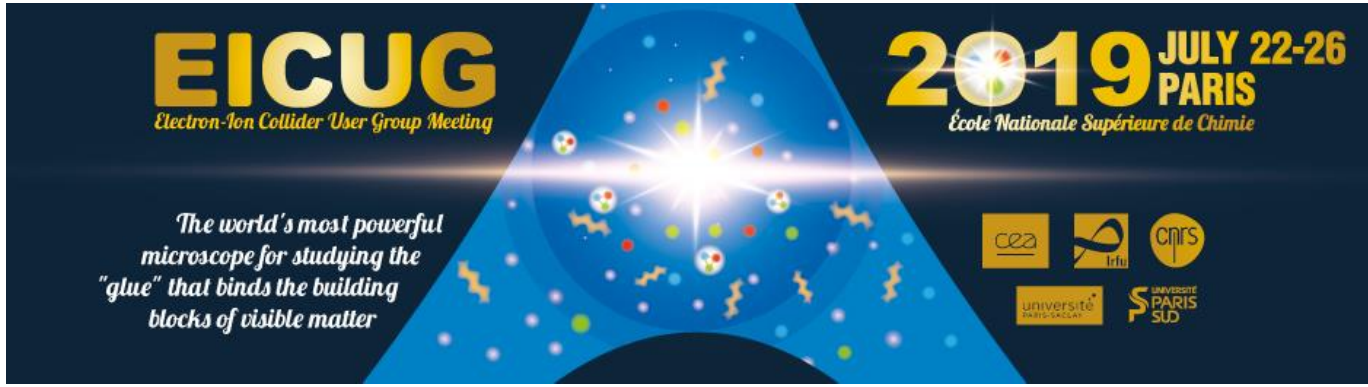


Mechanical properties of hadrons

Based on **[C.L., Mantovani, Pasquini, PLB776 (2018)]**
[C.L., EPJC78 (2018)]
[C.L., Moutarde, Trawinski, EPJC79 (2019)]
[Cosyn, Cotogno, Freese, C.L., EPJC79 (2019)]
[Cotogno, C.L., Lowdon, arXiv:1905.11969]

Cédric Lorcé

CPHT Centre de Physique Théorique

July 26, ENS Chimie, Paris, France

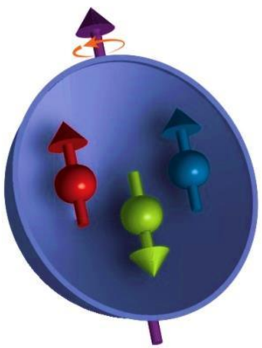
Origin of mass and spin?

Non-relativistic picture
dominated by constituents

	Mass	**Spin**

Until ~ 1980

Spectroscopy



$$M_N \sim \sum_Q M_Q + E_{\text{binding}}$$

~ 102 % ~ - 2 %

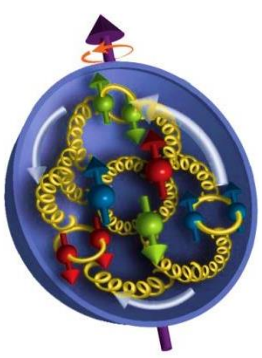
$$J_z^N \sim \sum_Q S_z^Q$$

~ 100 %

Relativistic picture
dominated by dynamics

Now

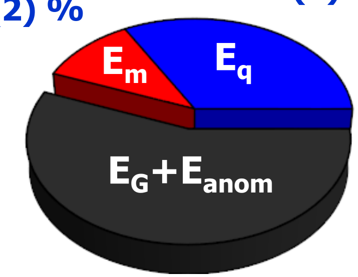
High-energy scattering



Higgs mechanism
+quark condensate

~ 13(2) %

~ 31(2) %



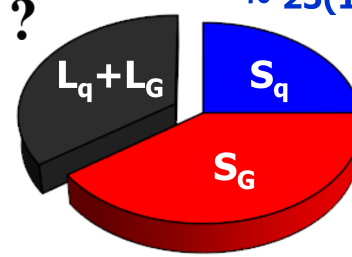
?

Gluon and quantum anomaly

Orbital angular momentum (OAM)

?

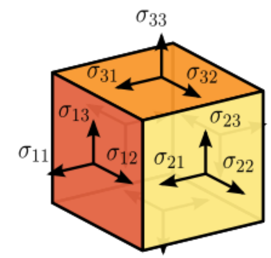
~ 25(10) %



~ 40(?) %

Energy-momentum tensor (EMT)

Mass, **spin** and **pressure** all encoded in



Energy density | Momentum density

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

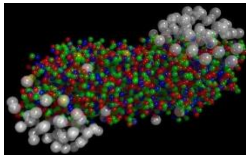
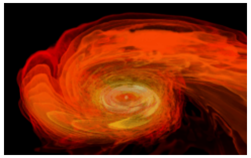
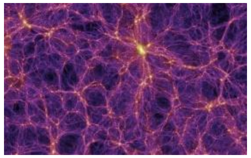
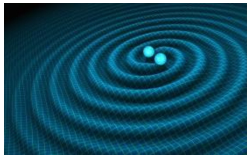
Energy flux | Momentum flux

Shear stress

Normal stress (pressure)

Key concept for

- **Nucleon mechanical properties**
- **Quark-gluon plasma**
- **Relativistic hydrodynamics**
- **Stellar structure and dynamics**
- **Cosmology**
- **Gravitational waves**
- **Modified theories of gravitation**
- **...**

Quantum chromodynamics (QCD)

Bare QCD energy-momentum tensor

$$T^{\mu\nu} = \overline{\psi}\gamma^{\mu}\,\tfrac{i}{2}\overset{\leftrightarrow}{D}{}^{\nu}\psi - G^{a\mu\alpha}G^{a\nu}{}_{\alpha} + \tfrac{1}{4}\,\eta^{\mu\nu}G^{2}$$

Renormalized trace of the QCD EMT

[Crewther (1972)]
[Chanowitz, Ellis (1972)]
[Nielsen (1975)]
[Adler, Collins, Duncan (1977)]
[Collins, Duncan, Joglekar (1977)]
[Nielsen (1977)]

$$T^{\mu}{}_{\mu} = \underbrace{\frac{\beta(g)}{2g}\,G^{2}}_{\text{Trace anomaly}} + (1+\gamma_{m})\,\overline{\psi}m\psi$$

Quark mass matrix

Gravitational form factors (GFFs)

See also talks by A. Freese, P. Schweitzer, and P. Sznajder and posters by A. Trawinski and P. Lowdon

Symmetrized variables $\quad P = \frac{p' + p}{2}, \qquad \Delta = p' - p, \qquad t = \Delta^2$

Spin-0

$a = q, g$

[Pagels (1966)]
[Donoghue, Leutwyler (1991)]
[Ji (1996)]

$$\langle p' | T_a^{\mu\nu}(0) | p \rangle = 2M \left[\frac{P^\mu P^\nu}{M} A_a(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{M} C_a(t) + M g^{\mu\nu} \bar{C}_a(t) \right]$$

Non-conserved

$$\Rightarrow \sum_a \bar{C}_a(t) = 0$$

Spin-1/2

[Kobzarev, Okun (1962)]
[Pagels (1966)]
[Ji (1996)]
[Bakker, Leader, Trueman (2004)]
[Leader, C.L. (2014)]

$$\langle p', s' | T_a^{\mu\nu}(0) | p, s \rangle = \overline{u}(p', s') \Gamma_a^{\mu\nu}(P, \Delta) u(p, s)$$

$$\Gamma_a^{\mu\nu}(P, \Delta) = \frac{P^\mu P^\nu}{M} A_a(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{M} C_a(t) + M g^{\mu\nu} \bar{C}_a(t) + \frac{P^{\{\mu} i\sigma^{\nu\}\lambda}\Delta_\lambda}{4M} J_a(t) + \frac{P^{[\mu} i\sigma^{\nu]\lambda}\Delta_\lambda}{4M} D_a(t)$$

Gravitational form factors (GFFs)

