I] Quantum Field Theory

× $E \ll mc^2$ (at low energies) : relativity is unimportant

$\Rightarrow$ NRQM (Non Relativistic Quantum Mechanics) works perfectly!

Rq. $N_{particle} = C^{te}$ (Number of particle is a conserved quantity)

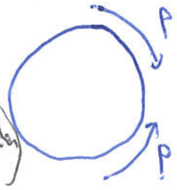
× $E \gtrsim mc^2$ (at high energies) : relativity is important (and things are more <u>complicated</u>)

$\Rightarrow$ Enough energy to pop out new particles of the vacuum.

• application :

LHC :

(Large Hadron Collider)



$E_{col.} \sim 13\,\text{TeV}$ $\quad (m_p \sim 1\,\text{GeV})$

process such as : $p + p \longrightarrow p + p + \pi^0$

$p + p \longrightarrow p + p + p + \bar{p}$

+ Higgs, ... (a more exotic particles)

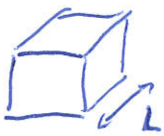
$\longrightarrow$ many-body problem

$\longrightarrow$ Need to construct a many-particle quantum theory to describe that.

1) Set-up QFT

a. Multi particle basis states (Fock space)

× System in a box



The allowed values of $\vec{k}$ for the wave function are discrete :

$$\vec{k} = \left(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L} \right)$$

• a state : $|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_{N_{TOT}}\rangle$ $\quad$ <u>ex</u>: $H|\dots\rangle = (\omega_{k_1} + \dots + \omega_{k_{N_{TOT}}})|\dots\rangle$

<u>BUT N_{TOT} is explicit</u>

• occupation nb representation (without specifying N_{TOT}) : $|\dots, n(\vec{k}), n(\vec{k}'), \dots\rangle$

× def Number op. $N(\vec{k})|\dots\rangle = n(\vec{k})|\dots\rangle$ $\quad$ nb of particles for each momentum

$\hookrightarrow$ count the occupation number for a given $\vec{k}$

$$\longrightarrow H = \sum_{\vec{k}} \omega_k N(\vec{k}) \quad , \text{ for } 1k : \; H = \omega N \text{ , does it ring a bell ?}$$

(2)

* (Single harmonic oscillator) $H_{SHO} = \omega \left(N + \frac{1}{2} \right)$
SHO

$N \uparrow$ excitation level of the oscillator

$\Rightarrow$ (1-1 correspondence with an infinite system of H.O.)

$$H \approx \sum_{i=1}^{\infty} H^i_{H.O}$$

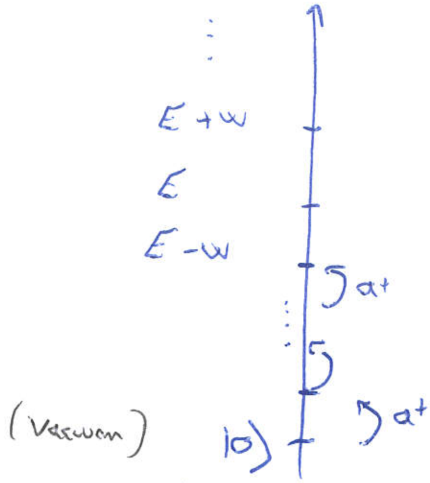
b. Simple Harmonic Oscillator

$$H = \frac{P^2}{2m} + \frac{1}{2}\omega^2 m X^2 = \frac{\omega}{2}\left(p^2 + q^2\right) \quad \text{with } [p,q] = -i$$

• def | lowering / raising op. $\begin{cases} a = \dfrac{q + ip}{\sqrt{2}} \\ a^+ = \dfrac{q - ip}{\sqrt{2}} \end{cases}$ with $[a, a^+] = 1 \longrightarrow H = \omega\left(\overbrace{a^+ a}^{N} + \frac{1}{2}\right)$ $\quad N|n\rangle = n|n\rangle$

• ↳ they generate a ladder of states

$$H|E\rangle = E|E\rangle$$
$$H a^+|E\rangle = (E + \omega) \; a^+|E\rangle$$
$$H a|E\rangle = (E - \omega) \; a|E\rangle$$



$N|0\rangle = 0 \; , \; a|0\rangle = 0$

c. Operator formalism for Fock space

$\longrightarrow$ We apply this formalism ($\underset{\text{annihilation}}{a_k}$ & $\underset{\text{creation op.}}{a^+_k}$) for each momentum :

1 particle : $|\vec{k}\rangle = a^+_k |0\rangle$

2 " : $|k, \vec{k}'\rangle = a^+_k a^+_{k'} |0\rangle$

$$H = \sum_{\vec{k}} \omega_k \, a^+_k a_k$$

- We set-up the operator formalism for a multi particle theory, based on SHO.
- Any observables may be written in terms of a, a^+, very useful.
- Need to construct the dynamics

d. Canonical Quantization & QFT

Continuum limit

Canonical quantization

• Classical Particle Mechanics:

state of a system described by coordinate $q_a(t)$

dynamics: $L(q_1,\dots,q_n,\dot q_1,\dots,\dot q_n,t) = \underbrace{T}_{\text{kinetic } E} - \overbrace{V}^{\text{potential } E}$

$$S \equiv \int_{t_i}^{t_f} L(t)\,dt$$

follow Hamilton's principle: (E.L eq)

$$\frac{\partial L}{\partial q_a} = \dot p_a \quad \text{with} \quad p_a \equiv \frac{\partial L}{\partial \dot q_a}$$

