NEW POSITIVITY BOUNDS FROM THE EFT HEDRON

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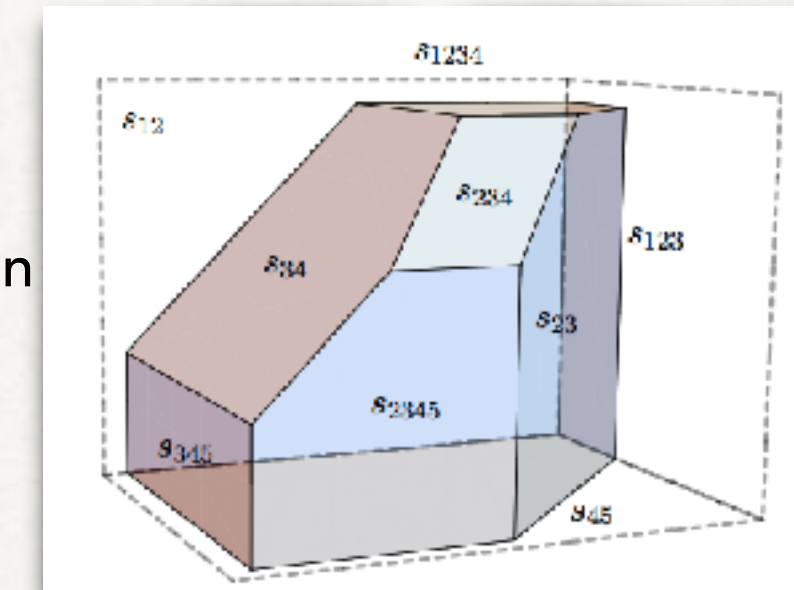
In recent years we in the study of perturbative scattering amplitudes we have seen the emergence of positive geometry, where the geometry is defined by a set of positivity conditions on the space in which the amplitude lives.

N=4 SYM

The Amplituehedron
Arkani-Hamed, Trnka

phi^3
The Associahedron
Arkani-Hamed, He,



The presence of this geometry can be traced back to the principle of locality and unitarity for which the scattering amplitude respects, or, it can be viewed as the unification of locality and unitarity!

→ If this is a reflection of a new fundamental principle of QFTs, we should be seeing its presence in a general context (where ?)
Can we impose the constraint that Lorentz invariance is not emergent (how?)

In the context of EFTs, it is long known that coefficients of leading higher dimension operators are constrained to be positive

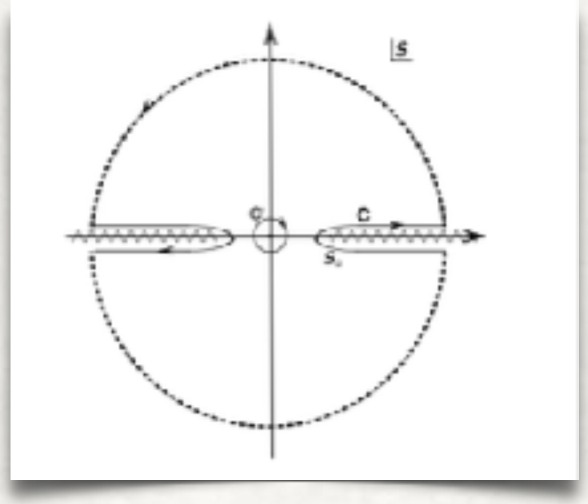
Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi + a(\partial^\mu\phi\partial_\mu\phi)^2$$

a>0

The coefficients can be captured through utilizing the analytic properties of the scattering amplitude (in the forward limit t=0). Through the Froissart bound

$$a = -\int \frac{ds}{s^3} M_4(s,0) = \int_{s_0}^{\infty} \frac{ds}{s^3} Dis[M(s,0)] = \int_{s_0}^{\infty} \frac{ds}{s^3} s\sigma$$



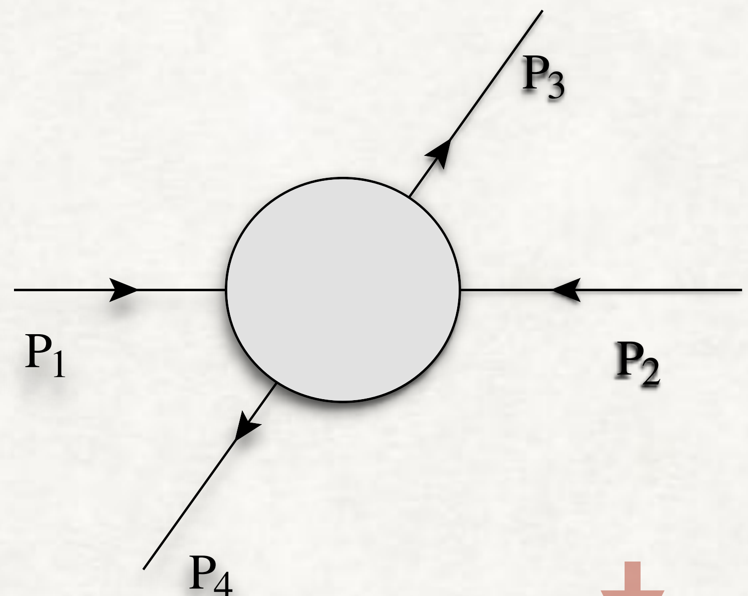
Utilizes the optical theorem. (If one assumes perturbative UV completion, there is an infinite positivity bound for s^n)

Can we probe the details (fine grained) of unitarity and Lorenz invariance in the UV ?

Can we differentiate the different operators that contribute for fixed derivative order ?

Let's consider the couplings of EFT operators in the context of scattering amplitudes

$$\int dx^4 \frac{1}{2}\phi\Box\phi + a_0\phi^4 + a_1(\partial\phi)^2\phi^2 + a_2(\partial\phi)^4 + \cdots$$



$s = (P_1 + P_2)^2$
$t = (P_1 + P_4)^2$

$$M^{IR}(s,t) = \{massless\ poles\} + \sum_{k,q} g_{k,q} s^{k-q} t^q,$$

We have an polynomial at the origin, with the Taylor coefficients identified with the coefficients of the EFT -> Are these coefficients well defined ?

We have an polynomial at the origin, with the Taylor coefficients identified with the coefficients of the EFT -> Are these coefficients well defined ?

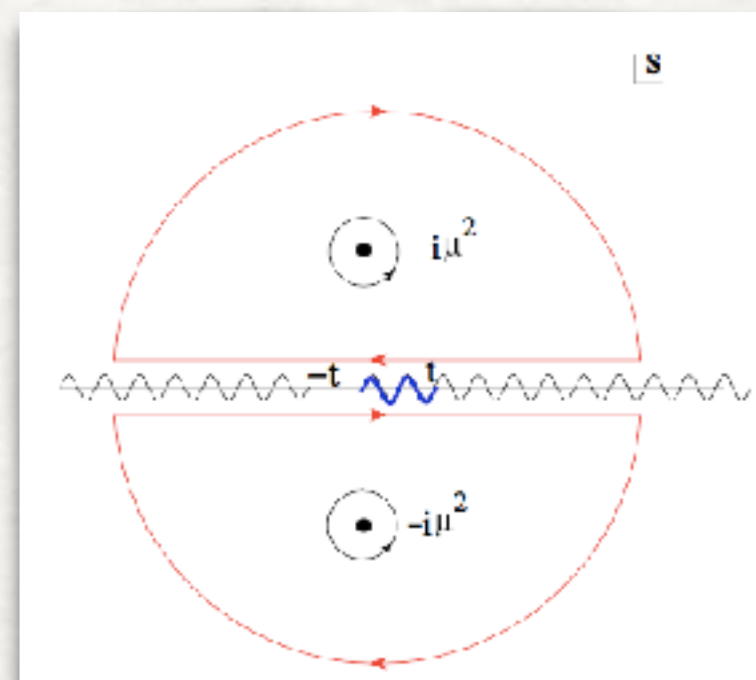
1. For theories whose UV completion is perturbative, (linear sigma model , string theory), the massless branch cut can be suppressed and we cleanly define:

$$g_{k,q} = \frac{1}{q!}\left[\frac{\partial^q}{\partial t^q}\frac{i}{2\pi}\int_{C_0}\frac{ds}{s^{k-q+1}}M(s,t)\right]\Big|_{t=0}$$

