I. Introduction

Physical cosmology is the branch of physic that aims to study the physical origin and evolution of our Univers. It includes the study of the nature of the Univers on large scale.

The origin of modern cosmology started with the publication of the general relativity in Einstein's paper in 1917.
Since arround 1990, several dramatic advance in observation have transformed cosmology from a largely speculative sience into a predictive science with precise agreement between theory and observation.

This observations matches the prediction of the cosmic inflation theory, a modified big bang theory and the so-called ΛCDM model of the Univers.

II Brief history of the Univers

Slide # 1 → The Big-Bang model of the Univers expension
Slide # 2 → Energy Bilan
Slide # 3 → Scale

III How to describe the Univers.

Slide # 5 →

To study the Univers we need to describe its geometry.
We considere the Univers at large scale and we ~~represent it~~ mesure the Univers metric to asses its geometry and its history and evolution.

The metric, in general relativity is defined by the length element : $ds^2 = c^2dt^2 - g_{ij}\,dx^i dx^j$ ⇐ g_{ij} metric tenso

$$ds^2 = c^2dt^2 - d\vec{r}^{\,2} = c^2dt^2 - dx^2 - dy^2 - dz^2$$

Massless particle, such as the photon, travel along nulle geodesic

$$ds^2 = 0 \Rightarrow c^2dt^2 = d\vec{r}^{\,2}$$

This length élément gives the interval, in both space and time, between 2 events.

a. La metric FRW (Friedmann-Lemaitre-Robertson-Walker)

To go further we need additional assumption:

1- The Univers looks isotrop to us → It looks the same in all direction

2- The Univers is homogenous: It is the same in all position and it behave the same for all observer.

This ~~#~~ two cosmological prinsiple imply a specific metric

$$ds^2 = c^2dt^2 - \left(\frac{dR^2_{(t)}}{1 - KR^2_{(t)}} + R^2_{(t)}d\Omega\right)$$

called the FLRW metric.

k is an integer that can be equal to +1, 0, −1, R(t) is a distance and may varies with time

The maximal symetry on homogeneity and isotropy even impose a stronger constraint. with a(t) the scalling factor

Let write down $R(t) = a(t)\chi$. The metric became

only dep. t.c (ignorance) $k = K a(t)$

$$ds^2 = c^2dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega\right]$$

For these metric, the curve $k=0$, $k>0$ or $k<0$

$(r,\theta,\varphi) = C^{te}$ and are called comoving coordinates. t is the proper time of the observer in the co-moving reference frame.

↳ define: Distance normalized by the scaling factor

independent of time

Depending on the value of k, the Univers geometry is either spherical ($k>0$), plane ($k=0$) or hyperbolic ($k<0$).

In an expanding Univers, the physica distance $R(t)$ between an observer, taken at the origin of its reference frame, and a receding galaxy is measured along the surface of constant $dt = 0$. If one consider the radial distance then $d\theta = d\varphi = 0$ (Isotropy)

Taking time derivative,

Hubble parameter

$$\frac{dR}{dt} = \frac{d}{dt}(a(t)\chi) = \dot{a}(t)\chi = \frac{\dot{a}(t)}{a(t_0)}a(t_0)\chi = H(t_0)R(t)$$

Physical distance

• The Univers is expending at a rate of $H(t)$.

In more general way, one can write the FLRW metric as

$$ds^2 = c^2dt^2 - a(t)^2\left[d\chi^2 + S_k(\chi)^2 d\Omega\right]$$

with $r = S_k(\chi) = \begin{cases} \sin\chi & k=+1 \\ \chi & k=0 \\ \mathrm{Sh}\,\chi & k=-1 \end{cases}$

b. Time and conformal time

For photo, and massless particle, the element length is minimal ($ds^2=0$) thus, with the radial distance ($d\Omega = 0$), one obtain

$$c^2dt^2 = a^2(t)\,d\chi^2 \Rightarrow c\,dt = a(t)\,d\chi$$

We then define the conformal time $d\tau = \frac{dt}{a(t)}$, compared to the proper time dt, one the obtain

$$d\tau = \frac{c\,dt}{a(t)} = d\chi \Rightarrow c\int\frac{dt}{d\tau} = \chi$$

• So that time and distance may be interchanged.