Spin-1

[Holstein (2006)]
[Abidin, Carlson (2008)]
[Taneja, Kathuria, Liuti, Goldstein (2012)]
[Cosyn, Cotogno, Freese, C.L. (2019)]
[Polyakov, Sun, arXiv:1903.02738]

$$\langle p', s'|T_a^{\mu\nu}(0)|p, s\rangle = 2M\,\epsilon_\alpha^*(p', s')\mathcal{M}_a^{\mu\nu;\alpha\beta}(P,\Delta)\epsilon_\beta(p, s)$$

$$\begin{aligned}\mathcal{M}_a^{\mu\nu;\alpha\beta}(P,\Delta) = -g^{\alpha\beta}&\left[\frac{P^\mu P^\nu}{M}\,\mathcal{G}_1^a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M}\left(\tfrac{1}{4}\mathcal{G}_3^a(t)\right) + M g^{\mu\nu}\left(-\tfrac{1}{2}\mathcal{G}_8^a(t)\right)\right]\\
&+ \frac{P^{\{\mu}g^{\nu\}[\alpha}\Delta^{\beta]}}{4M}\,\mathcal{G}_5^a(t) + \frac{P^{[\mu}g^{\nu][\alpha}\Delta^{\beta]}}{4M}\,\mathcal{G}_{10}^a(t)\\
&+ M g^{\alpha\{\mu}g^{\nu\}\beta}\left(\tfrac{1}{4}\mathcal{G}_7^a(t)\right)\\
&+ \frac{\Delta^\alpha\Delta^\beta}{2M^2}\left[\frac{P^\mu P^\nu}{M}\,\mathcal{G}_2^a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M}\left(\tfrac{1}{4}\mathcal{G}_4^a(t)\right) + M g^{\mu\nu}\left(\tfrac{1}{2}\mathcal{G}_9^a(t)\right)\right]\\
&+ \left[\frac{\Delta^{\{\mu}g^{\nu\}\{\alpha}\Delta^{\beta\}}}{4M} - g^{\alpha\{\mu}g^{\nu\}\beta}\frac{\Delta^2}{4M} - g^{\mu\nu}\frac{\Delta^\alpha\Delta^\beta}{2M}\right]\left(\tfrac{1}{2}\mathcal{G}_6^a(t)\right) + \frac{\Delta^{[\mu}g^{\nu]\{\alpha}\Delta^{\beta\}}}{4M}\,\mathcal{G}_{11}^a(t)\end{aligned}$$

Spin-*J*

[Boulware, Deser (1975)]
[Cotogno, C.L., Lowdon, arXiv:1905.11969]

$$\langle p', s'|T_a^{\mu\nu}(0)|p, s\rangle = \overline{\eta}(p', s')\left[2P^\mu P^\nu A_a(t) + iP^{\{\mu}S^{\nu\}\lambda}\Delta_\lambda G_a(t) + \cdots\right]\eta(p, s)$$

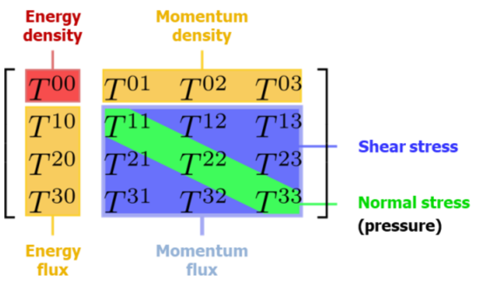
Lorentz generator

Mass decomposition

[C.L. (2018)]

Forward matrix element (spin J<1)

$$\langle P|T_a^{\mu\nu}(0)|P\rangle = 2P^\mu P^\nu A_a(t) + 2M^2\eta^{\mu\nu}\bar{C}_a(0)$$



Analogy with relativistic hydrodynamics

Perfect fluid element

$$\Theta_a^{\mu\nu} = (\varepsilon_a + p_a)u^\mu u^\nu - p_a\eta^{\mu\nu}$$

Four-velocity

$$u^\mu = P^\mu/M$$

Energy density

$$\varepsilon_a = \left[A_a(0) + \bar{C}_a(0)\right]\frac{M}{V}$$

Isotropic pressure

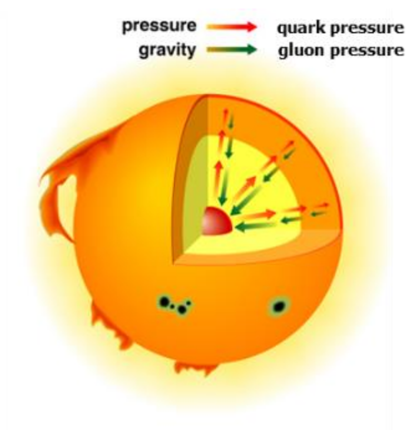
$$p_a = -\bar{C}_a(0)\,\frac{M}{V}$$

Nucleon mass decomposition $U_i = \varepsilon_i V$

$$M = \underbrace{U_q}_{\sim 44\%} + \underbrace{U_g}_{\sim 56\%}$$

$$p_q = -p_g \qquad \sim 11\%$$

$\mu = 2\,\mathrm{GeV}$

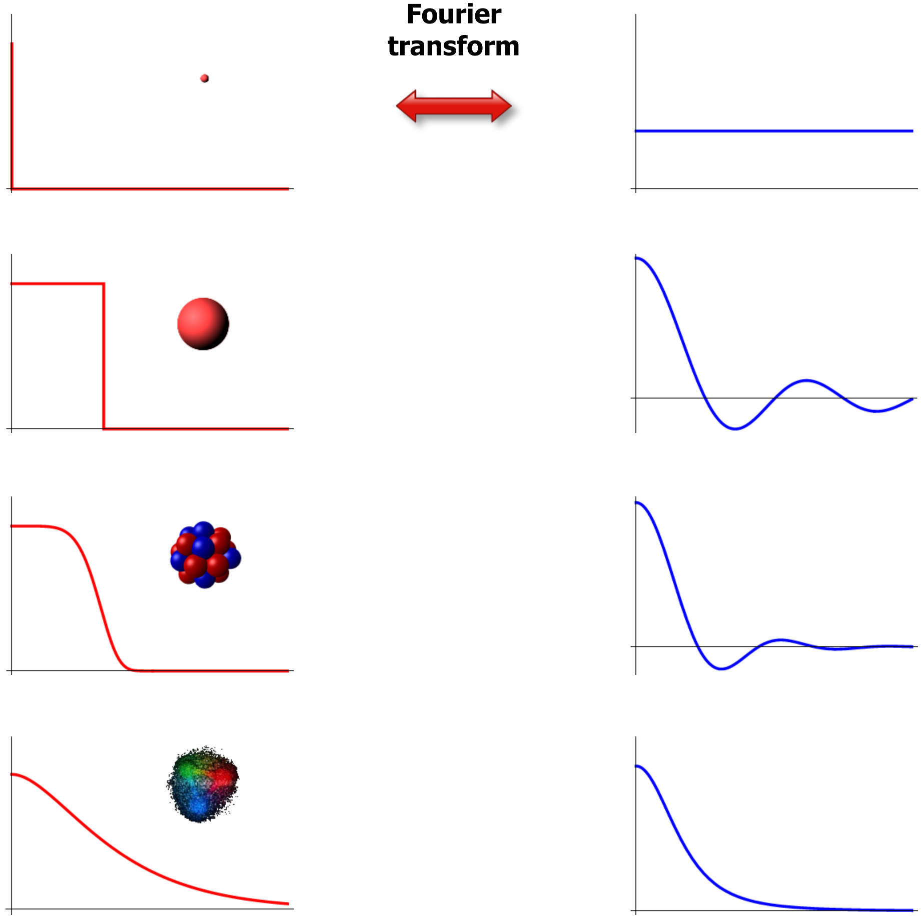


Spatial information

Spatial distribution | **Form factor**

$\rho(r)$ $\mathrm{FF}(\Delta)$

Fourier transform

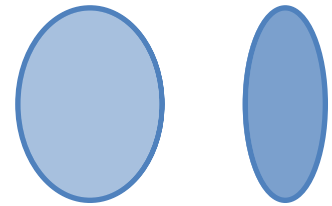


3D distribution in Breit frame

Lorentz factors

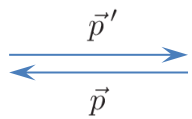
$$p^0 = \gamma M, \qquad p'^0 = \gamma' M$$

Initial ψ **Final** ψ'^*

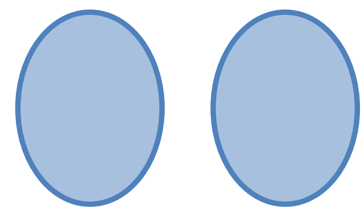


$$\Delta^0 \neq 0$$

Breit frame



$$\vec{P} = \vec{0} \quad \Rightarrow \quad \Delta^0 = \frac{\vec{P}\cdot\vec{\Delta}}{P^0} = 0$$



$$\Delta^0 = 0$$

$$\rho \sim \psi^* \psi$$

3D distribution

$$\langle T_a^{\mu\nu}\rangle(\vec{r}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{r}}\, \frac{\langle \frac{\vec{\Delta}}{2}|T_a^{\mu\nu}(0)|-\frac{\vec{\Delta}}{2}\rangle}{2P^0}$$