2. We can also have non-trivial massless branch cuts, then the we define the coefficients off from the real axes

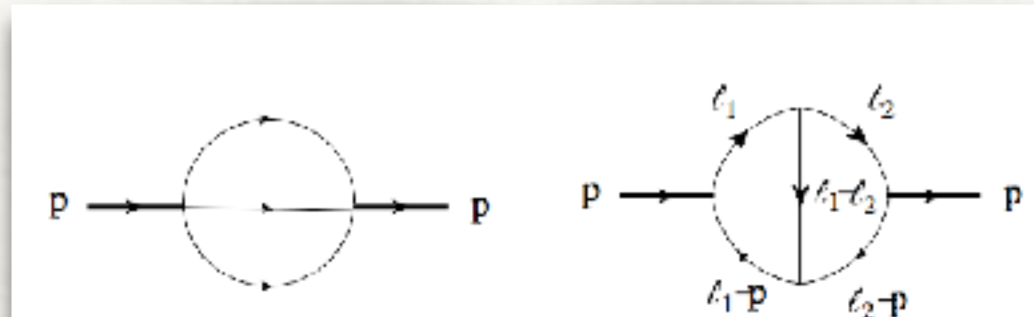
$$\mathsf{a}_{n,q} \equiv \left(\frac{1}{2\pi i}\right)^2 \oint \frac{dt}{(t-\epsilon)^{q+1}} \oint \frac{ds\, s^{\frac{1+(-1)^n}{2}}}{(s^2+\mu^4)^{\lfloor\frac{n-q}{2}\rfloor+1}} M(s,t)$$

$$\begin{pmatrix}\mathsf{a}_5\\ \mathsf{a}_6\\ \mathsf{a}_7\end{pmatrix} = \begin{pmatrix} \frac{\beta_1\bar{a}_2^2}{30\mu^{12}M^8} + \frac{\beta_2\bar{a}_2\bar{a}_4}{20\mu^8 M^{12}} + \frac{\beta_3\bar{a}_4^2}{5\mu^4 M^{16}} \\ \frac{\beta_1\bar{a}_2^2}{60\mu^{16}M^8} + \frac{\beta_2\bar{a}_2\bar{a}_4}{60\mu^{12} M^{12}} + \frac{\beta_3\bar{a}_4^2}{30\mu^8 M^{16}} \\ \frac{\beta_1\bar{a}_2^2}{105\mu^{20}M^8} + \frac{\beta_2\bar{a}_2\bar{a}_4}{140\mu^{16} M^{12}} + \frac{\beta_3\bar{a}_4^2}{105\mu^{12} M^{16}} \end{pmatrix}$$

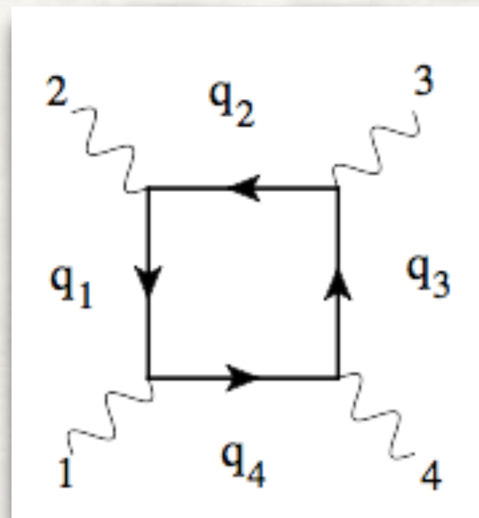


Where do we have control over the analytic property of the S-matrix ?

It is well known that non-analyticity of S-matrix does not always reflect particle production $\longrightarrow$ anomalous threshold singularities



Even at one-loops !



$$I_4[s,t] = \frac{3uv}{4\beta_{uv}}\left\{2\log^2\left(\frac{\beta_{uv}+\beta_u}{\beta_{uv}+\beta_v}\right) + \log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right)\log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right) - \frac{\pi^2}{2} + \sum_{i=u,v}\left[2\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{uv}+\beta_i}\right) - 2\mathrm{Li}_2\left(-\frac{\beta_{uv}-\beta_i}{\beta_i+1}\right) - \log^2\left(\frac{\beta_i+1}{\beta_{uv}+\beta_i}\right)\right]\right\}.$$

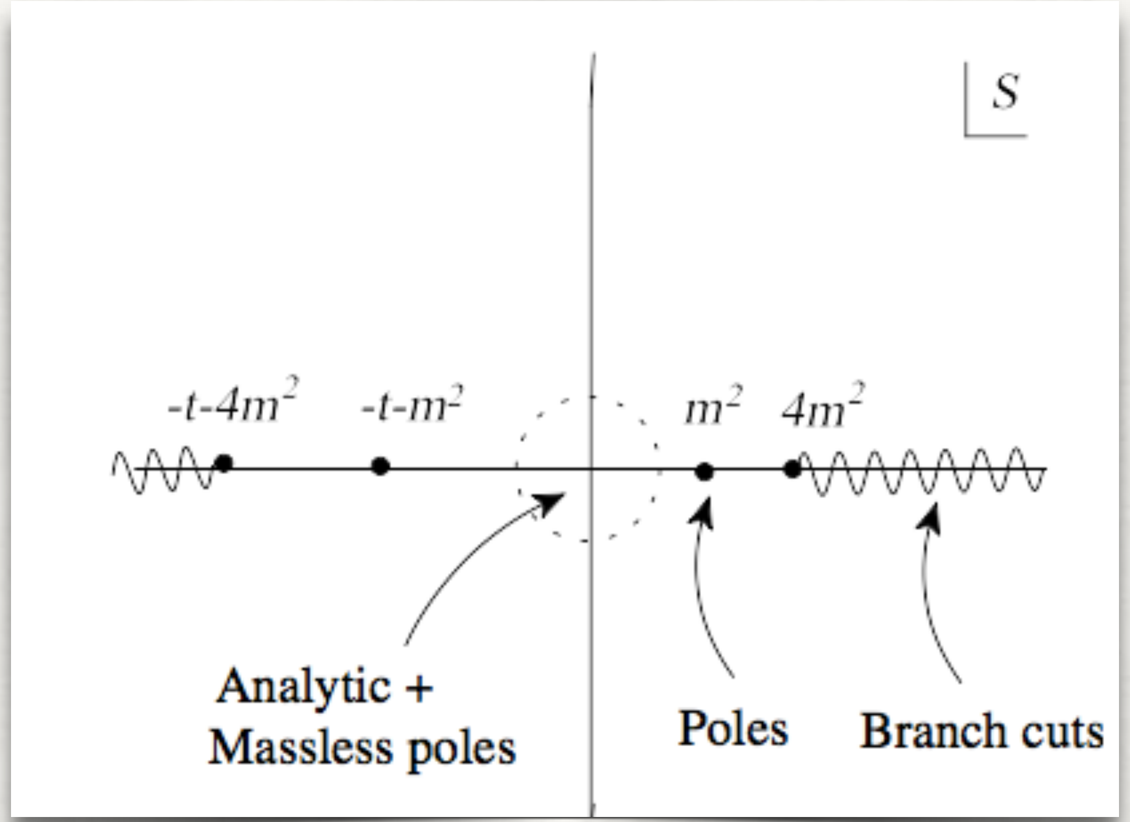
where $u = -\frac{4m^2}{s}$ and $v = -\frac{4m^2}{t}$, and

$$\beta_u = \sqrt{1+u}, \quad \beta_v = \sqrt{1+v}, \quad \beta_{uv} = \sqrt{1+u+v}\,.$$

Anomalous thresholds can be avoided if t<<m^2

With t<<m^2

The only non-analyticity on the complex s-plane with t held fixed lies on the real axes