Distance travelled by photons in the whole life time of the Univers define the "Horizon" which is simply the ellapsed conformal time

$$\chi_H = \int_0^t \frac{dt'}{a(t)} = \tau(t)$$

• The horizon allways grows with time

Cosmological redshift

~~For an observer mesuring the crest of a light wave at position r = 0 and at time $t_{now} = t_0$~~

For an observer observing, at $r = 0$ and $t_{now} = t_0$, the crest of a light wave emited in $r = R$ at time $t_{then} = t$ yields

$$c\int_{t_{then}}^{t_0} \frac{dt}{a(t)} = \int_R^0 \frac{dr}{\sqrt{1-kr^2}} = \int_\chi^0 d\chi' = \chi$$

In general, the wavelenght of light is not the same for two position and time due to the change of the metric while the Univers is expending.

When the wave was emited a the ~~frequency~~ time t_{then}, it has the fequency λ_{then}. The next crest of wave was emited at time $t = t_{then} + \frac{\lambda_{then}}{c}$ and the observer will mesure the next crest of wave at $t = t_{now} + \frac{\lambda_{now}}{c}$

$$= t_0 + \frac{\lambda_0}{c}$$

Since the subsequent crest is again emited from $r = R$ (compared to the age of the Univers => Comoving coordinates)

$$c\int_{t_{then}+\frac{\lambda_{then}}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} = \chi = c\int_{t_{then}}^{t_{now}} \frac{dt}{a(t)}$$

$$0 = -\int_{t_{then}+\frac{\lambda_{then}}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} + \int_{t_{then}}^{t_0} \frac{dt}{a(t)}$$

$$= \int_{t_{then}}^{t_{then}+\frac{\lambda_{then}}{c}} \frac{dt}{a(t)} + \int_{t_{then}+\frac{\lambda_{then}}{c}}^{t_0} \frac{dt}{a(t)} - \int_{t_e+\frac{\lambda_e}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} = \cancel{\int_{t_e}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)}} + \cancel{\int_{t_e+\frac{\lambda_e}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)}}$$

$$= \int_{t_e}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)} - \int_{t_0}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)} - \int_{t_e+\frac{\lambda_e}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} = \int_{t_e}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)} - \int_{t_0}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} = 0$$

This implies

$$\int_{t_0}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} = \int_{t_e}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)}$$

For very small variation in time, (a is almost a constant) $a_{now} = a_0$, $a = a_{then}$ we obtain

$$\frac{t_0 + \lambda_0/c - t_0}{a_0} = \frac{t_e + \lambda_e/c - t_e}{a} \Rightarrow \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a} = 1 + z$$

In an expanding univers the observe wavelength is shifted by a factor $\frac{a_0}{a} = 1+z$ where z is commonly called the cosmological redshift.

Be awar that it is not a "doppler effect" as the sourse was fixed (it do not have ~~proper~~ motion => Not that true in real life)

c. Cosmological distances

~~We defined the comoving distance~~ ~~such that, at~~
~~for which 2 bodyies, without proper motion~~

The FLRW metric is picked by hand as consequence of the cosmological principle. This have to be tested as much as possible. → assume isotropy and homogenity through space and time

- For example, the De-Sitter metric has higher symetry than the FLRW but it is already ruled out by observation.
- We suspect the Univers isn't exactly FLRW but we still do not know how important this is.

- The FLRW leads to "3 pillars" that are testable and verified
  - → Isotropy of the CMB better than 10^{-5} and the CMB itself
  - → The redshift and expanding Univers
  - → The inflation, that still need to be observed.

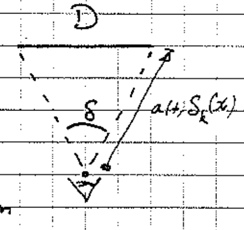
To mesure the Univers, we difined the comoving distance:
$S_k(\chi) = \frac{r}{a(t)}$ which is the physical distance scaled by the Univers scaling factor. In comoving coordinates 2 body without proper motion are allway at the same comoving distance from each other.

$$d_m = a(t)\, S_k(\chi)$$

But ~~fo~~ to measure and test the FLRW metric, the physical distance need to be observed.