[Polyakov (2003)]
[C.L., Mantovani, Pasquini (2018)]
[Polyakov, Schweitzer (2018)]
[C.L., Moutarde, Trawinski (2019)]

$$P^0 = \sqrt{\frac{\vec{\Delta}^2}{4} + M^2}$$

Anisotropic medium

[Polyakov (2003)]
[Goeke *et al.* (2007)]
[Polyakov, Schweitzer (2018)]
[C.L., Moutarde, Trawinski (2019)]

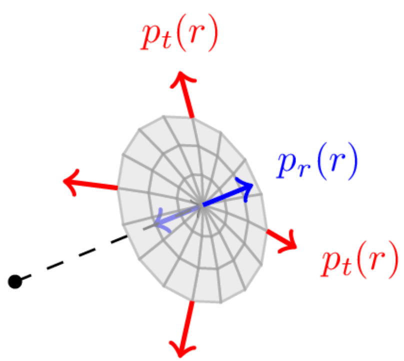
Breit frame amplitude (unpolarized) $t = -\vec{\Delta}^2$

$$\frac{\langle\frac{\vec{\Delta}}{2}|T^{\mu\nu}_a(0)|-\frac{\vec{\Delta}}{2}\rangle}{2P^0} = M\left\{\eta^{\mu 0}\eta^{\nu 0}\left[A_a(t)+\frac{t}{4M^2}\,B_a(t)\right]+\eta^{\mu\nu}\left[\bar{C}_a(t)-\frac{t}{M^2}\,C_a(t)\right]+\frac{\Delta^\mu\Delta^\nu}{M^2}\,C_a(t)\right\}$$

Analogy with relativistic hydrodynamics $r = |\vec{r}|$

Anisotropic fluid

$$\Theta^{\mu\nu}_a(\vec{r}) = [\varepsilon_a(r)+p_{t,a}(r)]\,u^\mu u^\nu - p_{t,a}(r)\eta^{\mu\nu} + [p_{r,a}(r)-p_{t,a}(r)]\,\frac{r^\mu r^\nu}{r^2}$$



Isotropic pressure

$$p_a(r) = \frac{p_{r,a}(r)+2p_{t,a}(r)}{3}$$

Pressure anisotropy

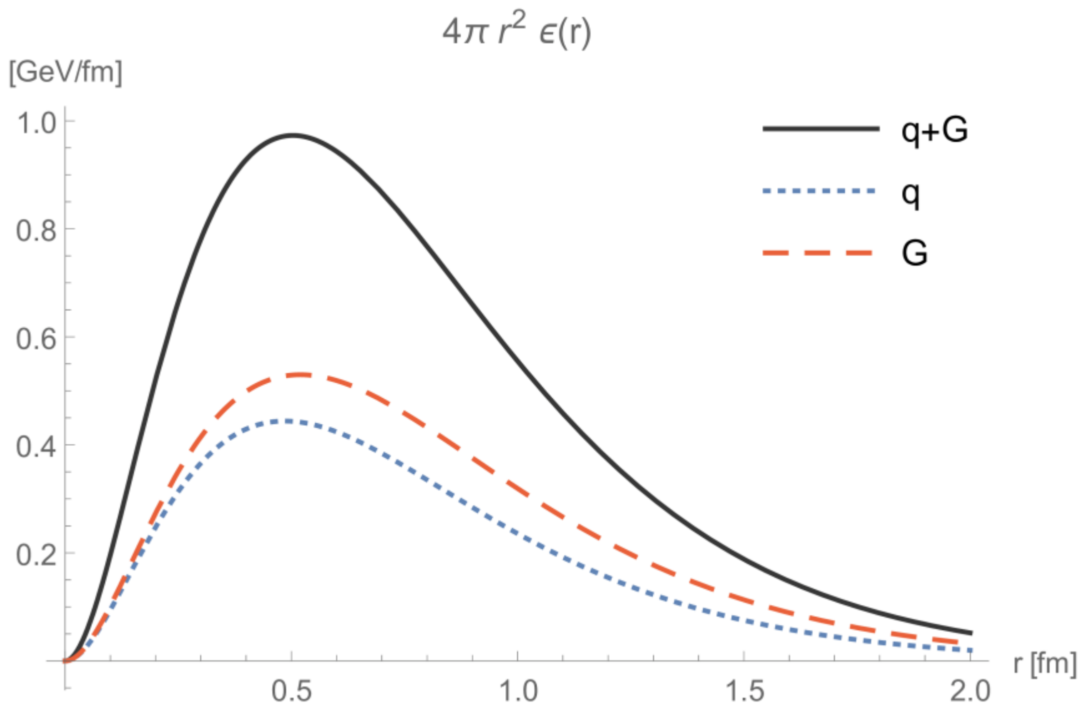
$$s_a(r) = p_{r,a}(r) - p_{t,a}(r)$$

Energy distribution

[C.L., Moutarde, Trawinski (2019)]

Multipole model for the GFFs

$$F(t) = \frac{F(0)}{(1 + t/\Lambda^2)^n}$$



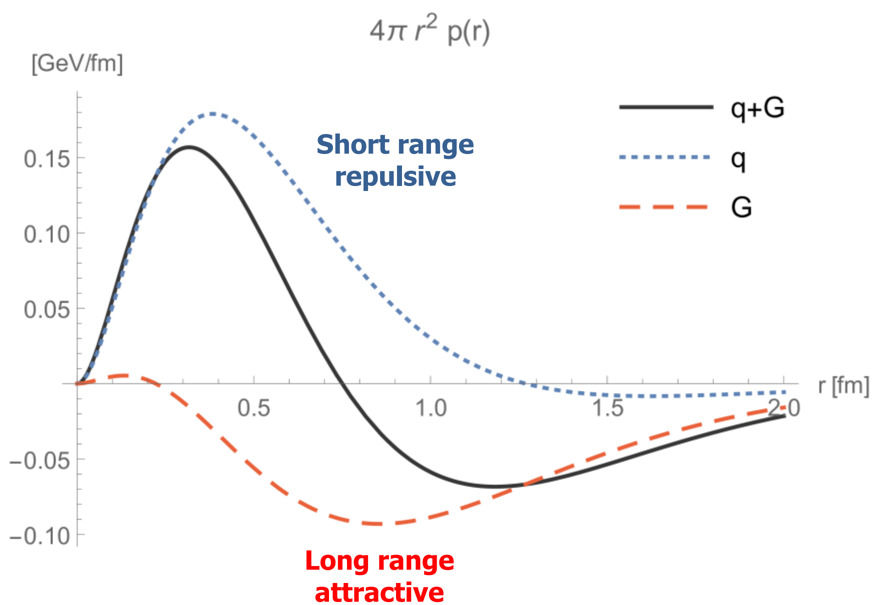
$$\sqrt{\langle r^2 \rangle_M} = 0.91\,\mathrm{fm}$$

$$\sqrt{\langle r^2 \rangle_Q} = 0.84 - 0.88\,\mathrm{fm}$$

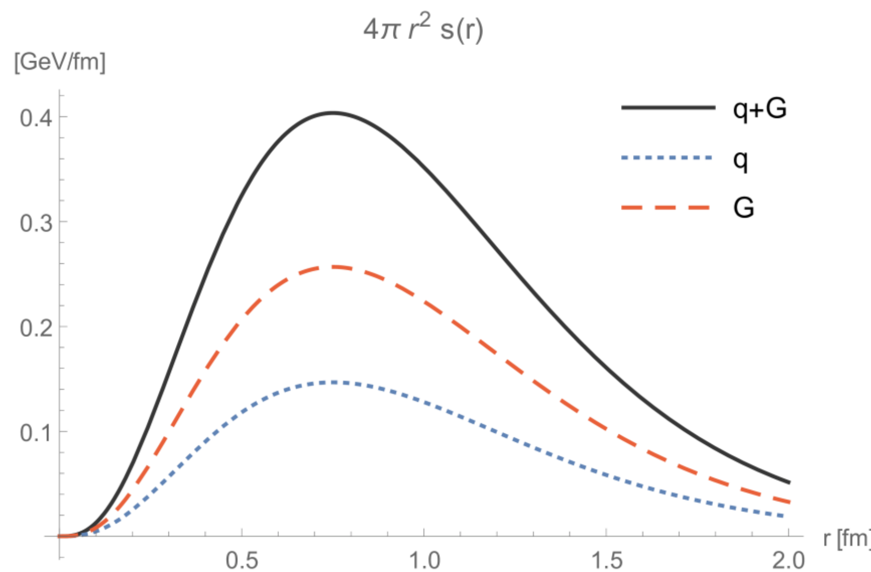
Pressure distribution

[C.L., Moutarde, Trawinski (2019)]