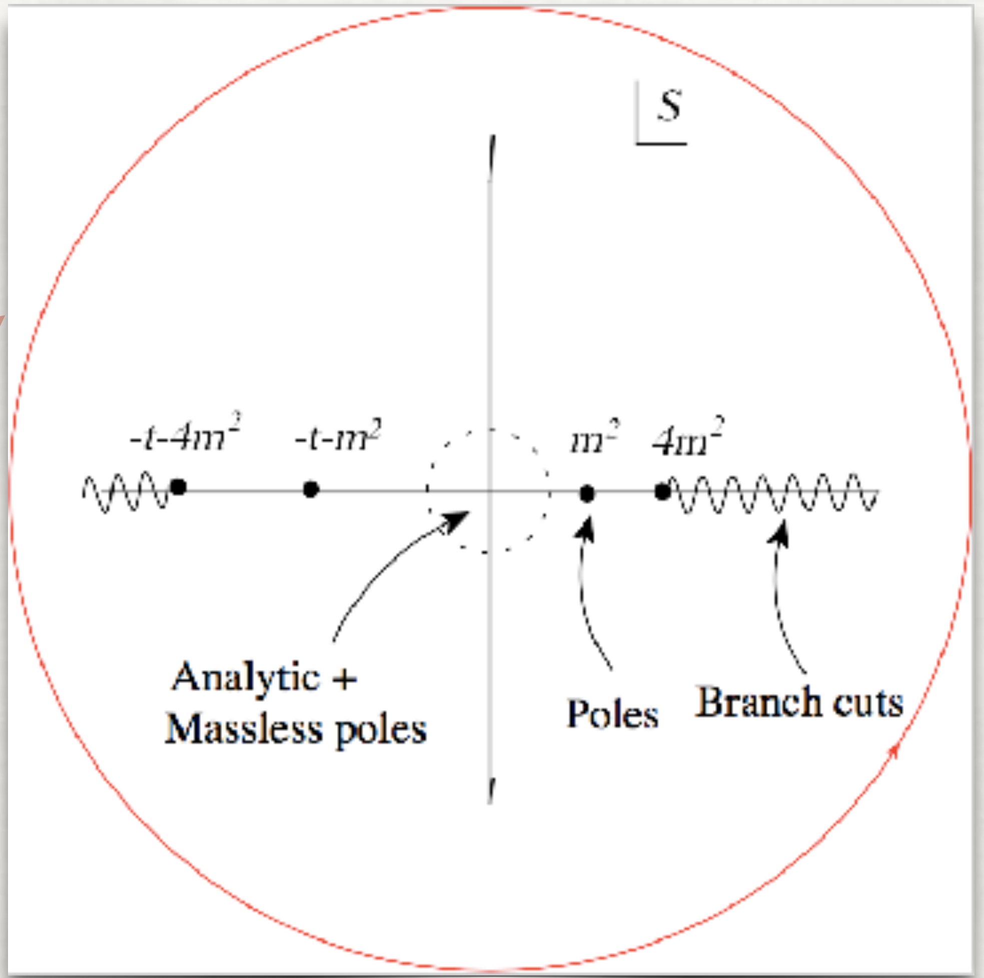
Importantly, the residues and discontinuity is positively expandable on the Gegenbauer polynomials (Legendre polynomials in 4D)

$$Res_{s=m^2} M(s,t) = \sum_a \mathsf{p}_a\, G^{\alpha}_{\ell_a}(\cos\theta), \quad \mathsf{p}_a > 0$$

$$Dis_{s\geq 4m^2} M(s,t) = \sum_\ell \mathsf{p}_\ell(s)\, G^{\alpha}_{\ell}(\cos\theta)\,, \quad \mathsf{p}_\ell(s) > 0$$

A renormalizable theory in the UV tell us that
M(s,t)<=s^2 at large s

$$I = \frac{i}{2\pi}\int_{\infty}\frac{ds}{s^{n+1}}M(s,t)$$ =0 for n>2



The vanishing of the contour tell us that the Taylor coefficients of the EFT is completely controlled by the residue and discontinuity Arkani-hamed, Huang, Huang

$$M^{IR}(s,t) = \{massless\ poles\} + \sum_{k,q} g_{k,q} s^{k-q} t^q,$$

$$\sum_q g_{k,q} t^q = \left(\sum_a \frac{\mathrm{p}_a G^{\alpha}_{\ell_a}(1+2\frac{t}{m_a^2})}{(m_a^2)^{k-q+1}} + \sum_b \int ds' \mathrm{p}_{b,\ell}(s') \frac{G^{\alpha}_{\ell}(1+2\frac{t}{s'})}{(s')^{k-q+1}} + \{u\}\right)$$

The fact that Gravity is UV completed, tell us that M(s,t)<=s^2 at large s

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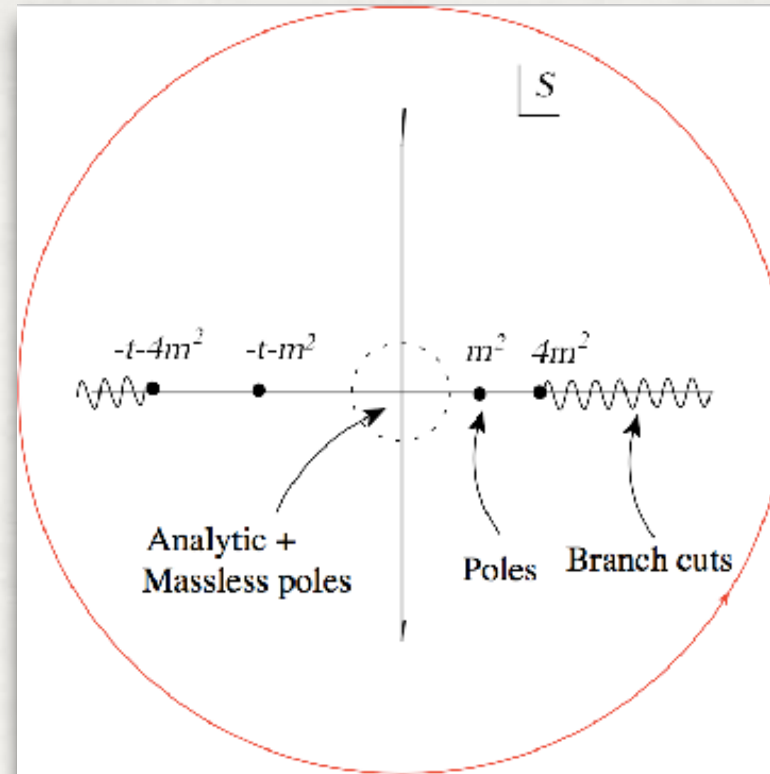

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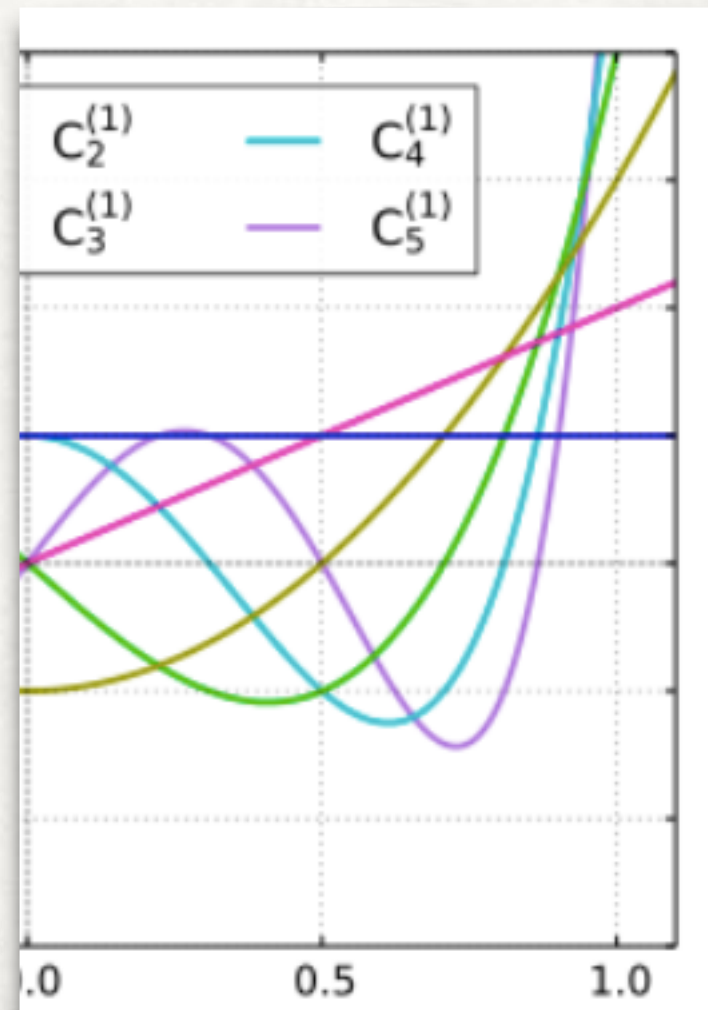
Let's focus on the forward limit, t=0

$$g_{k,0} = \left(\sum_a \frac{\mathsf{p}_a G^{\alpha}_{\ell_a}(1)}{(m_a^2)^{k+1}} + \sum_b \int ds' \mathsf{p}_{b,\ell}(s') \frac{G^{\alpha}_{\ell}(1)}{(s')^{k+1}} \right)$$