1- Angular distance

A physical object of known dimension is observed under an angle δ. However photon of that source where emitted at a time $t \ll t_0$ ~~also~~



$$D = a(t) S_k(\chi)\, \delta = \frac{a(t)}{a_0} a_0 S_k(t)\, \delta = d_A\, \delta$$

$$\Rightarrow d_A = \frac{a(t)}{a_0} a_0 S_k(t) = \frac{1}{1+z} d_m$$

2- Luminous Distance

Consider a source of kown intrinsic luminosity L. The mesured flux for a observer at the origin will be given by

$$F = \frac{L}{4\pi d_L^2}$$ with d_L the luminous distance from the source

In flat space, and because of the conservation of energy, the flux ~~decrease as $\frac{1}{d^2}$~~ from an isotropic source decay as $\frac{1}{d^2}$ as the radius of the sphere that delimite the distance travelled by photon so far is expending

Thus, ~~photo~~ by conservation of Energy, we know that all the photons emitted by the soure will eventually pass through a sphere of comoving distane $a_0 S_k(x)$ from the emitter.

But the flux is ~~d~~ diluted by two additional effects.

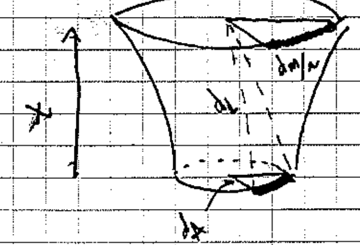
1) The individual photon redshift reduce their energy by a factor $(1+z)$

2) As photon are emitted at time interval δt from eache other, they will be observed with a time intervale of $(1+z)\,\delta t$ when they reach the observer

With this 2 effect on can write that $\frac{F}{L} = \frac{1}{(1+z)^2 4\pi [a_0 S_k(x)]^2}$

By identification $d_L = (1+z)\, a_0\, S_k(x) =$

$$\boxed{d_L = (1+z)^2 d_A}$$



IV The content of the Univers

The Friedman equation

So fare, we concentrate on the geomtry of the Univers. But its content also shape the Univers. Thus, now, we need to account for the content of the Univers.

So far, we havn't use the general relativity. Their are 2 aspect of general relativity one need to account for:

1- Gravitation can be described by a metric

2- General relativity connect the metric to the matter/energy.

As consequence geometry and energy have an close connection: This is translated into the Einstein equation

$$G_{\mu\nu} = \underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{Geometry}} = \underbrace{8\pi G T_{\mu\nu}}_{\text{Energy}}$$

$G_{\mu\nu}$: Einstein tensor

$R_{\mu\nu}$: The Ricci tensor

$R = g^{\mu\nu} R_{\mu\nu}$ the Ricci scalar

$T_{\mu\nu}$ = the energy momentum tensor. A symetric tensor that describe the constituent of the Univers

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$$

with $\Gamma^{\alpha}_{\mu\nu,\alpha} = \dfrac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}}$

An $\Gamma^{\mu}_{\alpha\beta} = \dfrac{g^{\mu\nu}}{2}\left(\underset{g_{\nu\alpha,\beta}}{g_{\alpha\nu,\beta}} + \underset{g_{\nu\beta,\alpha}}{g_{\beta\nu,\alpha}} - \underset{g_{\alpha\beta,\nu}}{g_{\alpha\beta,\nu}}\right)$ In FLRW, $\overset{\text{flat univers}}{g_{\mu\nu}} = \begin{bmatrix} 1 & & & 0 \\ & -a^2 & & \\ & & -a^2 & \\ 0 & & & -a^2 \end{bmatrix}$

$\Rightarrow \Gamma^{0}_{00} = 0$, $\Gamma^{0}_{0i} = \Gamma^{0}_{i0} = 0$ $\Gamma^{0}_{ij} = \delta_{ij}\dot{a}a$ $\Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \dfrac{\dot{a}}{a}\delta_{ij}$

$\Gamma^{k}_{ij} = 0$, $\Gamma^{i}_{00} = 0$.

If we make the calculation, on should find, for a flat Univers

$R_{00} = -3\dfrac{\ddot{a}}{a}$

$R_{ij} = \delta_{ij}\left[2\dot{a}^2 + \ddot{a}a\right]$

$R = g^{\mu\nu}R_{\mu\nu} = R_{00} - \dfrac{1}{a^2}\delta^{ij}R_{ij} = -6\left[\dfrac{\ddot{a}}{a} + \dfrac{\dot{a}^2}{a^2}\right]$

A perfect fluid can be completly defined by a rest frame energy density ρ and an isotropic rest frame Pressure. Thus energy-Tensor momentum of a perfect isotropic fluid

(remember, the univers is homogeneous and isotrop. It is also the case for its contents) became

$$g^{\mu\nu}T_{\beta\nu} = T^{\mu}_{\nu} = \begin{bmatrix} \rho & & & 0 \\ & -P & & \\ & & -P & \\ 0 & & & -P \end{bmatrix}$$

