Hadronic vacuum polarization contribution to the muon magnetic moment from lattice QCD

Laurent Lellouch

CPT Marseille
CNRS & Aix-Marseille U.

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Interaction with an external EM field: SM & BSM

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

$$\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m_\ell}\sigma_{\mu\nu}q^\nu F_2(q^2) - \gamma_5\sigma_{\mu\nu}q^\nu F_3(q^2)\right.$$

$$\left. + \gamma_5(q^2\gamma_\mu - 2m_\ell q_\mu)F_4(q^2)\right] u(p)$$

$F_1(q^2) \quad \rightarrow \quad$ Dirac form factor: $F_1(0) = 1$

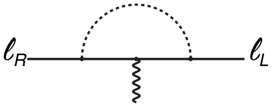
$F_2(q^2) \quad \rightarrow \quad$ Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \dfrac{g_\ell - \overbrace{2}^{g_\ell|_{\text{Dirac}}}}{2}$

$F_3(q^2) \quad \rightarrow \quad$ $\not{P}$, $\not{T}$, electric dipole moment: $F_3(0) = d_\ell/e_\ell$

$F_4(q^2) \quad \rightarrow \quad$ $\not{P}$, anapole moment: $\vec{\sigma}\cdot(\vec{\nabla}\times\vec{B})$

- $F_2(q^2)$ & $F_{3,4}(q^2)$ come from loops but UV finite once theory's couplings are renormalized (in a renormalizable theory)
- a_ℓ dimensionless
  - $\Rightarrow$ corrections including only ℓ and γ are mass independent, i.e. universal
  - $\rightarrow$ contributions from particles w/ $M \gg m_\ell$ are $\propto (m_\ell/M)^{2p} \times \ln^q(m_\ell^2/M^2)$
  - $\rightarrow$ contributions from particles w/ $m \ll m_\ell$ are e.g. $\propto \ln^2(m_\ell^2/m^2)$

Why are a_ℓ special?



$$\longrightarrow \quad \frac{a_\ell}{2m_\ell} eF^{\mu\nu}[\bar{\ell}_L \sigma_{\mu\nu} \ell_R]$$

- Loop induced $\Rightarrow$ sensitive to new dofs
- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to other measurements: EDMs, $b \to s\ell^+\ell^-$, $\mu \to e\gamma$, $B \to D^{(*)}\ell\nu_\ell$, EW precision observables, LHC direct searches, ...
- In SM, only source of chirality flips is $y_\ell \, \bar{\ell}_L H \ell_R$

$$m_\ell = y_\ell \langle H \rangle, \qquad a_\ell^{\text{weak}} \propto \frac{\alpha}{4\pi} \left(\frac{m_\ell}{M_W} \right)^2$$

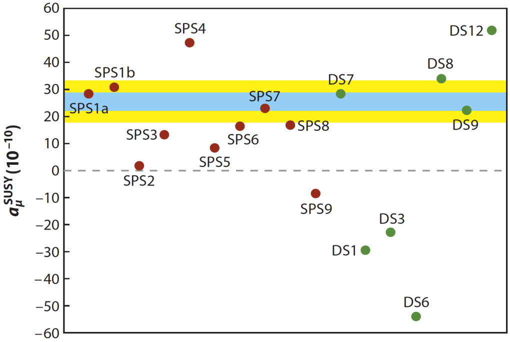
- BSM can be very different

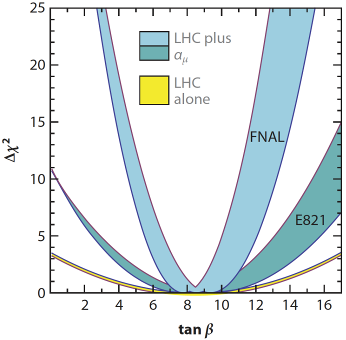
$$a_\ell^{\text{N}\Phi} \propto \left(\frac{\Delta^{\text{N}\Phi} m_\ell}{m_\ell} \right) \left(\frac{m_\ell}{M_{\text{N}\Phi}} \right)^2$$

Why is a_μ special?