The polynomials are positive in the forward limit



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Arkani-hamed, Huang, Huang

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What about higher order in t?

$$G^{\alpha}_{\ell}(1+\delta) = \sum_q v^{\alpha}_{\ell,q}\, \delta^q\,.$$

$$\boxed{g_{k,q} = \sum_a \mathsf{p}_a \frac{2^q u^{\alpha}_{\ell_a,k,q}}{(m_a^2)^{k+1}} \qquad \mathsf{p}_a > 0}$$

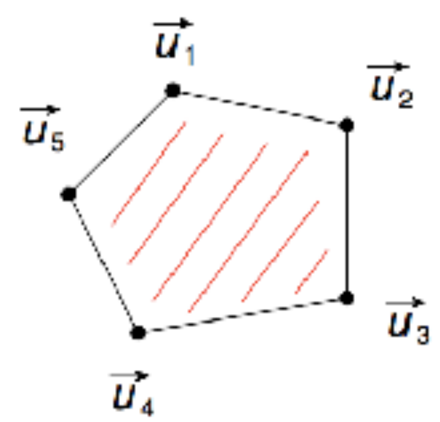
$$\begin{pmatrix} g_{k,0} \\ g_{k,1} \\ g_{k,2} \\ \vdots \end{pmatrix} = \sum_a \mathsf{p}_a \begin{pmatrix} u^{\alpha}_{\ell_a,k,0} \\ u^{\alpha}_{\ell_a,k,1} \\ u^{\alpha}_{\ell_a,k,2} \\ \vdots \end{pmatrix}$$

The couplings must sit inside the convex hull of Gegenbauer vectors !

CONVEX HULL

This is the primitive form of positive geometry

- The position of center of mass



$$\vec{A} = \frac{\sum_{i=1}^{5} m_i \vec{u}_i}{\sum_{i=1}^{5} m_i}$$

The center of mass is always “inside” the polygon because m>0

The inside of the polygon is inside the CONVEX HULL

$$A' = w_1 U_1' + \cdots + w_n U_n', \quad w_i > 0, \sum_{i=1}^{n} w_i = 1$$

$$\sum_q g_{k,q} t^q = \left(\sum_a \frac{\mathsf{p}_a G^\alpha_{\ell_a}(1 + 2\frac{t}{m_a^2})}{(m_a^2)^{k-q+1}} + \sum_b \int ds' \mathsf{p}_{b,\ell}(s') \frac{G^\alpha_\ell(1+2\frac{t}{s'})}{(s')^{k-q+1}} + \{u\} \right)$$

We can consider fixed k- (mass dimension)

$$\begin{array}{cccccc}
 & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\
t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\
t^1 & & g_{0,1} & g_{2,1} & g_{3,1} & \cdots \\
t^2 & & & g_{2,2} & g_{3,2} & \cdots \\
t^3 & & & & g_{3,3} & \cdots
\end{array}$$

$$\vec{g}_2 = \begin{pmatrix} g_{2,0} \\ g_{2,1} \\ g_{2,2} \end{pmatrix} \in \sum_a \mathsf{p}'_a v_{\ell_a} \quad \mathsf{p}'_a > 0$$

We can consider fixed q- (angular dependence)

$$\begin{array}{cccccc}
 & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\
t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\
t^1 & & g_{1,1} & g_{2,1} & g_{3,1} & \cdots \\
t^2 & & & g_{2,2} & g_{3,2} & \cdots \\
t^3 & & & & g_{3,3} & \cdots
\end{array}$$

$$\begin{pmatrix} g_{0,1} \\ g_{1,1} \\ g_{2,1} \end{pmatrix} \in \sum_a \mathsf{p}'_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \end{pmatrix} \qquad \mathsf{p}'_a > 0$$

$$
\begin{aligned}
M(s,t) &= -\sum_a p_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2} \\
&= \sum_a p_a \frac{1}{m_a^2}\left(1+\frac{s}{m_a^2}+\frac{s^2}{m_a^4}+\cdots\right)_{locality}\left(v_{\ell_a,0}+v_{\ell_a,1}\frac{t}{m_a^2}+v_{\ell_a,2}\frac{t^2}{m_a^4}\cdots\right)_{unitarity}
\end{aligned}
$$

These two convex hull reflects the constraint of locality+unitarity and unitarity+Lorentz invariance

Can we determine the boundary of this hull? Naively, no, there are infinite number of vectors (the complexity for n-vectors in d-dimensions is n^d/2). However, these vectors are special! There ordered determinants are all positive !

$$
(v_0^I, v_1^I, \cdots, v_8^I) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 \\
0 & 0 & \frac{3}{2} & \frac{15}{2} & \frac{45}{2} & \frac{105}{2} & 105 & 189 \\
0 & 0 & 0 & \frac{5}{2} & \frac{35}{2} & 70 & 210 & 525 \\
0 & 0 & 0 & 0 & \frac{35}{8} & \frac{315}{8} & \frac{1575}{8} & \frac{5775}{8} \\
0 & 0 & 0 & 0 & 0 & \frac{63}{8} & \frac{693}{8} & \frac{2079}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{231}{16} & \frac{3003}{16} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{429}{16}
\end{pmatrix}.
$$

$$
det[v_{\ell_1} v_{\ell_2} \cdots] > 0, \quad \forall\, \ell_1 > \ell_2 > \cdots
$$

$$
\begin{aligned}
M(s,t) &= -\sum_a p_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2} \\
&= \sum_a p_a \frac{1}{m_a^2}\left(1+\frac{s}{m_a^2}+\frac{s^2}{m_a^4}+\cdots\right)_{locality}\left(v_{\ell_a,0}+v_{\ell_a,1}\frac{t}{m_a^2}+v_{\ell_a,2}\frac{t^2}{m_a^4}\cdots\right)_{unitarity}
\end{aligned}
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$$
det\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ (x_1)^2 & (x_2)^2 & \cdots & (x_n)^2 \\ (x_1)^3 & (x_2)^3 & \cdots & (x_n)^3 \\ \vdots & \vdots & \vdots & \vdots \\ (x_1)^n & (x_2)^n & \cdots & (x_n)^n \end{pmatrix} = (x_1 x_2 \cdots x_n)\prod_{i<j}(x_j - x_i)
$$

The convex hull of vectors whose ordered determinants are all positive, correspond to cyclic polytopes

Let's say we have a set of vectors, with a well defined ordering. If its ordered determinant is positive:

$$det[U_{i_1}, U_{i_2}, \cdots, U_{i_k}] > 0, \quad \forall\, i_1 < i_2 < i_3 \cdots < i_k$$

The convex hull of U_i is a cyclic polytope

- It's boundaries are known

$$d = 2: \ (i, i+1), \quad d = 3: \ (0, i, i+1), (i, i+1, \infty), \quad d = 4: (i, i+1, j, j+1) \cdots$$

Exp: Let $A = aU_6 + bU_9$ with $a, b > 0$

$$\begin{aligned} Det[A, U_4, U_5, U_7, U_8] &= a\, Det[U_6, U_4, U_5, U_7, U_8] + b\, Det[U_9, U_4, U_5, U_7, U_8] \\ &= a\, Det[U_4, U_5, U_6, U_7, U_8] + b\, Det[U_4, U_5, U_7, U_8, U_9] \\ &= a\, (positive) + b\, (positive) \end{aligned}$$

It's boundaries are known

$$d=2:\ (i,i+1),\quad d=3:\ (0,i,i+1),(i,i+1,\infty),\quad d=4:(i,i+1,j,j+1)\cdots$$