In an expending Univers the conservation equation is given by

$$\frac{\partial T^{\mu}_{\nu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\alpha\mu}T^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\mu}T^{\mu}_{\alpha} = 0$$

For the componant $\nu = 0$

$\dfrac{\partial \rho}{\partial t} + 3\dfrac{\dot{a}}{a}(\rho + P) = 0$ that can be

rewritten as :

$$\boxed{a^{-3}\frac{\partial}{\partial t}(\rho a^3) = -3\frac{\dot{a}}{a}P}$$

$\dfrac{\partial \rho a^3}{\partial t} = \dfrac{\partial \rho}{\partial t}a^3 + 3\rho\dot{a}a^2 = -3\dfrac{\dot{a}}{a}$

$a^{-3}\dfrac{\partial \rho}{\partial t}a^3 + 3\rho\dfrac{\dot{a}}{a} = -\dfrac{3\dot{a}}{a}P$

$\dfrac{\partial \rho}{\partial t} + \dfrac{3\dot{a}}{a}\left[\rho + P\right] = 0$

This latter equation profide great information regarding the various fluid that composed the Univers.

⇒ For pressure less fluid (non relativistic matter)

$$P=0 \Rightarrow \frac{\partial}{\partial t}(\rho_m a^3) = 0 \Rightarrow \rho_m \propto a^{-3}$$

⇒ For radiation $P = \frac{\rho}{3}$ $\qquad \rho + P = (1+\frac{1}{3})\rho = \frac{4}{3}\rho$

$$a^{-3}\frac{\partial(\rho a^3)}{\partial t} = -\frac{\dot a}{a}\rho$$
$$a^{-3}\rho\frac{da^3}{dt} + \frac{d\rho}{dt} = -\frac{\dot a}{a}\rho$$
$$a^{-3}\rho\frac{3a^2\dot a}{a} + \frac{d\rho}{dt} = -\frac{\dot a}{a}\rho$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + 4\frac{\dot a}{a}\rho = a^{-4}\frac{\partial}{\partial t}(\rho a^4) = 0$$

$$\rho_r \propto a^{-4} \qquad \left\{ P_i = \frac{g_i}{(2\pi)^3}\int \frac{p^2}{3E(p)} f_i(\vec p)\, d p^3 \ \text{ et } \ f_i(p) = \frac{1}{e^{E(p)/T} \pm 1} \right.$$
$$g_i = 2 \text{ or } 3.$$

⇒ For negative pressor $P = -\rho$ $\qquad$ (Cosmological constant)

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \rho = C^{te}$$

Combining geometry and Energy-momentum tensor, one obtain the ~~fir~~ friedman equation.

In a flat Univers:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

If one want to generalize to ~~g~~ the various possible geometry for homogeneous and Isotropic Univers, one get:

Consider cosmological C^{te} (DE) as a fluid of $\omega = \frac{P}{\omega} = -1$

$c = 1$

$$(1) \quad \left(\frac{\dot a}{a}\right)^2 = \sum \frac{8\pi G}{3}\rho_i - \frac{k\cancel{c^2}}{a^2} + \frac{\Lambda \cancel{c^2}}{3}$$

$$(2) \quad \frac{\ddot a}{a} = -\sum\frac{4\pi G}{3}\left(\rho_i + \frac{3P}{\cancel{c^2}}\right) + \frac{\Lambda\cancel{c^2}}{3} \qquad c = 1$$

$$(1) \Rightarrow \ 2\sum\frac{4\pi G\rho_i}{3} = \left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3}$$

$$(2) \Rightarrow \frac{\ddot a}{a} + \sum\frac{4\pi G P}{c^2} - \frac{\Lambda c^2}{3} = -\sum\frac{4\pi G\rho}{3}$$

$$(1) \ \& \ (2) \Rightarrow 2\frac{\ddot a}{a} + 2\sum\frac{4\pi G P}{c^2} - \frac{2\Lambda c^2}{3} = -\left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3}$$

$$\frac{2\ddot a}{a} + \frac{8\pi G P}{\cancel{c^2}} - \Lambda c^2 + \frac{k\cancel{c^2}}{a^2} + \frac{\dot a^2}{a} = 0$$

$$(3) \quad -2\frac{\ddot a}{a} - \frac{\dot a^2}{a^2} + \Lambda\cancel{c^2} - \frac{k}{a^2} = 8\pi G P$$