$m_e : m_\mu : m_\tau = 0.0005 : 0.106 : 1.777\,\mathrm{GeV}$ $\tau_e : \tau_\mu : \tau_\tau =$ “∞” $: 2.\cdot 10^{-6} : 3.\cdot 10^{-15}$ s

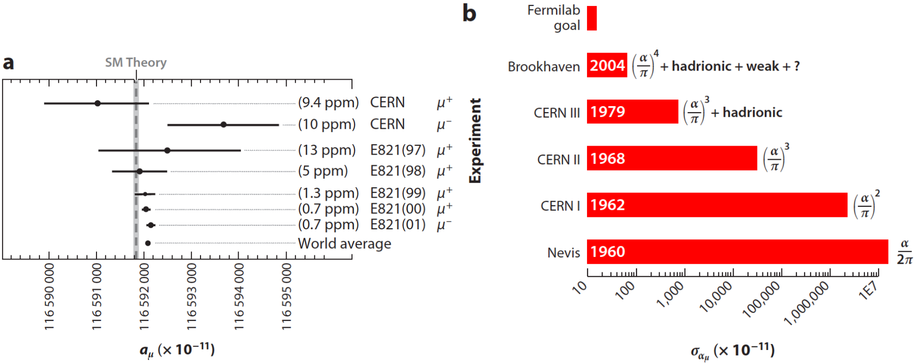
- a_μ is $(m_\mu/m_e)^2 \sim 4. \times 10^4$ times more sensitive to new Φ than a_e
- a_τ is even more sensitive to new Φ, but is too shortly lived





(Miller et al ’12)

a_μ experimental summary



(Miller et al '12)

Two new experiments plan to reduce error on a_μ to $\sim$ 0.14 ppm

- New $g-2$ (E989) @ Fermilab: has started taking data fall 2017
- $g-2$/EDM (E34) @ J-PARC: should start taking data $\geq$ 2021

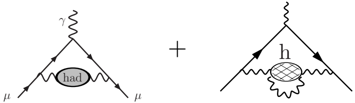
Standard model calculation of a_μ

$$\begin{aligned} a_\mu^{\text{SM}} &= a_\mu^{\text{QED}} + a_\mu^{\text{had}} + a_\mu^{\text{weak}} \\ &= O\left(\frac{\alpha}{\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{g_2}{4\pi}\right)^2\left(\frac{m_\mu}{M_W}\right)^2\right) \\ &= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right) \end{aligned}$$

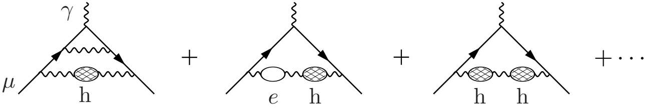
- **QED:** computed to $O(\alpha^5)$ (Aoyama, Kinoshita, Nio '96-'15)
  - 12,672 diagrams at $O(\alpha^5)$
  - $a_\mu^{\text{QED}}(a_e) = 0.001165847188 41(7)_m(17)_{\alpha^4}(6)_{\alpha^5}(28)_{\alpha(a_e)}$ (Aoyama et al '18)
- **Weak:** computed to 2 loops (Gnendiger et al '15 and refs therein)
  - $a_\mu^{\text{weak}} = 0.000000001536(10)$
- **Hadronic:** non-perturbative QCD because $q^2 = 0$ and $m_\mu \ll 1\,\text{GeV}$
  - $a_\mu^{\text{had}} \stackrel{?}{=} a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{weak}} = 0.00000007219(63)$
  - clearly right order of magnitude

Hadronic contributions to a_μ: diagrams

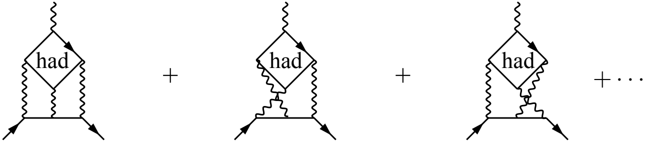
$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{NLO-HVP}} + a_\mu^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$



$$\to a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$$



$$\to a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$



$$\to a_\mu^{\text{HLbyL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

HVP from dispersion relations (DR) and $e^+e^- \to$ hadrons & HLbyL from DR, data and models

SM prediction vs experiment

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{11}$	Ref.
QED [5 loops]	116584718.841 ± 0.034	[Aoyama et al '18]
HVP LO	6933 ± 25	[KNT '18]
	6931 ± 34	[DHMZ '17]
	6881 ± 41	[Jegerlehner '17]
HVP NLO	-98.7 ± 0.9	[Kurz et al '14]
		[Kurz et al '14, Jegerlehner '16]
HVP NNLO	12.4 ± 0.1	[Kurz et al '14]
		[Jegerlehner '16]
HLbyL	105 ± 26	[Prades et al '09]
	$54 \pm 14 \pm ??$	[RBC '16]
Weak (2 loops)	153.6 ± 1.0	[Gnendiger et al '15]
SM Tot [0.31 ppm]	116591824 ± 36	[w/ KNT '18]
[0.37 ppm]	116591822 ± 43	[w/ DHMZ '17]
[0.42 ppm]	116591772 ± 49	[Jegerlehner '17]
Exp [0.54 ppm]	116592091 ± 63	[Bennett et al '06]
Exp − SM	267 ± 72	[KNT '18]
	269 ± 76	[DHMZ '17]
	319 ± 80	[Jegerlehner '17]