• Classical Field Theory: (ex: classical electrodynamics)

observables (ex: $\vec E$) are defined at each point in space-time $\phi_a(\vec x,t)$

$$\longrightarrow L(t) = \int d^3x\, \underbrace{\mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))}_{\text{density}}$$

$$\longrightarrow S = \int_{t_i}^{t_f} d^4x\, \mathcal{L}(\vec x,t)$$

$$\longrightarrow \text{E.O.M}: \frac{\partial \mathcal{L}}{\partial \phi_a} = \partial_\mu \Pi_a^\mu \quad \text{with} \quad \Pi_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}$$

ex: $\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right] \xrightarrow{\text{E.O.M}} (\partial_\mu\partial^\mu + m^2)\phi(x) = 0$: Klein-Gordon eq.

free scalar field

• Quantum Particle Mechanics:

classical g.al coord

$$\begin{cases} q_a(t) \\ p_a(t) \end{cases} \longrightarrow \begin{cases} \hat q_a(t) \\ \hat p_a(t) \end{cases}$$

operator-valued function

work in Heisenberg picture

respecting: $[\hat p_a(t), \hat q_b(t)] = -i\delta_{ab}$

$[\hat q_a, \hat q_b] = [\hat p_a, \hat p_b] = 0$

• QFT: To quantize an classical theory we do exactly what we did to quantize classical particle mechanics

$$\overset{\text{classical}}{\begin{cases} \phi(x) \\ \Pi^\mu(x) \end{cases}} \longrightarrow \begin{cases} \hat\phi \\ \hat\Pi^\mu \end{cases} \quad \text{op. valued functions}$$

$$[\hat\phi_a(\vec x,t), \hat\Pi_b^0(\vec y,t)] = i\delta_{ab}\,\delta^{(3)}(\vec x-\vec y)$$

$$[\hat\phi_a, \hat\phi_b] = [\hat\Pi_a^0, \hat\Pi_b^0] = 0$$

* Schrödinger picture:

$|\,\rangle$ f(t)

op. ind. of t

$$i\frac{d}{dt}|\psi(t)\rangle_S = H|\psi(t)\rangle_S \Rightarrow |\psi(t)\rangle_S = e^{-iH(t-t_0)}|\psi(t_0)\rangle_S$$

* Heisenberg picture:

$|\,\rangle$ ind t

op. = f(t)

$$i\frac{d}{dt}O_H(t) = [O_H(t), H] \quad \text{and} \quad |\psi(t)\rangle_H = e^{iH(t-t_0)}|\psi(t)\rangle_S \longrightarrow \text{op. satisfy Heisenberg eq.}$$

example : scalar field solution of the K-G eq:

$$\phi(x) = \int d^3k \left[a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right]$$ with $[a_k, a_{k'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}')$

commutators of creat°/annihilat° op.

↳ sum over all momenta of creat°/an op.

$$\longrightarrow H = \int d^3k \, \omega_k \left[a_k^\dagger a_k + \frac{1}{2} \delta^{(3)}(0) \right]$$

∞ but only ΔE has a physical meaning (we renormalize with $E(\text{ground state}) = \infty$) $|0\rangle$

× quantizing classical field theory ⟶ forced upon us a particle interpretation of the field

$$\phi(\vec{x}, 0)|0\rangle = \int \frac{d^3k}{2\omega_k} e^{-i\vec{k}\cdot\vec{x}} |k\rangle$$

field op act on vacuum

pops out a linear combination of momentum eigenstates

- metaphor with hammer on vacuum
- creation of particle at $\vec{x}$
- when it acts on n particles it produces states with $n+1$ & $n-1$.

2) S-matrix, perturbation theory and Feynman rules / diagrams

a. The interacting picture

(× So far only free fields
× We now consider a more interesting case when fields interact between each other)

× free fields ⟶ interacting fields

× We would like (as before) $\phi = f(a, a^\dagger)$ because we know how it acts on the states of the theory

× But sol° to Heisenberg eq. are no longer plane waves but much more complicated

× Trick called interaction picture :

$$H = H_0 + H_I$$

(free fields) (contains interactions)

def: $|\psi(t)\rangle_I \equiv e^{iH_0 t} |\psi(t)\rangle_S$ (if $H_I = 0 \Rightarrow I = H$)

E.O.M of fields op. : $i\frac{d}{dt} O_I(t) = [O_I(t), H_0]$

Great! because fields in an I.P. evolve just like free field in the H.P.

⟶ we can use all the free fields results we obtained so far!