$$\frac{d(1)}{dt} = 2\left[\frac{\ddot a a - \dot a^2}{a^2}\right]\frac{\dot a}{a} = \frac{8\pi G\dot\rho}{3} + \frac{2ak}{a^6} = \frac{2ak\dot a}{a^3} + \frac{8\pi G\rho}{3} = \frac{k}{a^2}\frac{2\dot a}{a} + 8\pi G$$

$$2\frac{\ddot a\dot a}{a^2} - \frac{2\dot a^2}{a^3} = \frac{8\pi G\dot\rho}{3} = 2\frac{\ddot a}{a}\times\frac{\dot a}{a} - \frac{\dot a^2}{a^2}\frac{1}{a} = 8\pi G\dot\rho$$

$$= 2\frac{\dot a}{a}\left[\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right] = 8\pi G\dot\rho + \frac{k}{a^2}\frac{2\dot a}{a}$$

$$= 2\frac{\dot a}{a}\left[-\frac{4\pi G}{3}\rho_i + 4\pi G P - \frac{8\pi G}{3}\rho + \frac{k}{a^2}\right] = \ldots$$

$$-\frac{4\pi G}{3}\rho_i - 4\pi G P - \frac{8\pi G}{3}\rho + \frac{k}{a^2} = \frac{k}{a^2} + \frac{4\pi G}{3}\rho\frac{a}{\dot a}$$

$$-4\pi G\rho_i - 4\pi G P = \frac{4\pi G}{3}\dot\rho\frac{a}{\dot a} \Rightarrow (4)\quad -3\frac{\dot a}{a}(\rho + P) = \dot\rho$$

The ~~first~~ equation ~~translate the expension~~ link the Univers expension ~~rate~~ to the contente of the Univers

The second give the acceleration of the expension.

For the ~~perfect~~ fluid we have seen earlier, one can generalise their equation of state as

$$\omega = \frac{P}{\rho}$$

Thus $\frac{\ddot a}{a} = -\frac{4\pi G\rho}{3}(1+3\omega) + \cancel{\frac{\Lambda}{3}}$

Neglecting the cosmological constant and for a flat univers, one obtain an acceleration of the expension

$$\cancel{\frac{\ddot a}{a} = -\frac{1}{2}\frac{8\pi G\rho}{3}(1+3\omega)} = -\frac{1}{2}\frac{\dot a^2}{a^2}(1+3\omega)$$

of the Univers for $1+3\omega < 0 \Rightarrow \boxed{\omega < -\frac{1}{3}}$

The equation (4) is called the fluid equation. It tell us how the energy density varies in an expending Univers

$$-3\frac{\dot a}{a}(1+\omega) = \frac{\dot\rho}{\rho} \Rightarrow \int \cancel{\frac{3\dot a}{a}(1+\omega)}$$

$$-3\frac{da}{dt}\frac{1}{a}(1+\omega) = \frac{d\rho}{dt}\frac{1}{\rho} \Rightarrow -3(1+\omega)\int^{a_0}\frac{da}{a} = \int_{\rho_0}^{\rho}\frac{d\rho}{\rho}$$

$$\cancel{\rho =} -3(1+\omega)\ln\frac{a_0}{a_i} = \ln\frac{\rho_0}{\rho_i}$$

$$\boxed{\rho_i\left(\frac{a}{a_i}\right)^{-3(1+\omega)} = \rho(a)}$$

$$\rho_0 = a_0^{-3(1+\omega)}$$

$$\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+\omega)} \Rightarrow \frac{\rho}{a^{-3(1+\omega)}} = \frac{\rho_0}{a_0^{-3(1+\omega)}} = \rho_0$$

2- The cosmological parameters

We have difined $H = \frac{\dot a}{a}$ the hubble parameters.