Consider in Euclidean spacetime (Blum '02)

$$\Pi_{\mu\nu}(Q) =$$

$$= \int d^4x\, e^{iQ\cdot x}\langle J_\mu(x)J_\nu(0)\rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2\right)\Pi(Q^2)$$

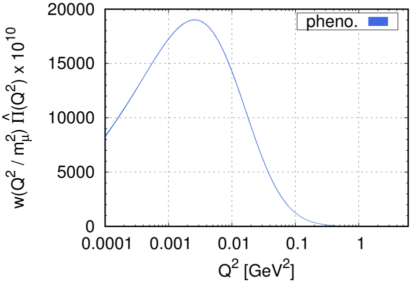
w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

Then (Lautrup et al '69, Blum '02)

$$a_\ell^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dQ^2}{m_\ell^2}\, w(Q^2/m_\ell^2)\hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$

Integrand peaked for $Q \sim (m_\ell/2)$



(HVP from Jegerlehner, "alphaQEDc17" (2017))

Low-Q^2 challenges in finite volume (FV)

A. Must subtract $\Pi_{\mu\nu}(Q=0)\neq 0$ in FV that contaminates $\Pi(Q^2)\sim\Pi_{\mu\nu}(Q)/Q^2$ for $Q^2\to 0$ w/ very large FV effects

B. On-shell renormalization requires $\Pi(0)$ which is problematic (see above)

C. Need $\hat{\Pi}(Q^2)$ interpolation due to $Q_{\text{min}}=2\pi/T\sim 135\,\text{MeV}>\frac{m_\mu}{2}\sim 50\,\text{MeV}$ for $T\sim 9\,\text{fm}$

$$\Downarrow$$

- Compute on $T\times L^3$ lattice

$$C_L(t)=\frac{a^3}{3}\sum_{i=1}^{3}\sum_{\vec{x}}\langle J_i(x)J_i(0)\rangle$$

- Decompose ($C_L^{I=1}=\frac{9}{10}C_L^{ud}$)

$$\begin{aligned}C_L(t) &= C_L^{ud}(t)+C_L^{s}(t)+C_L^{c}(t)+C_L^{\text{disc}}(t)\\ &= C_L^{I=1}(t)+C_L^{I=0}(t)\end{aligned}$$

- Define (Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...) (ad A, B, C)

$$\hat{\Pi}_L^f(Q^2)\equiv\Pi_L^f(Q^2)-\Pi_L^f(0)=\frac{1}{3}\sum_{i=1}^{3}\frac{\Pi_{ii,L}^f(0)-\Pi_{ii,L}^f(Q)}{Q^2}-\Pi_L^f(0)=2a\sum_{t=0}^{T/2}\text{Re}\left[\frac{e^{iQt}-1}{Q^2}+\frac{t^2}{2}\right]\text{Re}\,C_L^f(t)$$

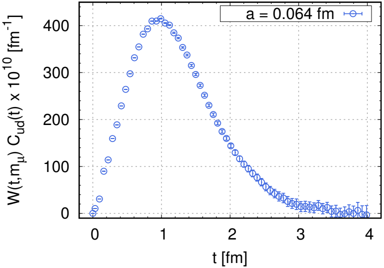
Our lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_L^f(t)$:

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \le Q^2_{\max}) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m_\ell^2}\right) \sum_{t=0}^{T/2} W(tm_\ell, Q^2_{\max}/m_\ell^2)\, \mathrm{Re}C_L^f(t)$$

where

$$W(\tau, x_{\max}) = \int_0^{x_{\max}} dx\, w(x) \left(\tau^2 - \frac{4}{x}\sin^2\frac{\tau\sqrt{x}}{2}\right)$$

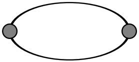


$(144 \times 96^3,\, a \sim 0.064\ \text{fm},\, M_\pi \sim 135\ \text{MeV})$

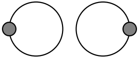
Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ have physically light π

E. Two types of contributions



quark-connected (qc)



quark-disconnected (qd)

where qd contributions are $\mathrm{SU}(3)_f$ and Zweig suppressed but very challenging

F. $\langle J_\mu^{ud}(x) J_\nu^{ud}(0)\rangle_{\mathrm{qc}}$ & disc. have very poor signal at large $\sqrt{x^2}$ + need high-precision results

$\rightarrow$ very high statistics + many algorithmic improvements + rigorous bounds
$\rightarrow$ 9M / 39M conn./disc. measurements