Radial pressure



Pressure anisotropy



Stability condition

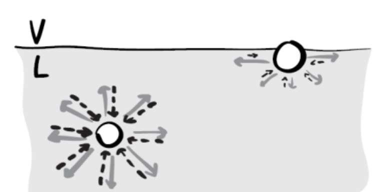
$$\int_0^\infty \mathrm{d}r\, r^2\, p(r) = 0$$

[von Laue (1911)]

Surface tension

$$\gamma = \int \mathrm{d}r\, s(r)$$

[Bakker (1928)]
[Kirkwood, Buff (1949)]
[Marchand *et al.* (2011)]

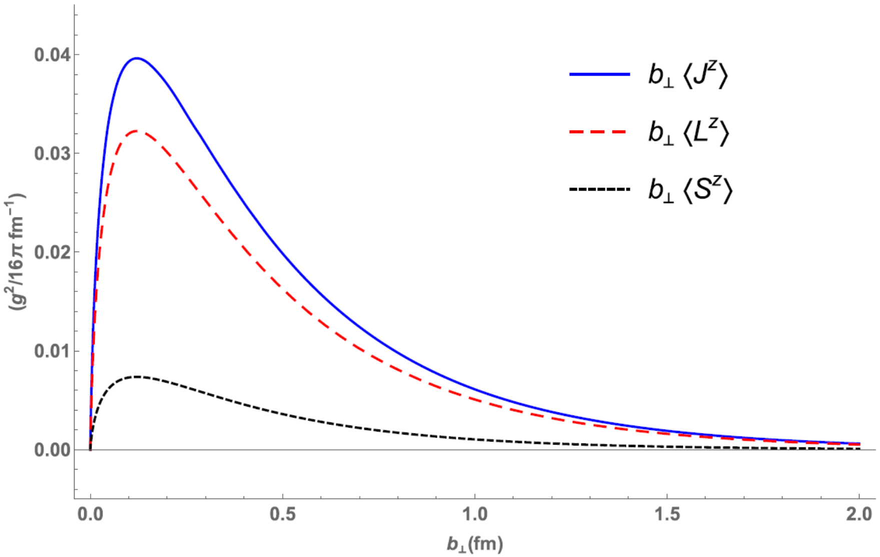


Angular momentum distribution

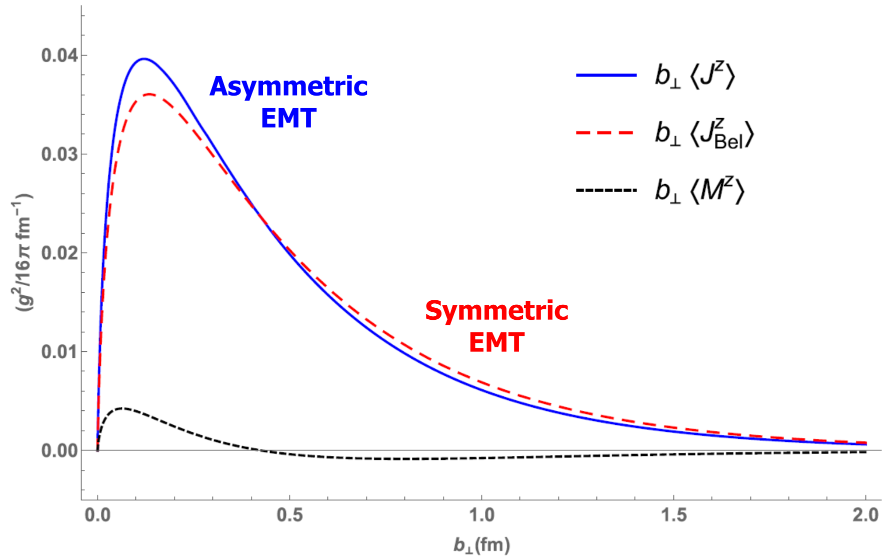
[C.L., Mantovani, Pasquini, PLB776 (2018)]

See also poster by A. Amor

Spin and OAM



Kinetic vs. Belinfante

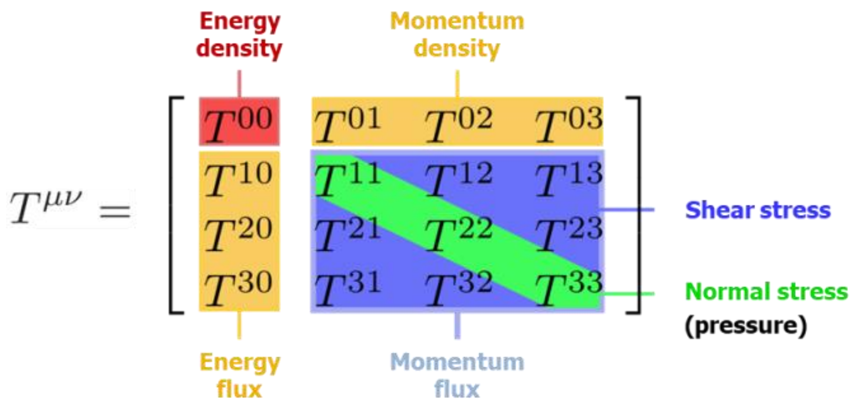


Scalar diquark model — **Kinetic** / **Canonical** — **First analytic proof!**

$$\boxed{\langle L^z \rangle(b_\perp) = \mathcal{L}^z(b_\perp)}$$

$$= \frac{1}{4\pi}\int_0^1 \mathrm{d}x\,(1-x)\,|\Psi^+_-(x,\vec{b}_\perp)|^2$$

Summary



$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

- **Key information about the structure encoded in EMT**
- **Breit frame defines the spatial distribution at rest**
- **Mechanical properties can be expressed in terms of spatial moments**
  Mass, pressure, moment of inertia, ...
- **Much more to say !**

[C.L., Mantovani, Pasquini, PLB776 (2018)]
[C.L., EPJC78 (2018)]
[C.L., Moutarde, Trawinski, EPJC79 (2019)]
[Cosyn, Cotogno, Freese, C.L., EPJC79 (2019)]
[Cotogno, C.L., Lowdon, arXiv:1905.11969]

You’re all welcome!

https://indico.cern.ch/e/LC2019

Backup slides

Quantum chromodynamics (QCD)

Classical QCD energy-momentum tensor

$$T^{\mu\nu} = \overline{\psi}\gamma^{\mu}\,\tfrac{i}{2}\overleftrightarrow{D}^{\nu}\psi - G^{a\mu\alpha}G^{a\nu}{}_{\alpha} + \tfrac{1}{4}\,\eta^{\mu\nu}G^{2}$$

Renormalized trace of the QCD EMT

$$T^{\mu}{}_{\mu} = \underbrace{\frac{\beta(g)}{2g}\,G^{2}}_{\text{Trace anomaly}} + (1+\gamma_m)\,\overline{\psi}\underset{\substack{\uparrow\\ \text{Quark mass matrix}}}{m}\psi$$

[Crewther (1972)]
[Chanowitz, Ellis (1972)]
[Nielsen (1975)]
[Adler, Collins, Duncan (1977)]
[Collins, Duncan, Joglekar (1977)]
[Nielsen (1977)]

Poincaré invariance

$$\partial_{\mu}T^{\mu\nu} = 0 \quad \Rightarrow \quad \sum_{i=q,g} A_i(0) = 1, \qquad \sum_{i=q,g} \bar{C}_i(t) = 0$$

$$\partial_{\mu}\underbrace{J^{\mu\alpha\beta}}_{x^{\alpha}T^{\mu\beta} - x^{\beta}T^{\mu\alpha} + S^{\mu\alpha\beta}} = 0 \quad \Rightarrow \quad \sum_{i=q,g} B_i(0) = 0, \qquad D_q(t) = -G^{q}_{A}(t)$$

[Kobzarev, Okun (1962)]
[Teryaev (1999)]
[Brodsky, Hwang, Ma, Schmidt (2001)]
[Leader, C.L. (2014)]
[Teryaev (2016)]
[Lowdon, Chiu, Brodsky (2017)]

Target polarization

Spin-0 $1 = 1$

Spin-1/2

Dipole (vector)

$$u(p,s)\overline{u}(p,s') = (\not{p} + M)\,\frac{\delta_{s's}\mathbb{1} - \mathcal{S}^{\mu}_{s's}(p)\gamma_\mu\gamma_5}{2}$$

[Bouchiat, Michel (1958)]