Consider fixed mass dimension we have an infinite number of positivity bounds. Exp. k=2

$$\begin{array}{cccccc}
 & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\
t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\
t^1 & & g_{1,1} & g_{2,1} & g_{3,1} & \cdots \\
t^2 & & & g_{2,2} & g_{3,2} & \cdots \\
t^3 & & & & g_{3,3} & \cdots
\end{array}$$

$$\vec{g}_2=\begin{pmatrix} g_{2,0}\\ g_{2,1}\\ g_{2,2}\end{pmatrix}\rightarrow Det[\vec{g}_2, v_\ell, v_{\ell+1}]>0$$

Defining things projectively, we organize the couplings as $\mathbf{X}=(1,x,1)$ where $x=\frac{g_{2,1}}{g_{2,0}}$

The couplings are subject to

$$\langle\mathbf{X},1,2\rangle>0\rightarrow\frac{8}{3}>x,\quad\langle\mathbf{X},2,3\rangle>0\rightarrow\frac{19}{4}>x,\quad\langle\mathbf{X},3,4\rangle>0\rightarrow\frac{122}{15}>x$$
$$\cdots,\ \langle\mathbf{X},\infty,0\rangle>0\rightarrow x>0$$

In summary

$$\boxed{g_{2,0}>0,\quad g_{2,1}>0,\quad \frac{8}{3}>\frac{g_{2,1}}{g_{2,0}}>0}$$

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$$d=2:\ (i,i+1),\quad d=3:\ (0,i,i+1),(i,i+1,\infty),\quad d=4:(i,i+1,j,j+1)\cdots$$

Consider fixed mass dimension we have an infinite number of positivity bounds.

Exp. k=3

$$\mathbf{X}=(1,x,x,1),\ \text{where now } x=\frac{g_{3,1}}{g_{3,0}}=\frac{g_{3,2}}{g_{3,0}}$$

subject to

$$\langle 0,\mathbf{X},3,4\rangle>0\to x>-\frac{3}{7}\ ,\langle 0,\mathbf{X},4,5\rangle>0\to x>-\frac{3}{28}\ ,\langle 0,\mathbf{X},5,6\rangle>0\to x>-\frac{1}{24}$$
$$\cdots,\ \ \langle 0,\mathbf{X},\infty-1,\infty\rangle>0\to x>0$$
$$\langle \mathbf{X},0,1,\infty\rangle>0\to x>0\ ,\langle \mathbf{X},1,2,\infty\rangle>0\to 6>x\ ,\langle \mathbf{X},3,4,\infty\rangle>0\to \frac{120}{13}>x\,,$$

$$\longrightarrow\quad \boxed{g_{3,0}>0,\ \ g_{3,1}>0,\ \ 6>\frac{g_{3,1}}{g_{3,0}}>0}$$

It's boundaries are known

$$d=2:\ (i,i+1),\quad d=3:\ (0,i,i+1),(i,i+1,\infty),\quad d=4:(i,i+1,j,j+1)\cdots$$

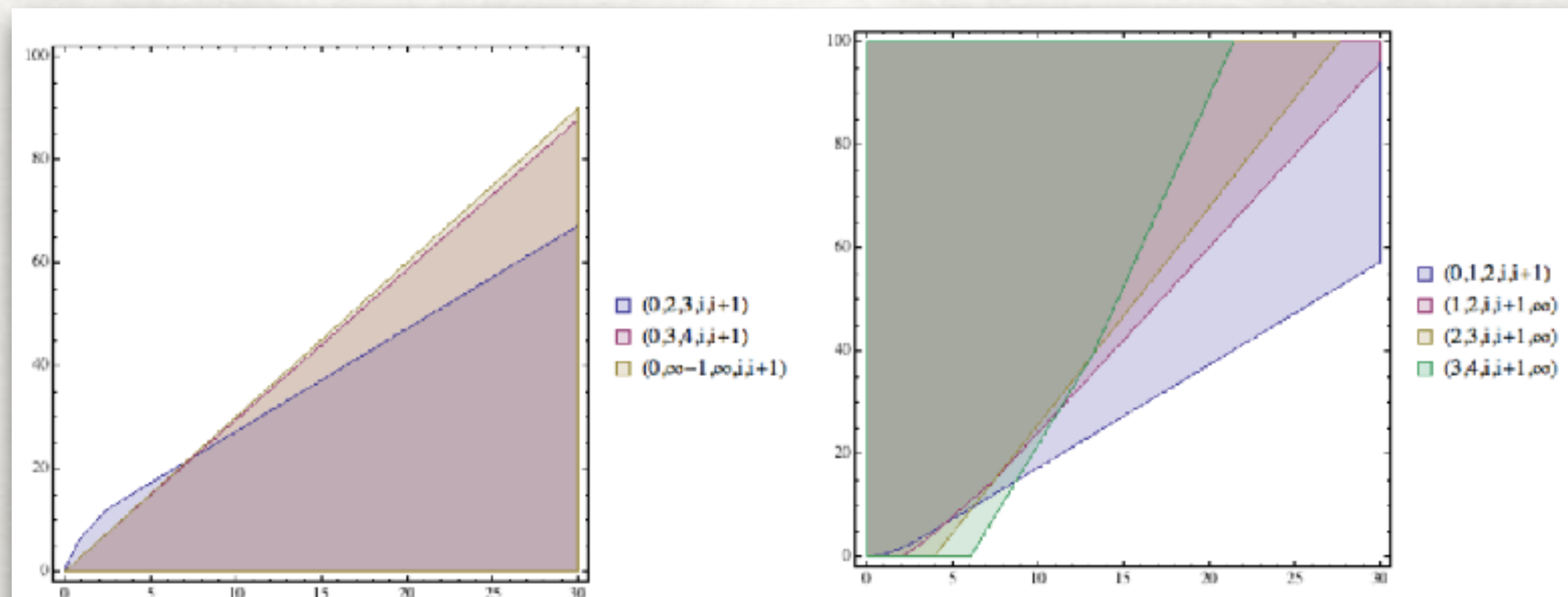
Consider fixed mass dimension we have an infinite number of positivity bounds.

Exp. k=5

the couplings are organized as $\mathbf{X}=(1,x,y,y,x,1)$.

$$x=\frac{g_{5,1}}{g_{5,0}}=\frac{g_{5,4}}{g_{5,0}}\text{ and }y=\frac{g_{5,2}}{g_{5,0}}=\frac{g_{5,3}}{g_{5,0}}$$

the boundaries are $(0,i,i+1,j,j+1)$ and $(\infty,i,i+1,j,j+1)$.



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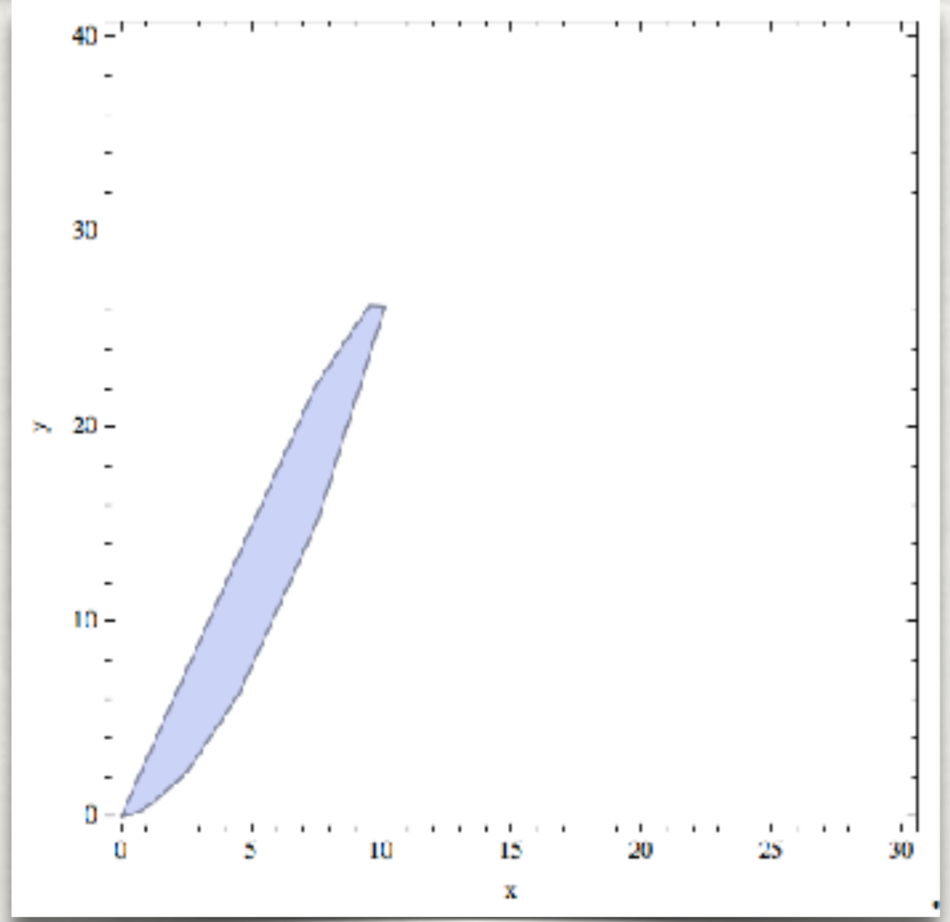


For external spinning states, we can analyze constraints on
R^2, R^2F, F^4, ..