We the can rewrite the Friedman equation as follow

$$(1)\quad H^2 = \sum_i \frac{8\pi G\rho_i}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$\dot H = \frac{\ddot a a - \dot a^2}{a^2} = \frac{\ddot a}{a} - \frac{\dot a^2}{a^2} \Rightarrow \frac{\ddot a}{a} = \frac{\ddot a}{a} - \frac{\dot a^2}{a^2} + \frac{\dot a^2}{a^2} = \dot H + H^2$$

$$(2)\quad \dot H + H^2 = -\sum_i \frac{4\pi G}{3}(\rho_i + 3P_i) + \frac{\Lambda}{3} \qquad (4)\quad -3H(\rho + P) = \dot\rho$$

We now define the critical densit $\rho_c = \frac{3H^2}{8\pi G}$ corresponding to the density one would expect for a univers dominated by matter $\Rightarrow \Lambda = 0$ & $k = 0$

We can the rewrite the Friedman equations as follow

$$(1)\quad 1 = \sum_i \frac{8\pi G\rho_i}{3H^2} - \frac{k}{a^2H^2} + \frac{\Lambda}{3H^2} = \sum_i \Omega_i + \Omega_k + \Omega_\Lambda$$

with $\Omega_i = \frac{\rho_i}{\rho_c}$ the energy density parameter of the fluid i

$$\Omega_k = -\frac{k}{H^2} \quad \text{the curvature density}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} \quad \text{the energy density parameters}$$

$\Lambda = 8\pi G \rho$

Ther is an other useful way do rewrite the Friedmann equation

Let assume that we normalise the scaling factor today to $1 \Rightarrow a(t_0) = 1$

Thus $\rho(a) = \rho_0 \left(\frac{a}{a_0}\right)^{-3(w+1)} = \rho_0 a^{-3(w+1)} \Rightarrow \frac{\rho_0}{a^{+3(1+w)}}$

and the Friedmann equation became

$$H^2 = \sum_i \frac{8\pi G \rho_i}{a^{-3(1+w_i)}} + \frac{K}{a^2} + \frac{\Lambda}{3}$$

With

- non relativistic mater $\quad w_m = 0$
- radiation $\quad w_r = \frac{1}{3}$
- Dark energy (cosmological constant) $\quad w_\Lambda = -1$

$$H^2 = \frac{8\pi G \rho_{r,0}}{3a^4} + \frac{8\pi G \rho_{m,0}}{3a^3} + \frac{8\pi G \rho_{\Lambda,0}}{3} - \frac{k}{a^2}$$

$$H^2 = H_0^2 \left[\frac{\Omega_r^0}{a^4} + \frac{\Omega_m^0}{a^3} + \Omega_\Lambda^0 + \frac{\Omega_k^0}{a^2} \right]$$

And as we also defined the redshift $\frac{a_0}{a} = (1+z)$ with a_0 normalized to 1 on are $\frac{1}{a} = 1+z$ and consequently

$$H^2 = H_0^2 \left[\Omega_r^0 (1+z)^4 + \Omega_m^0 (1+z)^3 + \Omega_\Lambda^0 + \Omega_k^0 (1+z)^2 \right]$$

3. Measuring the cosmological parameters

$$\chi = \int_{t_z}^{t_0} \frac{dt}{a} = \int_{a_z}^{1} \frac{da}{a^2} \frac{1}{H} = \int_0^z \frac{+dz}{H}$$

$dt = \frac{dt}{da} da = \frac{da}{\dot a}$

$\frac{da}{a \dot a} \frac{a}{a} = \frac{da}{Ha^2}$

$a = \frac{1}{1+z} \Rightarrow \frac{da}{dz} = -\frac{1}{(1+z)^2}$

$\Rightarrow \frac{da}{a^2} = \frac{da}{dz}\frac{dz}{a^2} = -\frac{(1+z)^2}{(1+z)^2}$

Do you recall the angular distance?

$$d_A = \frac{1}{1+z} d_m = \frac{1}{1+z} \, a \, S_k(\chi)$$

$$= \frac{1}{(1+z)^2} S_k \left[\int_0^z \frac{dz}{H} \right] = \frac{1}{(1+z)^2} S_k \left[\int_0^z \frac{dz}{\sqrt{\Omega_r (1+z)^4 + \Omega_m^0 (1+z)^3 + \dots}} \right]$$

Thus mesuring angular distance provide information on the cosmological parameters. But, they are all degenerated!

Similarely for the luminous distance

$$d_L = 1+z^2\, d_A = S_k\left[\int_0^z \frac{dz}{H_0\sqrt{\Omega_r^0(1+z)^3+\Omega_m^0(1+z)^3+\Omega_\Lambda^0+\Omega_k(1+z)^2}}\right]$$

Combining the two measurement may remove some degenerancy. But, 5 unknown against 2 equation !