Spin-1

[Joos, Kramer (1964)]
[Zwanziger (1965)]

$$\epsilon_\beta(p,s)\epsilon^*_\alpha(p,s') = -\delta_{s's}\,\frac{1}{3}\left(g_{\beta\alpha} - \frac{p_\beta p_\alpha}{M^2}\right) + \mathcal{S}^{\mu}_{s's}(p)\,\frac{i}{2M}\,\varepsilon_{\mu\beta\alpha\nu}p^\nu - \mathcal{T}^{\mu\nu}_{s's}(p)g_{\mu\beta}g_{\nu\alpha}$$

Dipole (vector) Quadrupole (tensor)

Onshell constraints $p_\mu \mathcal{S}^{\mu}_{s's}(p) = 0, \quad p_\mu \mathcal{T}^{\mu\nu}_{s's}(p) = 0, \quad \mathcal{T}^{[\mu\nu]}_{s's}(p) = 0, \quad g_{\mu\nu}\mathcal{T}^{\mu\nu}_{s's}(p) = 0$

Generalization to off-forward case $\Rightarrow$

[C.L. (2018)]
[Cosyn, Cotogno, Freese, C.L., arXiv:1903.00408]

Textbook decomposition

Forward matrix element

$$\langle P|T^{\mu\nu}(0)|P\rangle = 2P^\mu P^\nu$$

$$\langle P'|P\rangle = 2P^0\,(2\pi)^3\,\delta^{(3)}(\vec{P}'-\vec{P})$$

Trace decomposition

[Shifman, Vainshtein, Zakharov (1978)]
[Luke, Manohar, Savage (1992)]
[Donoghue, Golowich, Holstein (1992)]
[Kharzeev (1996)]
[Bressani, Wiedner, Filippi (2005)]
[Roberts (2017)]
[Krein, Thomas, Tsushima (2017)]

$$\begin{aligned} 2M^2 &= \langle P|T^\mu{}_\mu(0)|P\rangle \\ &= \underbrace{\langle P|\frac{\beta(g)}{2g}\,G^2|P\rangle}_{\sim 89\%} + \underbrace{\langle P|(1+\gamma_m)\,\overline{\psi}m\psi|P\rangle}_{\sim 11\%} \end{aligned}$$

- ✓ Manifestly covariant
- ✓ Reminiscent of Gell-Mann–Oakes–Renner formula for pion

- ✗ Depends on state normalization
- ✗ No spatial extension
- ✗ No clear relation to energy

Ji's decomposition

[Ji (1995)]

Separation of quark and gluon contributions

$$T^{\mu\nu} = \underbrace{\bar{T}^{\mu\nu}}_{\text{Traceless}} + \underbrace{\hat{T}^{\mu\nu}}_{\text{Pure trace}} \qquad \bar{T}^{\mu\nu} = \bar{T}^{\mu\nu}_q + \bar{T}^{\mu\nu}_g \qquad \hat{T}^{\mu\nu} = \hat{T}^{\mu\nu}_m + \hat{T}^{\mu\nu}_a$$

Forward matrix elements

$$\langle P|\bar{T}^{\mu\nu}_i(0)|P\rangle = 2\left(P^\mu P^\nu - \tfrac{1}{4}\,\eta^{\mu\nu}M^2\right)A_i(0)$$
$$\langle P|\hat{T}^{\mu\nu}_i(0)|P\rangle = \tfrac{1}{2}\,\eta^{\mu\nu}M^2\left[A_i(0) + 4\bar{C}_i(0)\right]$$

$$\langle O\rangle = \frac{\langle P|\int \mathrm{d}^3r\, O(r)|P\rangle}{\langle P|P\rangle}$$

Ji's decomposition

[Gao *et al.* (2015)]

$$M = \underbrace{M_q}_{\sim 31\%} + \underbrace{M_g}_{\sim 34\%} + \underbrace{M_m}_{\sim 13\%} + \underbrace{M_a}_{\sim 22\%}$$

$\mu = 2\,\mathrm{GeV}$

$$M_q = \langle \bar{T}^{00}_q\rangle|_{\vec{P}=\vec{0}} - \frac{3}{1+\gamma_m}\,\langle \hat{T}^{00}_m\rangle|_{\vec{P}=\vec{0}}$$
$$M_g = \langle \bar{T}^{00}_g\rangle|_{\vec{P}=\vec{0}}$$
$$M_m = \frac{4+\gamma_m}{1+\gamma_m}\,\langle \hat{T}^{00}_m\rangle|_{\vec{P}=\vec{0}}$$
$$M_a = \langle \hat{T}^{00}_a\rangle|_{\vec{P}=\vec{0}}$$

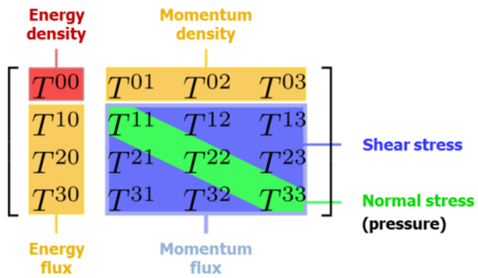
- ✓ **Proper normalization**
- ✓ **Clear relation to energy distribution**
- ! **Scale-dependent interpretation in the rest frame**
- ✗ **Pressure effects not taken into account**

New decomposition

[C.L. (2018)]

Forward matrix element

$$\langle P|T_i^{\mu\nu}(0)|P\rangle = 2P^\mu P^\nu A_i(0) + 2M^2\eta^{\mu\nu}\bar{C}_i(0)$$



Analogy with relativistic hydrodynamics

Perfect fluid element $\quad \Theta_i^{\mu\nu} = (\varepsilon_i + p_i)u^\mu u^\nu - p_i\,\eta^{\mu\nu}$

Four-velocity

$$u^\mu = P^\mu/M$$

Energy density

$$\varepsilon_i = [A_i(0) + \bar{C}_i(0)]\,\frac{M}{V}$$

Isotropic pressure

$$p_i = -\bar{C}_i(0)\,\frac{M}{V}$$

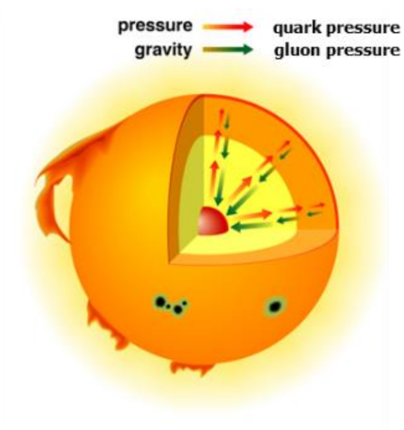
Nucleon mass decomposition $\quad U_i = \varepsilon_i V$