We simply replace the gegenbauer polynomials with

$$f_l^{\{h_s\}}(\cos\theta) = d^l_{h_1-h_2,h_3-h_4}(\theta), \quad f_l^{\{h_u\}}(\cos\theta) = d^l_{h_1-h_3,h_2-h_4}(\theta)$$

Arkani-hamed, Huang, Huang

$$h{=}1: \begin{pmatrix} \frac{1}{4} & \frac{5}{4} & \frac{15}{4} & \frac{35}{4} & \frac{35}{2} & \frac{63}{2} & \frac{105}{2} & \frac{165}{2} \\ 0 & \frac{3}{4} & \frac{21}{4} & 21 & 63 & \frac{315}{2} & \frac{693}{2} & 693 \\ 0 & 0 & \frac{7}{4} & \frac{63}{4} & \frac{315}{4} & \frac{1155}{4} & \frac{3465}{4} & \frac{9009}{4} \\ 0 & 0 & 0 & 15 & \frac{165}{4} & \frac{495}{2} & \frac{2145}{2} & \frac{15015}{4} \\ 0 & 0 & 0 & 0 & \frac{495}{64} & \frac{6435}{64} & \frac{45045}{64} & \frac{225225}{64} \\ 0 & 0 & 0 & 0 & 0 & \frac{1001}{64} & \frac{15015}{64} & \frac{15015}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1001}{32} & \frac{17017}{32} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1989}{32} \end{pmatrix} \qquad h{=}2: \begin{pmatrix} \frac{1}{16} & \frac{9}{16} & \frac{45}{16} & \frac{165}{16} & \frac{495}{16} & \frac{1287}{16} & \frac{3003}{16} & \frac{6435}{16} \\ 0 & \frac{5}{16} & \frac{55}{16} & \frac{165}{8} & \frac{715}{8} & \frac{5005}{16} & \frac{15015}{16} & \frac{5005}{2} \\ 0 & 0 & \frac{33}{32} & \frac{429}{32} & \frac{3003}{32} & \frac{15015}{32} & \frac{15015}{8} & \frac{51051}{8} \\ 0 & 0 & 0 & \frac{91}{32} & \frac{1365}{32} & \frac{1365}{4} & \frac{7735}{4} & \frac{69615}{8} \\ 0 & 0 & 0 & 0 & \frac{455}{64} & \frac{7735}{64} & \frac{69615}{64} & \frac{440895}{64} \\ 0 & 0 & 0 & 0 & 0 & \frac{1071}{64} & \frac{20349}{64} & \frac{101745}{32} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4845}{128} & \frac{101745}{128} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10659}{128} \end{pmatrix}$$

All ordered determinants are positive!

$$\det[v_{\ell_1} v_{\ell_2} \cdots] > 0, \quad \forall\, \ell_1 > \ell_2 > \cdots$$

Consider the configuration (-2, +2, +2, -2) where we have

$$\langle 14\rangle^4[23]^4\left(\sum_{i,j} g_{i,j} z^i t^j\right) \tag{8}$$

The exchanged spin begins with spin-4

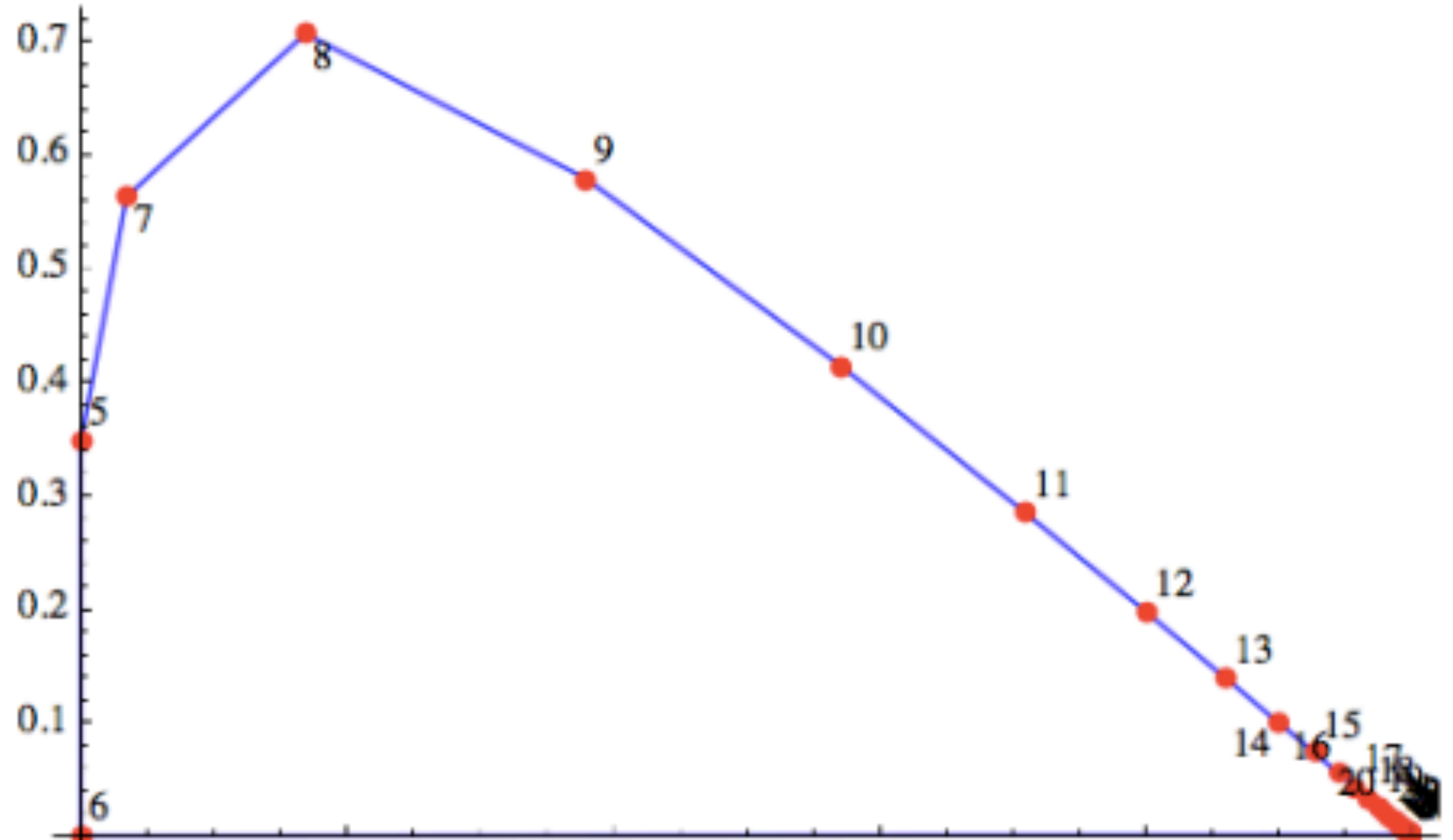
- (z^2, t^2): The space is one-dimensional, and the bound is simply

$$-\frac{11}{36} < \frac{g_{2,0}}{g_{0,2}}$$

- (z^4, z^2t^2, t^4): The critical spin is $s_c = 6$, spin-4 is inside the hull, i.e. not a vertex. The boundaries are:

$$\langle X, i, i+1\rangle > 0 \text{ for}, \quad i \geq 7, \langle X, 6, 5\rangle > 0, \quad \langle X, 5, 7\rangle > 0 \tag{9}$$