× all the complications have been relegated to the E.O.M of states :

$$i\frac{d}{dt}|\psi(t)\rangle_I = H_I(t)\,|\psi(t)\rangle_I$$

× we want to connect the far past with far future, after collision has taken place :

$$|i\rangle = |\psi(-\infty)\rangle \xrightarrow[S]{\text{scattering}} t \qquad |f\rangle = |\psi(\infty)\rangle$$

$$|f\rangle = \underset{\substack{\uparrow \\ \text{scattering op.}}}{S}|i\rangle$$

× def : S-matrix element : $\langle f|S|i\rangle$: amplitude to find the system in some given state $|f\rangle$ starting from $|i\rangle$.

we integrate both side

$$|\psi(t)\rangle = |i\rangle - i\int_{-\infty}^{t} dt_1\, H_I(t_1)\underbrace{|\psi(t_1)\rangle}_{\text{we don't know that}}$$

so we replace by

$$|\psi(t)\rangle = |i\rangle - i\int_{-\infty}^{t} dt_1\, H_I(t_1)\,|i\rangle + (-i)^2\int_{-\infty}^{t} dt_2\, H_I(t_1)\,H_I(t_2)\underbrace{|\psi(t_2)\rangle}$$

by iterations → ∞ times taking the limit $t \to \infty$:

$$S = \sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n\; H_I(t_1)\cdots H_I(t_n)$$

$$\left[\text{Dyson's formula : } S = \underset{\text{ordering product}}{T}\left(e^{-i\int d^4x\, \mathcal{H}_I(x)}\right)\right]$$

6. Perturbativity, Feynman amplitudes

- time dep. of op. is trivial (free field E.O.M)
- ——— states is complicated ! → is going to be taken into account perturbatively

× Let's try to get an idea of how perturbation theory is going to work in I.P. :

× Let's take $H_I = g\,\psi^\dagger\psi\,\phi$

complex field   scalar field

$$\phi(x) = \int d^3k\left[\underset{\text{annihilate}}{a_k}e^{-ikx} + \underset{\text{create}}{a_k^\dagger}e^{ikx}\right]$$

meson

$\psi(x) =$ —— b_k ——— $c_k^\dagger$ ——   create anti-nucleon   (if spin $\frac{1}{2}$ → nucleon)

ann

$\psi^\dagger(x) =$ —— c_k ——— $b_k^\dagger$ ——

anni   create nucleon

× small $g \gg g^2 \gg g^3 \ldots$ $\left(H_I(x_1) \gg H_I(x_1) H_I(x_2) \gg \ldots \right)$

perturbation theory: imposing a cut makes sense

× first order in g:

$$\langle f | H_I(x_1) | i \rangle = \langle f | g\, \psi^\dagger(x_1) \psi(x_1) \phi(x_1) | i \rangle$$

$$\begin{pmatrix} c \\ b^\dagger \end{pmatrix} \begin{pmatrix} b \\ c^\dagger \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

~ Feynman Amplitude

set of rules called Feynman rules

• int. doesn't conserve N_{TOT} (we made it!) and it contributes to a number of processes.

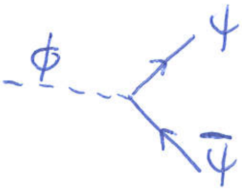
ex: $b^\dagger c^\dagger a$:

↳ anni. a ϕ particle

→ create a ψ anti-particle $(\bar{\psi})$

→ create a ψ

if $|i\rangle = |\phi\rangle$:

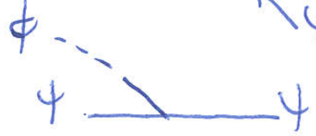


Feynman diagram (ϕ decays)

this is a number!

ex: $c^\dagger c a$:

if $|i\rangle = |\phi, \psi\rangle$:



(absorption of a ϕ by ψ)

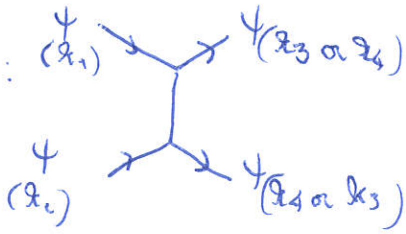
× second order in g: H acts twice on the $|i\rangle$

$$\langle f | H_I(x_1) H_I(x_2) | i \rangle = g^2 \langle f | \psi^\dagger(x_1) \psi(x_1) \phi(x_1) \psi^\dagger(x_2) \psi(x_2) \phi(x_2) | i \rangle$$

• Let's consider the scattering process: $\psi + \psi \to \psi + \psi$

$$\begin{cases} |i\rangle = |\vec{k}_1(N), \vec{k}_2(N)\rangle = b^\dagger(k_1)\, b^\dagger(k_2) |0\rangle \\ |f\rangle = \;—\;—\; (k_3) — (k_4) — \end{cases}$$

2 FD :



⟷ F. Amplitude in g^2:

$$\mathcal{A} = g^2 \left[\frac{1}{(k_3 - k_1)^2 - m^2} + \frac{1}{(k_4 - k_2)^2 - m^2} \right]$$