$$M = \underbrace{U_q}_{\sim 44\%} + \underbrace{U_g}_{\sim 56\%}$$

$\mu = 2\,\mathrm{GeV}$

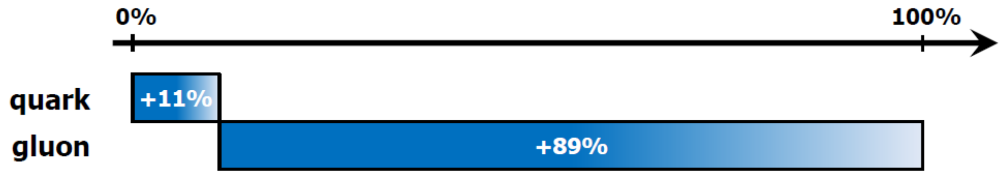
$$p_q = -p_g$$

~ 11%

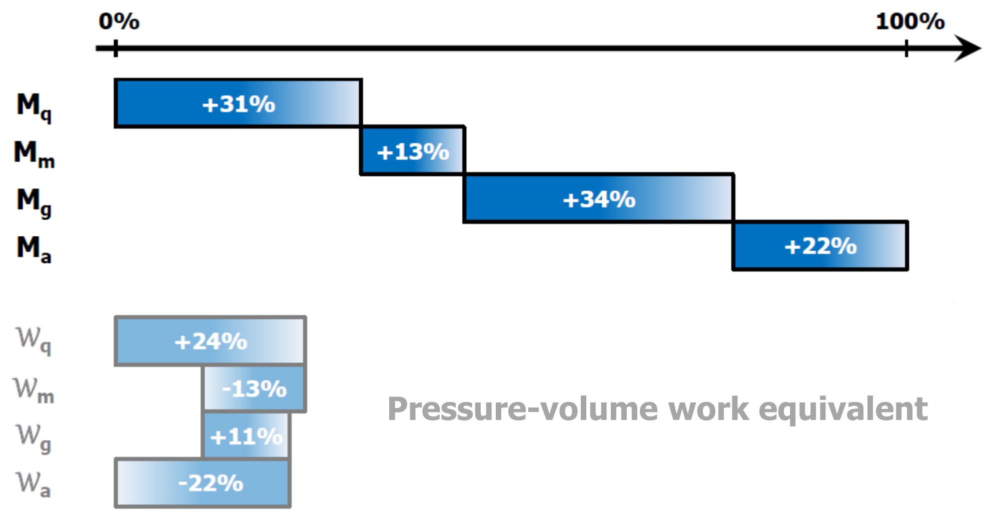


In short

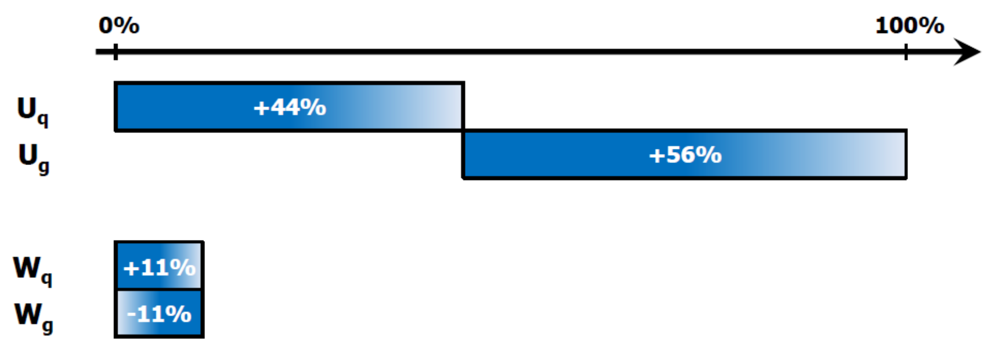
Trace decomposition



Ji's decomposition

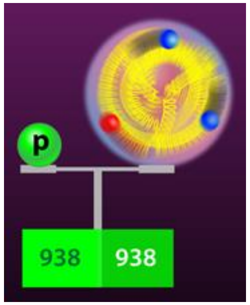


New decomposition



$\mu = 2\,\mathrm{GeV}$

Mass decomposition and balance equations



1st moment

$$\int \mathrm{d}^3r\, \langle T_a^{\mu\nu}\rangle(\vec{r}) = \frac{\langle\vec{0}|T_a^{\mu\nu}(0)|\vec{0}\rangle}{2M}$$

Forward amplitude

$$\langle p,s'|T_a^{\mu\nu}(0)|p,s\rangle = \delta_{s's}\left\{2p^\mu p^\nu\left[\mathcal{G}_1^a(0)+\tfrac{1}{6}\mathcal{G}_7^a(0)\right]-2M^2g^{\mu\nu}\left[\tfrac{1}{2}\mathcal{G}_8^a(0)+\tfrac{1}{6}\mathcal{G}_7^a(0)\right]\right\}$$
$$-\,\mathcal{T}_{s's}^{\mu\nu}(p)\,2M^2\,\tfrac{1}{2}\mathcal{G}_7^a(0)$$

Poincaré symmetry

$$\langle p,s'|\sum_a T_a^{\mu\nu}(0)|p,s\rangle = 2p^\mu p^\nu \quad\Longrightarrow\quad \sum_a \mathcal{G}_1^a(0)=1$$

Mass decomposition

$$\sum_a U_a = M \qquad U_a = \frac{\langle p|T_a^{00}(0)|p\rangle}{2M} = \left[\mathcal{G}_1^a(0)-\tfrac{1}{2}\mathcal{G}_8^a(0)\right]M$$

Balance equations

$$\sum_a W_a = 0 \qquad W_a = \frac{\delta^{ij}}{3}\,\frac{\langle p|T_a^{ij}(0)|p\rangle}{2M} = \left[\tfrac{1}{2}\mathcal{G}_8^a(0)+\tfrac{1}{6}\mathcal{G}_7^a(0)\right]M$$

$$\sum_a W_a^{ij} = 0 \qquad W_a^{ij} = \frac{\langle p|T_a^{ij}(0)|p\rangle}{2M} - \delta^{ij}W_a = \mathcal{T}^{ij}(p)\left[-\tfrac{1}{2}\mathcal{G}_7^a(0)\right]M$$

[C.L. (2018)]
[Cosyn, Cotogno, Freese, C.L., arXiv:1903.00408]

Angular momentum sum rule

2nd moment

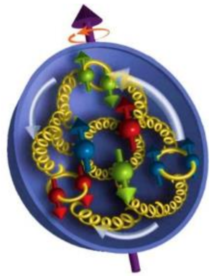
$$\int \mathrm{d}^3r\, r^j \langle T_a^{\mu\nu}\rangle(\vec{r}) = \left[-i\nabla_\Delta^j \frac{\langle \frac{\vec{\Delta}}{2}|T_a^{\mu\nu}(0)|-\frac{\vec{\Delta}}{2}\rangle}{2P^0}\right]_{\vec{\Delta}=\vec{0}}$$

Orbital and total angular momentum

[Cosyn, Cotogno, Freese, C.L., arXiv:1903.00408]
[Polyakov, Sun, arXiv:1903.02738]

$$L_a^i = \varepsilon^{ijk}\int \mathrm{d}^3r\, r^j \langle T_a^{0k}\rangle(\vec{r}) = \frac{s^i}{2}\left[\mathcal{G}_5^a(0) + \tfrac{1}{2}\mathcal{G}_7^a(0) + \mathcal{G}_{10}^a(0)\right]$$

$$J_a^i = \varepsilon^{ijk}\int \mathrm{d}^3r\, r^j \langle T_a^{\{0k\}}\rangle(\vec{r}) = \frac{s^i}{2}\left[\mathcal{G}_5^a(0) + \underbrace{\tfrac{1}{2}\mathcal{G}_7^a(0)}\right]$$



Missing in
[Taneja, Kathuria, Liuti, Goldstein (2012)]

Angular momentum sum rule

$$\sum_a J_a^i = s^i \quad \Rightarrow \quad \sum_a \mathcal{G}_5^a(0) = 2$$

[Abidin, Carlson (2008)]

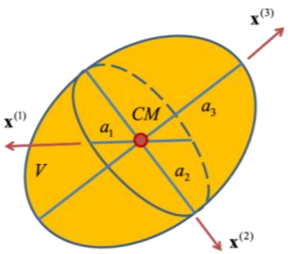
☞ **Generalization to arbitrary spin** **[Cotogno, C.L., Lowdon, arXiv:1905.xxxxx]**

Mass radius and inertia tensor

3rd moment

$$\int \mathrm{d}^3r\, r^i r^j \langle T^{00}_a\rangle(\vec r) = \left[-\nabla^i_\Delta \nabla^j_\Delta \frac{\langle \frac{\vec\Delta}{2}|T^{00}_a(0)|-\frac{\vec\Delta}{2}\rangle}{2P^0}\right]_{\vec\Delta=\vec 0}$$

Inertia tensor



$$I^{ij}_a = \int \mathrm{d}^3r \left(\vec r^{\,2}\delta^{ij} - r^i r^j\right)\langle T^{00}_a\rangle(\vec r)$$

$$= \frac{1}{M}\left[2\delta^{ij}\left(\mathcal{A}_a(0) + \tfrac{1}{3}\mathcal{B}_a(0)\right) + \mathcal{T}^{ij}\mathcal{B}_a(0)\right]$$

$$\mathcal{A}_a(t) = -\frac{1}{4}\left[\mathcal{G}^a_1(t) + 2\mathcal{G}^a_3(t) + \tfrac{1}{2}\mathcal{G}^a_8(t)\right] + 2M^2\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{G}^a_1(t) - \tfrac{1}{2}\mathcal{G}^a_8(t)\right]$$

$$\mathcal{B}_a(t) = -\mathcal{G}^a_1(t) - \mathcal{G}^a_3(t) + \mathcal{G}^a_5(t) + \tfrac{1}{2}\left[\mathcal{G}^a_6(t) + \tfrac{1}{2}\mathcal{G}^a_7(t) + \mathcal{G}^a_8(t) - \mathcal{G}^a_9(t)\right]$$

Mass radius

$$\langle \vec r^{\,2}\rangle = \frac{\delta^{ij}}{2M}\sum_a I^{ij}_a = \frac{1}{M^2}\sum_a\left[3\mathcal{A}_a(0) + \mathcal{B}_a(0)\right]$$

Mass quadrupole moment

$$Q^{ij}_a = \frac{\delta^{ij}}{3}(\delta^{kl} I^{kl}_a) - I^{ij}_a = -\frac{1}{M}\mathcal{T}^{ij}\mathcal{B}_a(0)$$

[Cosyn, Cotogno, Freese, C.L., arXiv:1903.00408]