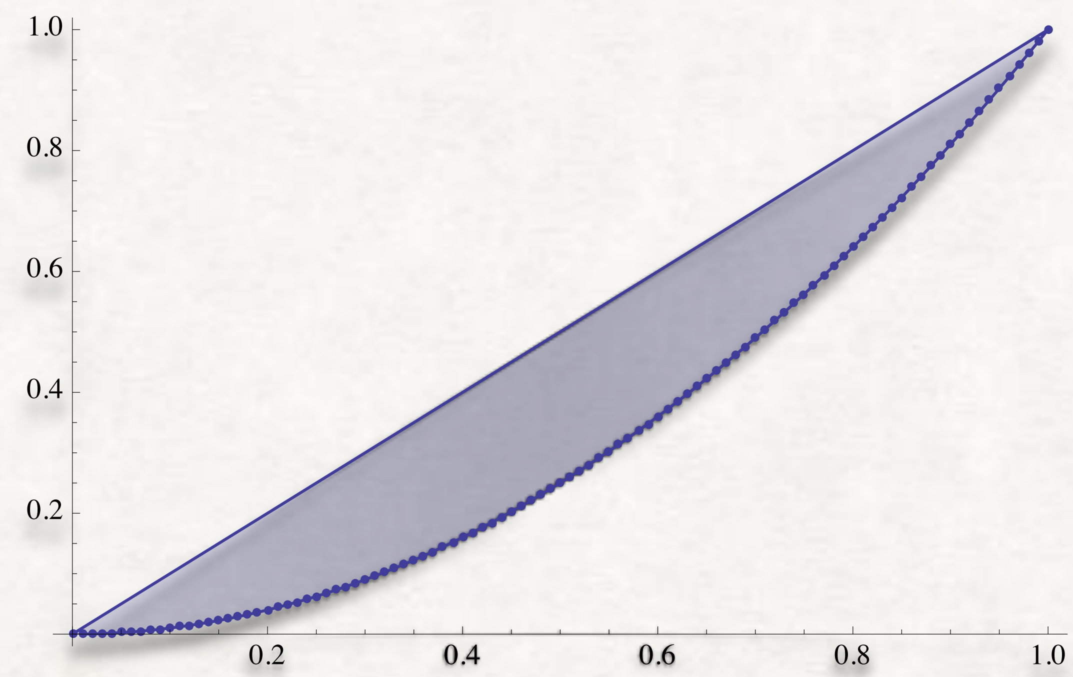
Out[175]=



Instead of fixed mass-dimensions, lets consider fixed degree in angle.

Take the forward limit, collect the coefficients of successive powers in s, we find that the couplings live in the convex hull of points on a moment curve

$$\begin{pmatrix} g_2 \\ g_3 \\ g_4 \\ \vdots \end{pmatrix} = \sum_a \mathsf{p}_a \vec{m}_a, \quad \vec{m}_a = \begin{pmatrix} 1 \\ \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \vdots \end{pmatrix}.$$



$$i \in even: \quad \text{Det}\begin{bmatrix} g_0 & g_1 & \cdots & g_{\frac{i}{2}} \\ g_1 & g_2 & \cdots & g_{\frac{i}{2}+1} \\ \vdots & \vdots & \vdots & \vdots \\ g_{\frac{i}{2}} & g_{\frac{i}{2}+1} & \cdots & g_i \end{bmatrix} \geq 0, \qquad i \in odd: \quad \text{Det}\begin{bmatrix} g_1 & g_2 & \cdots & g_{\frac{i+1}{2}} \\ g_2 & g_3 & \cdots & g_{\frac{i+3}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ g_{\frac{i+1}{2}} & g_{\frac{i+3}{2}} & \cdots & g_i \end{bmatrix} \geq 0$$

The coupling constants are in the convex hull, iff the Hankel matrix is a positive matrix

Now let's consider the EFT for generic QFTs. At low energies we only have massless photons, with an infinite set of higher dimensional operators (in F)

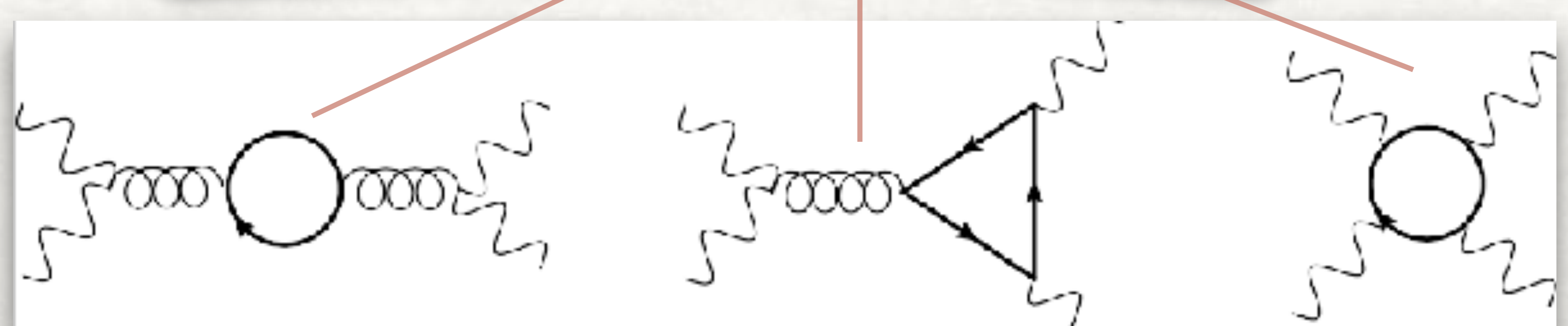
$$\mathcal{L} = -\frac{1}{4}F^2 + \sum_{i=0} c_i D^{2i} F^4 \qquad c_i\left(\frac{qg}{m}, M_{pl}\right)$$

The Wilson coefficients are functions of the charge, gauge coupling and mass. It is more natural to parameterize in dimensionless ratio

$$z = \frac{gqM_{pl}}{m}$$

with the coefficients parametrized as:

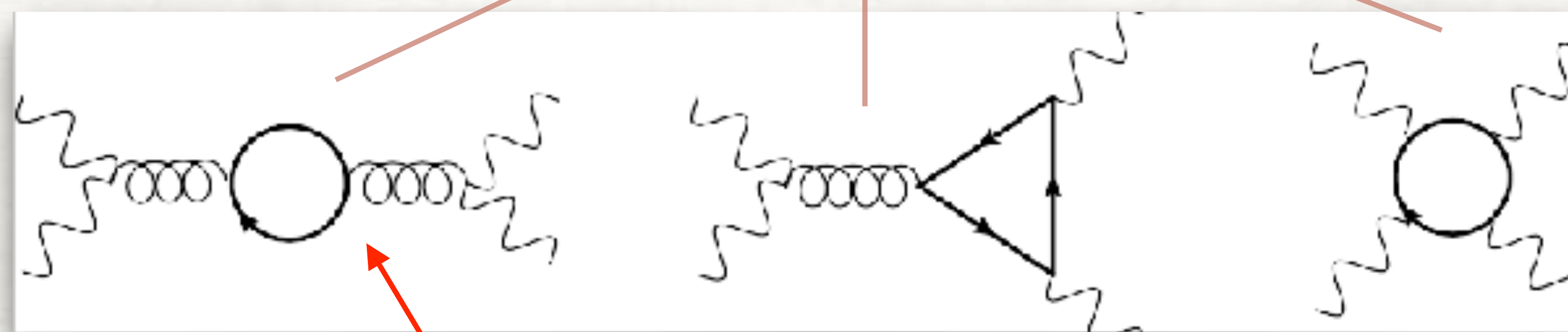
$$c_i\left(\frac{qg}{m}, M_{pl}\right) = \frac{1}{M_{pl}^4}\left(\alpha_i\, z^0 + \beta_i\, z^2 + \gamma_i\, z^4\right)\frac{1}{m^{2i}}$$



$\alpha\beta\gamma$ are calculable coefficients. Could unitarity constrain z ??