G. Must control $\langle J_\mu(x) J_\nu(0)\rangle$ at $\sqrt{x^2} \gtrsim 2/m_\mu$ $\rightarrow L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$

H. Need controlled continuum limit $\rightarrow$ have 6 a's: $0.134 \rightarrow 0.064\,\mathrm{fm}$

More challenges

I. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0, +\infty[$, but $\frac{\pi}{a} \sim 9.7\,\mathrm{GeV}$ for $a \sim 0.064\,\mathrm{fm}$
$\rightarrow$ match onto perturbation theory

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_\ell(Q_{\text{max}})\,\hat{\Pi}^f(Q^2_{\text{max}}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

J. Include c quark for higher precision and good matching onto perturbation theory $\rightarrow$ done

K. Even in our large volumes w/ $L \gtrsim 6.1\,\mathrm{fm}$ & $T \geq 8.7\,\mathrm{fm}$, finite-volume (FV) effects can be significant

$\rightarrow$ correct using 1-loop SU(2) χPT (Aubin et al '16)

L. Our $N_f = 2+1+1$ calculation has $m_u = m_d$ and $\alpha = 0$

$\Rightarrow$ missing effects compared to HVP from dispersion relations that are relevant at %-level precision

$\rightarrow$ use phenomenology (F.Jegerlehner (& M. Benayoun), *private communication*)

Systematic errors and results for $a_\mu^{\text{LO-HVP}}$

- Stat. error: jackknife
- $a \to 0$: from 4 (3) cuts on a for conn. (disc.)
- bounds: from $t_c = 3.000(2.600) \pm 0.134\,\mathrm{fm}$ vs $t_c = 2.866(2.466) \pm 0.134\,\mathrm{fm}$ for conn. (disc.)
- PT match: from $Q^2_{\text{max}} = 2\,\mathrm{GeV}^2$ vs $Q^2_{\text{max}} = 5\,\mathrm{GeV}^2$
- $\delta a \simeq 0.4\% \Rightarrow \delta_a a_\mu^{\text{LO-HVP}} \simeq 0.8\%$
- FV: $a^{\text{LO-HVP}}_{\mu,I=1}(\infty) - a^{\text{LO-HVP}}_{\mu,I=1}(L{=}6\,\mathrm{fm}) = (13.5 \pm 13.5) \times 10^{-10}$ from χPT
- IB: $\Delta_{\text{IB}} a_\mu^{\text{LO-HVP}} = (7.8 \pm 5.1) \times 10^{-10}$ from pheno.

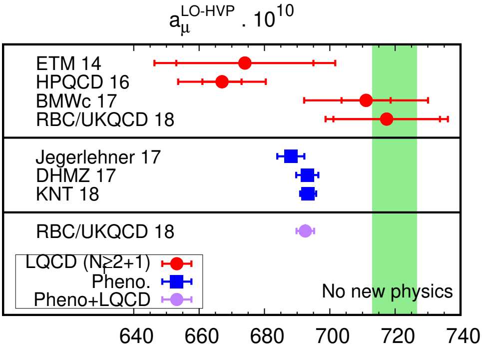
Contrib.	$a_\mu^{\text{LO-HVP}} \times 10^{10}$
$I = 1$	583(7)(7)(0)(0)(5)(14)
$I = 0$	121(3)(4)(0)(0)(1)
Total	711(8)(8)(0)(0)(6)(13)(5)

Error on total:

- Stat. = 1.1%
- LQCD syst. = 1.2%
- FV = 2.3%
- IB = 0.8%
- Total = 2.7%

Compare w/ upper bound (Bell et al '69) using Π_1 from BMWc, PRD96 $= 792(24)$

Comparison



- "No New Physics" scenario: $= (720 \pm 7) \times 10^{-10}$
- BMWc '17 consistent w/ "No new physics" scenario & pheno.
- Total uncertainty of 2.7% is $\sim 6\times$ pheno. error
- BMWc '17 is larger than other $N_f = 2 + 1 + 1$ results
  $\rightarrow$ difference w/ HPQCD '16 is $\sim 1.9\sigma$

What next?

- Need to reduce our error by 10!

$\rightarrow$ Increase statistics by $\times 50 \div 100$ (need new methods)

$\rightarrow$ Understand and control FV effects much better

$\rightarrow$ Compute QED and $m_d \neq m_u$ corrections (see RBC/UKQCD '17-'18, ETM '17)

$\rightarrow$ Need high precision scale setting

$\rightarrow$ Detailed comparison to phenomenology to understand where we agree and why if we don't

$\rightarrow$ Combine LQCD and phenomenology to improve overall uncertainty (RBC/UKQCD '18), only if the two agree statistically with comparable errors