Link with GPDs

2nd Mellin moment

$$\int_{-1}^{1} \mathrm{d}x\, x\, V_a^{\mu}(x) = \frac{T_a^{\mu+}(0)}{2(P^+)^2}$$

$$V_q^{\mu}(x) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-}\, \overline{\psi}(-\tfrac{z^-}{2})\gamma^{\mu}\mathcal{W}\psi(\tfrac{z^-}{2})$$

$$V_G^{\mu}(x) = \frac{\delta^{\mu}_{\{\alpha}\delta^{+}_{\beta\}}}{2xP^+}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-}\,\mathrm{Tr}\Big[\mathcal{W}F^{\alpha\lambda}(-\tfrac{z^-}{2})\mathcal{W}F_{\lambda}{}^{\beta}(\tfrac{z^-}{2})\Big]$$

Twist-2

$$\int_{-1}^{1} \mathrm{d}x\, xH_2^a(x,\xi,t) = \mathcal{G}_5^a(t)$$

$$\int_{-1}^{1} \mathrm{d}x\, xH_5^a(x,\xi,t) = -\frac{t}{4M^2}\,\mathcal{G}_6^a(t) + \tfrac{1}{2}\mathcal{G}_7^a(t)$$

Twist-3

$$\int_{-1}^{1} \mathrm{d}x\, xG_6^q(x,\xi,t) = -\frac{1}{2}\,[\mathcal{G}_5^q(t) + \mathcal{G}_{10}^q(t)]$$

[Taneja, Kathuria, Liuti, Goldstein (2012)]
[Cosyn, Cotogno, Freese, C.L., arXiv:1903.00408]
[Polyakov, Sun, arXiv:1903.02738]

Angular momentum relations

Spin-1

$$J_z^a = \int_{-1}^{1} \mathrm{d}x\, \frac{x}{2}\,[H_2^a(x,0,0) + H_5^a(x,0,0)]$$

$$L_z^q = \int_{-1}^{1} \mathrm{d}x\, x\,\big[-G_6^q(x,0,0) + \tfrac{1}{2}H_5^q(x,0,0)\big]$$

Spin-1/2

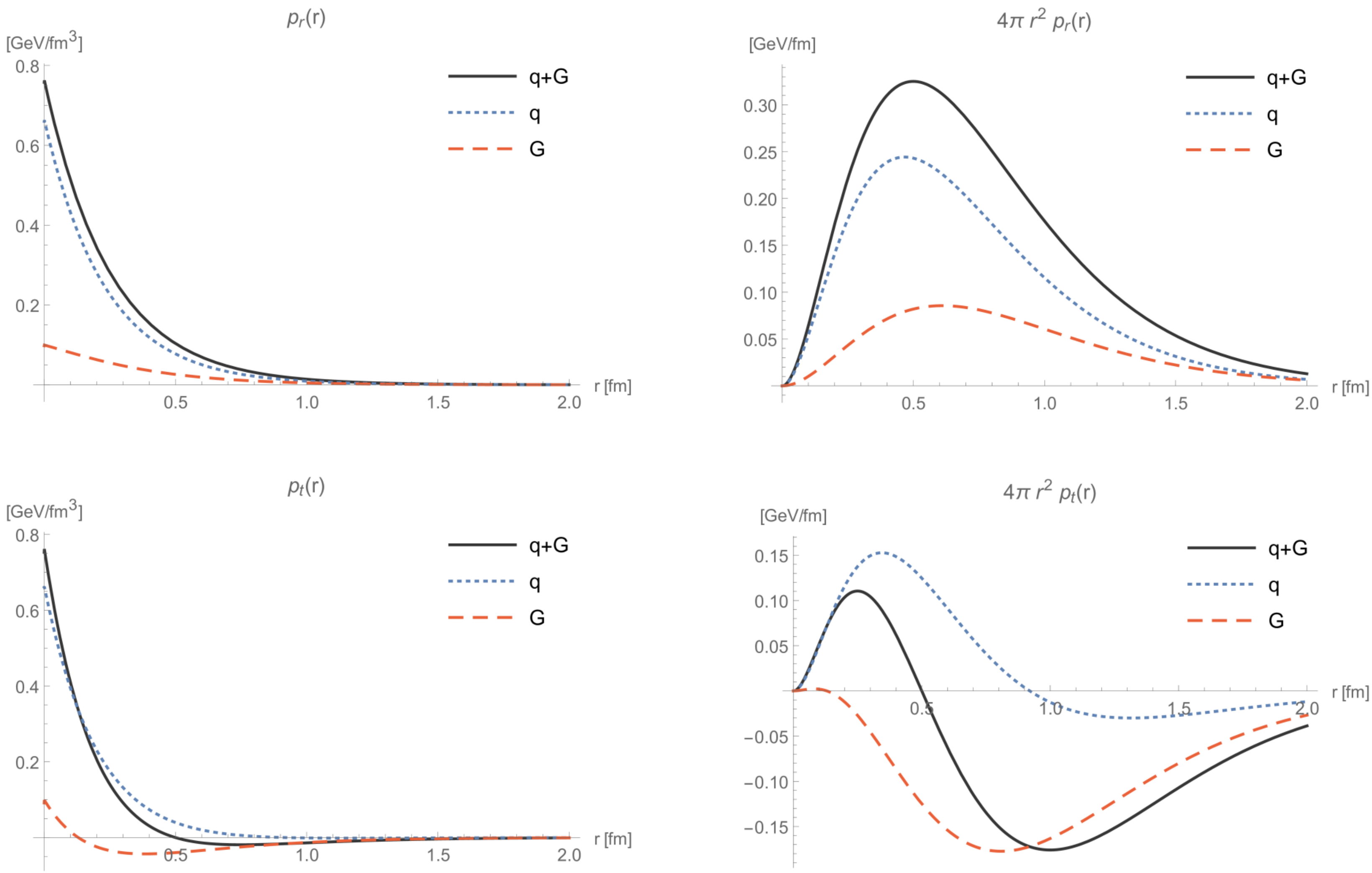
$$J_z^a = \int_{-1}^{1} \mathrm{d}x\, \frac{x}{2}\,[H^a(x,0,0) + E^a(x,0,0)]$$

$$L_z^q = -\int_{-1}^{1} \mathrm{d}x\, xG_2^q(x,0,0)$$

[Ji (1996)]
[Penttinen, Polyakov, Shuvaev, Strikman (2000)]
[Kiptily, Polyakov (2004)]
[Hatta, Yoshida (2012)]

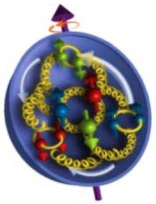
Pressure distribution

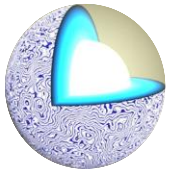
[C.L., Moutarde, Trawinski (2019)]



What can we learn?

