{\partial g_k} + n \gamma_\phi \right] \langle \phi(x_1) \dots \phi(x_n) \rangle = 0$$

↳ RGE f. correlators.

Dimensional analysis:

$$b^{nd_\phi} \langle \phi(bx_1) \dots \phi(bx_n) \rangle_{g, ba} = \langle \phi(x_1) \dots \phi(x_n) \rangle_a$$

ie. in infinitesimal form:

$$\left[ a \frac{\partial}{\partial a} + \sum_i \kappa_i \cdot \partial_i + n d_\phi \right] \langle \phi(x_1) \dots \phi(x_n) \rangle = 0$$

$$\Rightarrow \left[ \sum_i \kappa_i \cdot \partial_i + \sum_k \beta_k(g) \frac{\partial}{\partial g_k} + n \Delta_\phi \right] \langle \phi(x_1) \dots \phi(x_n) \rangle = 0$$

## Two important consequences

(i) "Trace anomaly"

$$\langle \tilde{T}_n^{\mu}(\omega) \phi(x_1) \dots \phi(x_n) \rangle =$$

$$= \left[ \sum_k \frac{\beta_k(g)}{g_k} g_k \frac{\partial}{\partial g_k} + n \gamma_{\phi} \right] \langle \phi(x_1) \dots \phi(x_n) \rangle$$

$$\Delta_{\phi} = d_{\phi} + \gamma_{\phi}, \quad \gamma_k(g) = -a \frac{d}{da} \log g_k.$$

" $g_k \frac{\partial}{\partial g_k}$ "  $\rightarrow$  zero-momentum insertion of  $-\int d^D x a^{d_k - D} g_k O_k(x)$

(ii) large-distance behaviour

$$Z_{\phi}(b\alpha)^{-n/2} b^{nd_{\phi}} \langle \phi(bx_1) \dots \phi(bx_n) \rangle_{g(\alpha/b), a} =$$

$$= Z_{\phi}(\alpha)^{-n/2} \langle \phi(x_1) \dots \phi(x_n) \rangle_{g(\alpha), a}$$

$\lim_{b \rightarrow \infty} g(b\alpha)$  describes the long-distance behaviour of the theory.

## 3. Fixed pt

$$\beta_k(g^*) = 0$$

conformal dynamics characterized by power law scaling:

$$\langle \phi(bx_1) \dots \phi(bx_n) \rangle_* \propto b^{-n \Delta_\phi^*} \langle \phi(x_1) \dots \phi(x_n) \rangle_*$$

$$\Delta_\phi^* = d_\phi + \gamma_\phi^* \quad \gamma_\phi^* = \gamma_\phi(g^*)$$

Neighbourhood of a fixed pt.  $\delta g_k = g_k - g_k^*$

$$-a \frac{d}{da} (\delta g_k) = \beta_k(g^* + \delta g) = \sum_l \left. \frac{\partial \beta_k}{\partial g_l} \right|_* \delta g_l + O(\delta g^2)$$

$$M_{kl} = \left. \frac{\partial \beta_k}{\partial g_l} \right|_* \quad \leftarrow \text{defined at a given fixed pt.}$$

Basis of eigenvectors of  $M$ :

$$a \frac{d}{da} u_k = \gamma_k u_k + O(u^2)$$

$$\Rightarrow u_k(a) = \left( \frac{a}{a'} \right)^{\gamma_k} u_k(a')$$

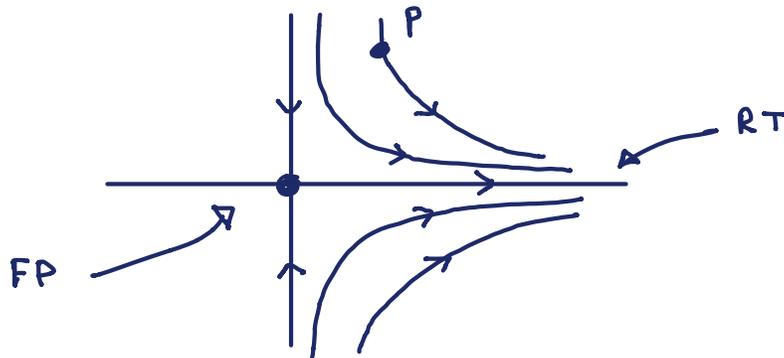
$\gamma_k > 0$ : relevant coupling,  $|u_k| \nearrow$  as  $a \nearrow$

$\gamma_k < 0$ : irrelevant coupling,  $|u_k| \rightarrow 0$  as  $a \nearrow$ .

$\gamma_k = 0$ : marginal, compute beyond linear approx.

Physics at large distances i.e.  $aE \ll 1$

is determined by specifying the value of the relevant couplings ONLY.



### Scheme dependence

$$g_k(a) = g_k(\tilde{g}(a))$$

$$\begin{aligned} \Rightarrow \beta_k(g) &= -a \frac{d}{da} g_k(a) = \sum_m \frac{\partial g_k}{\partial \tilde{g}_m} \left( -a \frac{d}{da} \tilde{g}_m(a) \right) \\ &= \sum_m \frac{\partial g_k}{\partial \tilde{g}_m} \tilde{\beta}_m(\tilde{g}) \end{aligned}$$

$$\beta_k(g^*) = 0 \Leftrightarrow \tilde{\beta}_k(\tilde{g}^*) = 0 \quad \text{where } g_k^* = g_k(\tilde{g}^*)$$

i.e. existence of the fixed pt is scheme independent.

Taking one more derivative wrt.  $\tilde{g}_n$

$$\sum_l \frac{\partial g_l}{\partial \tilde{g}_n} \frac{\partial \beta_k}{\partial g_l} = \sum_m \frac{\partial^2 g_k}{\partial \tilde{g}_n \partial \tilde{g}_m} \tilde{\beta}_m + \sum_m \frac{\partial g_k}{\partial \tilde{g}_m} \frac{\partial \tilde{\beta}_m}{\partial \tilde{g}_n}$$

Evaluating the eq. above at the fixed pt. and defining

9.

$$S_{en} = \left. \frac{\partial S_e}{\partial \tilde{g}_n} \right|_*$$

$$\sum_l M_{kl} S_{en} = \sum_m S_{km} \tilde{M}_{mn}$$

$\Rightarrow M$  &  $\tilde{M}$  have the same e.vals.