4. Special case

a. Radiation dominated Univers

In the early Univers, when the scale factor $a(t)$ is much smaller, radiation dominates over matter

$a \ll 1 \Rightarrow a^{-4} \gg a^{-3} \gg 1 \Rightarrow \rho_r \gg \rho_m$

The first friedmann equation approximate to

$$H^2 = H_0^2 \frac{\Omega_r}{a^4} \Rightarrow a^2\dot{a}^2 = H_0^2\,\Omega_r \Rightarrow a\dot{a} = H_0\sqrt{\Omega_r}$$

Thus $\frac{1}{2}\frac{da^2}{dt} = H_0\sqrt{\Omega_r} \Rightarrow a(t) = \sqrt{H_0\Omega_r^{1/2}}\cdot t^{1/2}$

$\frac{2}{2}\int a\,da = H_0\sqrt{\Omega_r}\int dt \Rightarrow$ The scalling factor grows as the square root of the time

b. matter dominated flat Univers

Curently radiation energy density is negligible. Neglecting cosmological constant in flat Univers, one obtain

$$H^2 = H_0 \frac{\Omega_m}{a^3} \qquad \dot{H}+H^2 = -\frac{4\pi G\rho}{3} \Rightarrow \frac{\dot{H}}{H^2}+1 = -\frac{1}{2}\Omega$$

$\dot{H} = \frac{H^2}{2} \Rightarrow \frac{dH}{H^2}$

The Friedmann equation is now $\frac{\dot{a}}{a} \propto a^{-3/2}$ which can be solved imediatly to $a(t) \propto t^{2/3}$

$$\frac{da}{dt}\,a^{-5/2} \Rightarrow \int a^{-3/2}da = \int dt$$

$$-\frac{2}{3}a^{-3/2} = t \Rightarrow a \propto t^{2/3}$$

We can notice that $\dot{a}(t) > 0 \;\forall t > 0$ thus the Univers always expand itself. But as $\ddot{a} = \frac{3}{4\sqrt{t}}$, the expension slow down with incrasing time

For a flat Univers we also have $\int_0^{t_0} dt = \int_0^\infty \frac{dz}{H(z)(1+z)}$

$$\int_0^{t_0} dt = \int_0^1 \frac{da}{\dot{a}} = \int_0^1 \frac{da}{aH} = \int_\infty^0 \frac{dz}{H(1+z)}$$

Thus the age of the Univers is calculated to be

$$t_0 = \int_0^\infty \frac{dz}{H_0(1+z)^{5/2}} = \int_1^\infty \frac{du}{H_0 u^{5/2}} = -\frac{2}{3H_0}u^{-3/2}\Big|_1^\infty = \frac{2}{3H_0}$$

With ~~cosmological~~ Hubble parameter beeing $\frac{1}{H_0}$ = ~~[illegible]~~ 13,6Gyr
The age of the Univers would be arround 9Gyr.
However, observation have shown that the oldest observed stars are 11 to 18 Gyr old. This already implies that matter isn't dominating the Univers!

V The perturbation theory

So fare, we assumed the Univers was homogeneous and isotropic so ~~does~~ the content of the Univers.
But, when we ~~look at~~ at small scale, we distinguish structure (the galaxy). At small scale, the Univers isn't that homogenous! (But still, on average, the Univers remain homogoneous, even at small scale)
Those structures originate from priomordial fluctuat° of energy density. Those perturbation then evolved with the expending Univers to give rise to Halos, galaxy, stars...

To deal with those structure, which embed themself important cosmological information, we used perturbation theory onto the Friedmam equation

We start by ~~modif~~ perturbating the metric and the Energy momentum tensor:

$$g_{\mu\nu} = \langle g_{\mu\nu} \rangle + a^2 h_{\nu\mu} \qquad T_\mu^\nu = \langle T_\mu^\nu \rangle + \delta T_\mu^\nu$$

This leads to new Friedmann equation

VI Conclusion

Cosmology ~~is based~~ and FRW is constructed upon the cosmological principle that assume the Univers is homogenous and isotrop. Are those hypothesis resonable? This have to be tested and, so far, it was verified.

- The Univers is expanding: we then conclude it was smaller and hotter in the past
- Current obseration lead us to the Λ-CDM model of the Univers ⇒ 95% of it energy density remain Unknown, but to know, ~~man~~ no observation have ruled out this model

- Still many thing needs to be understood in Λ-CDM model
  - Why does the Univers is so flat $\Omega_k = 0 \pm 0{,}005$
  - Where does the primordial perturbation come from
  - What are made of DE & DM.