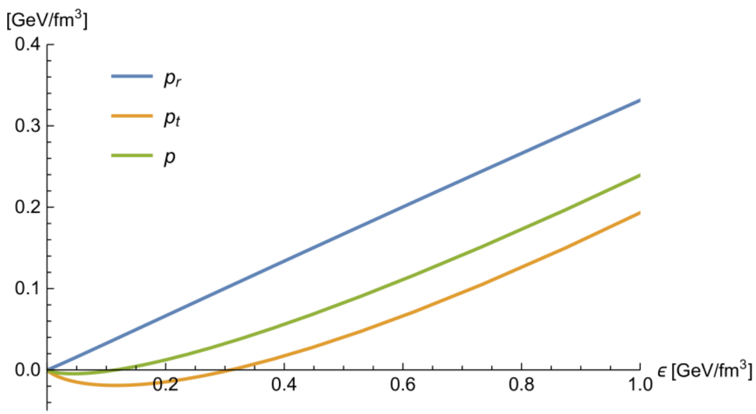
Nucleon



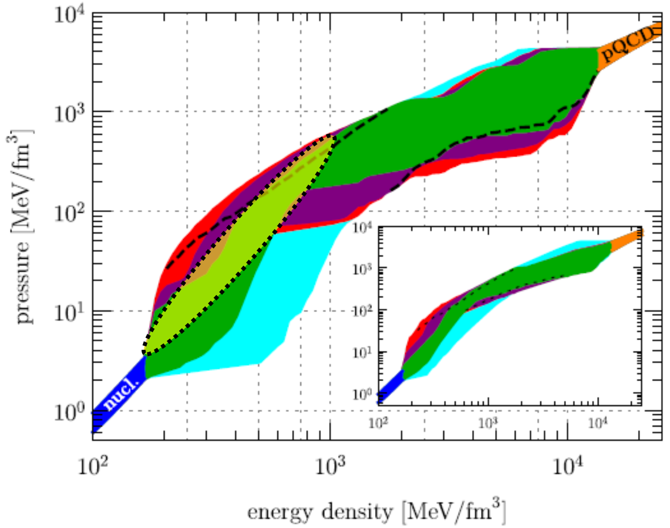


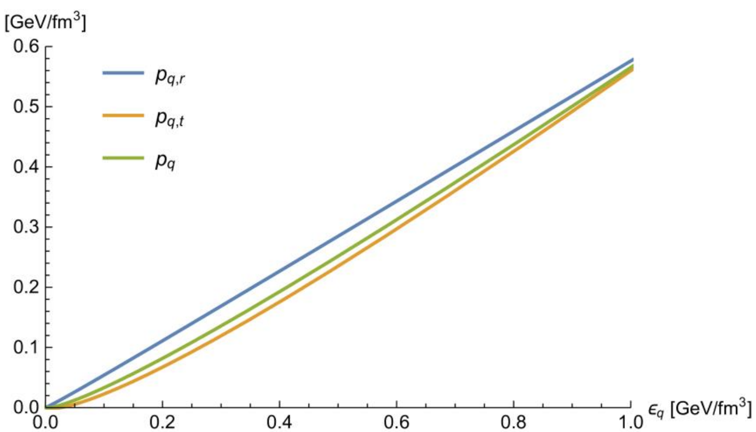
Neutron star

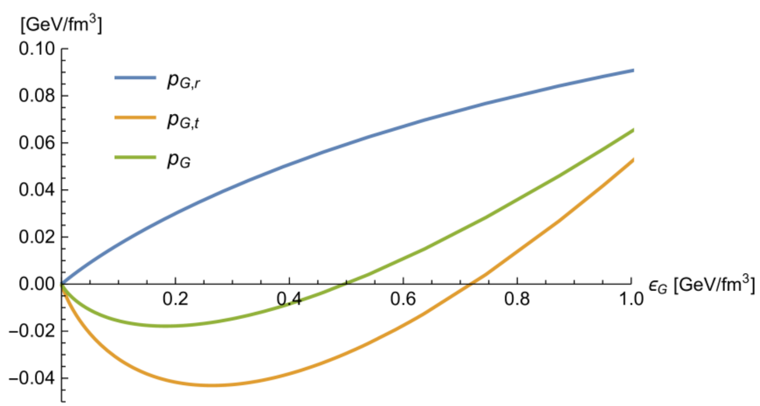
Equation of state



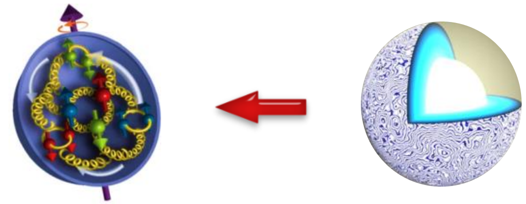
[Annala *et al.* (2018)]







What can we learn?

Nucleon  **Neutron star**

Stability constraints

[Wald (1984)]
[Herrera, Santos (1997)]
[Poisson (2004)]
[Abreu, Hernandez, Nunez (2007)]
[Hawking, Ellis (2011)]

Mechanical regularity

(i) $\varepsilon(0) < \infty$, $p(0) < \infty$ and $s(0) = 0$;

(ii) $\varepsilon(r) > 0$ and $p_r(r) > 0$;

(iii) $\frac{\mathrm{d}\varepsilon(r)}{\mathrm{d}r} < 0$ and $\frac{\mathrm{d}p_r(r)}{\mathrm{d}r} < 0$.

Speed of sound

(iv) $0 \leq v_{sr}^2(r) \leq 1$ and $0 \leq v_{st}^2(r) \leq 1$;

(v) $|v_{st}^2(r) - v_{sr}^2(r)| \leq 1$;

(vi) $\Gamma(r) = \frac{\varepsilon(r)+p_r(r)}{p_r(r)}\, v_{sr}^2 > \frac{4}{3}$.

Energy conditions

$$\begin{aligned}
&\varepsilon(r) + p_i(r) \geq 0, \\
&\varepsilon(r) + p_i(r) \geq 0 \quad \text{and} \quad \varepsilon(r) \geq 0, \\
&\varepsilon(r) + p_i(r) \geq 0 \quad \text{and} \quad \varepsilon(r) + 3\,p(r) \geq 0, \\
&\qquad \varepsilon(r) \geq |p_i(r)|,
\end{aligned}$$

3D distribution in Breit frame

Kinetic OAM

$$\langle L^i\rangle(\vec{x}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{x}} \left[s^i L(t) + [(\vec{\Delta}\cdot\vec{s})\Delta^i - \vec{\Delta}^2 s^i]\,\frac{\mathrm{d}L(t)}{\mathrm{d}t}\right]_{t=-\vec{\Delta}^2}, \qquad L(t) \equiv \frac{A(t)+B(t)+D(t)}{2}$$

Spin

$$\langle S^i\rangle(\vec{x}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{x}} \left[\frac{s^i}{2}\,G_A(t) - \frac{(\vec{\Delta}\cdot\vec{s})\Delta^i}{4}\,\frac{\mathrm{d}G(t)}{\mathrm{d}t}\right]_{t=-\vec{\Delta}^2}, \qquad \frac{\mathrm{d}G(t)}{\mathrm{d}t} \equiv \frac{1}{2P^0}\left[\frac{G_A(t)}{P^0+M} + \frac{G_P(t)}{M}\right]$$

Belinfante total AM

$$\langle J^i_{\mathrm{Bel}}\rangle(\vec{x}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{x}} \left[s^i J(t) + [(\vec{\Delta}\cdot\vec{s})\Delta^i - \vec{\Delta}^2 s^i]\,\frac{\mathrm{d}J(t)}{\mathrm{d}t}\right]_{t=-\vec{\Delta}^2}, \qquad J(t) \equiv \frac{A(t)+B(t)}{2}$$

$$\boxed{\int \mathrm{d}^3x\,[\langle L^i\rangle(\vec{x}) + \langle S^i\rangle(\vec{x})] = \int \mathrm{d}^3x\,\langle J^i_{\mathrm{Bel}}\rangle(\vec{x})} \qquad \int \mathrm{d}^3x \leftrightarrow \vec{\Delta} = \vec{0}$$

But $\langle L^i\rangle(\vec{x}) + \langle S^i\rangle(\vec{x}) \neq \langle J^i_{\mathrm{Bel}}\rangle(\vec{x})$ **!!!!**

3D distribution in Breit frame

Kinetic OAM

$$\langle L^i\rangle(\vec{x}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{x}} \left[s^i L(t) + [(\vec{\Delta}\cdot\vec{s})\Delta^i - \vec{\Delta}^2 s^i]\, \frac{\mathrm{d}L(t)}{\mathrm{d}t} \right]_{t=-\vec{\Delta}^2}, \qquad L(t) \equiv \frac{A(t)+B(t)+D(t)}{2}$$

Spin

$$\langle S^i\rangle(\vec{x}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{x}} \left[\frac{s^i}{2}\, G_A(t) - \frac{(\vec{\Delta}\cdot\vec{s})\Delta^i}{4}\, \frac{\mathrm{d}G(t)}{\mathrm{d}t} \right]_{t=-\vec{\Delta}^2}, \qquad \frac{\mathrm{d}G(t)}{\mathrm{d}t} \equiv \frac{1}{2P^0} \left[\frac{G_A(t)}{P^0+M} + \frac{G_P(t)}{M} \right]$$

Belinfante total AM

$$\langle J^i_{\mathrm{Bel}}\rangle(\vec{x}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{x}} \left[s^i J(t) + [(\vec{\Delta}\cdot\vec{s})\Delta^i - \vec{\Delta}^2 s^i]\, \frac{\mathrm{d}J(t)}{\mathrm{d}t} \right]_{t=-\vec{\Delta}^2}, \qquad J(t) \equiv \frac{A(t)+B(t)}{2}$$

Superpotential

$$\langle M^i\rangle(\vec{x}) = -\int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{x}} \left[\frac{(\vec{\Delta}\cdot\vec{s})\Delta^i - \vec{\Delta}^2 s^i}{2}\, \frac{\mathrm{d}G_A(t)}{\mathrm{d}t} + \frac{(\vec{\Delta}\cdot\vec{s})\Delta^i}{4}\, \frac{\mathrm{d}G(t)}{\mathrm{d}t} \right]_{t=-\vec{\Delta}^2}$$

$$\boxed{\langle L^i\rangle(\vec{x}) + \langle S^i\rangle(\vec{x}) = \langle J^i_{\mathrm{Bel}}\rangle(\vec{x}) + \langle M^i\rangle(\vec{x})} \qquad \int \mathrm{d}^3x\, \langle M^i\rangle(\vec{x}) = 0$$