The coefficients parametrized as:

$$c_i\left(\frac{qg}{m}, M_{pl}\right) = \frac{1}{M_{pl}^4}\left(\alpha_i\, z^0 + \beta_i\, z^2 + \gamma_i\, z^4\right)\frac{1}{m^{2i}}$$



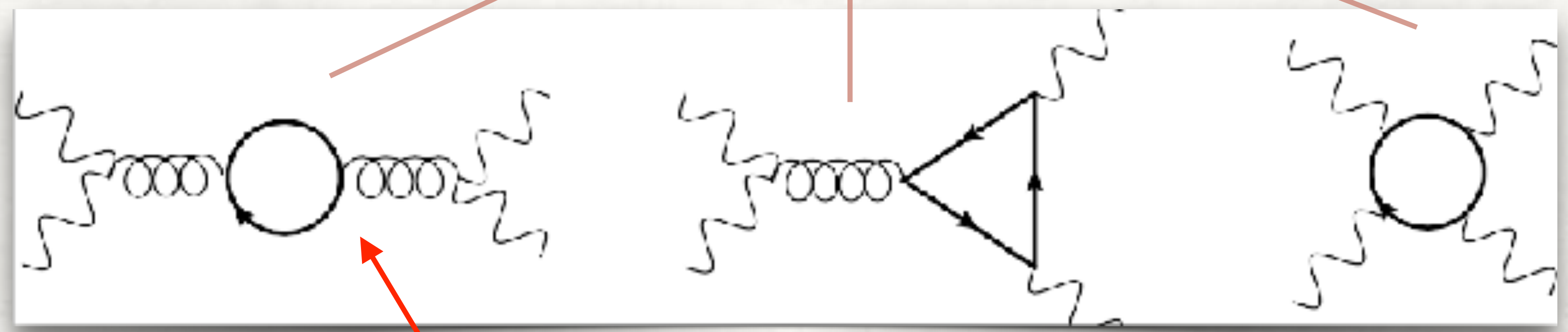
The leading coefficient is determined by the largest z. Could unitarity provide a bound for z?

- In D=4, there is a one-loop UV divergence with F^4 counter term, whose finite value is determined from the UV physics and thus un-known.
- Since these coefficients ARE derived from local loop corrections, whose boundary behaves as s^2 (due to gravitons). Any Hankel matrix constraint that does not involve g_2 must be trivially satisfied with no bearing on z (indeed it is)
- Hankel matrix constraint that does involve g_2 will likely constrain the value of the aforementioned normalization constant, not z.

See C. Cheung and G. Remmen

Instead, lets's consider 3D :

$$c_i\left(\frac{qg}{m}, M_{pl}\right) = \frac{1}{mM_{pl}^2}\left(\alpha_i\, z^0 + \beta_i\, z^2 + \gamma_i\, z^4\right)\frac{1}{m^{2i}}$$



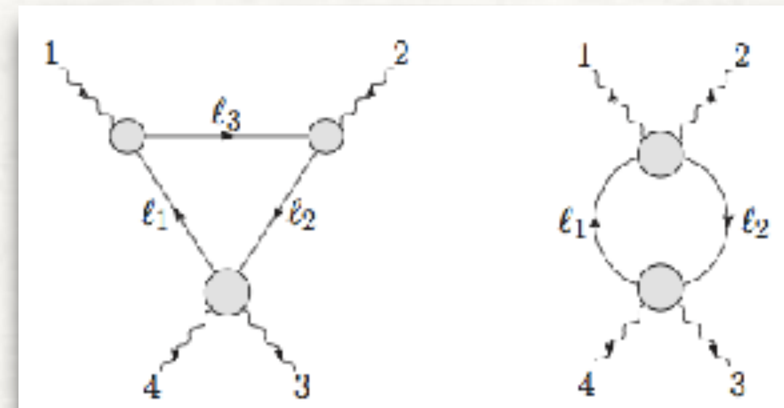
- In D=3, there no one-loop UV divergence. The leading contributions in M_pl IS calculable
- The leading coefficient is determined by the largest z and lightest mass
- The t-channel pole is not physical (removable) See Brando Bellazzini, Matthew Lewandowski, Javi Serra 1902.03250
- Hankel matrix constraint that does involve g_2 will constrain the value z.

Lets consider the 3D EFT instead

$$\mathcal{L} = -\frac{1}{4}F^2 + \sum_{i=0} c_i D^{2i} F^4 \qquad c_i\left(z = \frac{qg\sqrt{M_{pl}}}{m},\ m,\ M_{pl}\right)$$

We consider the one-loop EFT of massive charged scalars and fermions.

$$M_4(s,t) = 4\,\mathcal{C}(34;1,2)I_{\text{tri}}(s,m^2) + \mathcal{C}(12;34)I_{\text{bub}}(s,m^2) + (s\leftrightarrow t) + (s \leftrightarrow u), \qquad ($$



The integral coefficients are given as

$$\mathcal{C}(34;1,2) = -\frac{z^2m^4tu\,(4m^2+s)}{4s^2M_{pl}^2} - \frac{z^4m^6\,(4m^2-s)\,[2m^2\,(t^2+u^2)+stu]}{2\,(4m^2u+st)\,(4m^2t+su)\,M_{pl}^2},$$

$$\mathcal{C}(12;34) = \frac{8z^4s^2m^4 + z^2m^2[4m^2(3s^2-32tu)-s^3]}{32s^2M_{pl}^2} + \frac{(16m^4+s^2)(3s^2-8tu) - 8m^2(t^3+u^3-11stu)}{2048s^2M_{pl}^2}$$

$$\mathcal{C}(34;1,2) = \frac{z^2m^4\,(16m^2tu+3stu+t^3+u^3)}{16s^2M_{pl}^2} + \frac{z^4m^6\,[32m^4(t^2+u^2)+8m^2(t^3+u^3)-s^2(s^2+2tu)]}{8\,(4m^2u+st)\,(4m^2t+su)\,M_{pl}^2}, \qquad (35)$$

$$\mathcal{C}(12;34) = \frac{(4m^2-s)\,[4m^2(8tu-3s^2)+5stu+t^3+u^3]}{2048s^2M_{pl}^2} + \frac{8z^4s^2m^4+z^2m^2\,[4m^2(32tu-3s^2)+5stu+t^3+u^3]}{32s^2M_{pl}^2}.$$

Scalars

Fermions

Let's consider the limit s-> infinity,

$$\text{scalar}: \quad \frac{1-z^2}{7680\pi m M_{pl}^2}s^2 + \mathcal{O}(s^{\frac{3}{2}}),$$
$$\text{fermion}: \quad \frac{1-z^2}{2560\pi m M_{pl}^2}s^2 + \mathcal{O}(s^{\frac{3}{2}}).$$

Without any extra massless D.O.F, the cancellation of s^2 growth requires the presence of z >1, the WGC in 3D!

For multiple U(1) we have

$$M_4(s,t;u,v) = \sum_{i,j,k,l} u_i v_j v_k u_l M_4(1_i, 2_j, 3_k, 4_l),$$

$$\text{scalar}: \quad \frac{2-|\vec{z}\cdot u|^2-|\vec{z}\cdot v|^2}{7680\pi m M_{pl}^2}s^2 + \mathcal{O}(s^{\frac{3}{2}}),$$
$$\text{fermion}: \quad \frac{2-|\vec{z}\cdot u|^2-|\vec{z}\cdot v|^2}{2560\pi m M_{pl}^2}s^2 + \mathcal{O}(s^{\frac{3}{2}}).$$

There must exists states whose convex hull contains the unit circle

$$|\vec{z}\cdot u|^2 + |\vec{z}\cdot v|^2 > 2$$

Expanding in 1/m, in the forward limit we have the EFT description

$$M_4(s,0)|_{s/m_i^2\ll 1} = c_0\frac{s^{\frac{3}{2}}}{M_{pl}^2}+\sum_{i,n}\frac{(c_{n,4}z_i^4+c_{n,2}z_i^2+c_{n,0})}{m_i^{2n-3}M_{pl}^2}\, s^n$$

With exact expression for EFT Taylor coefficients to all order in derivatives

$$c_{n,4} = \frac{(n^2+n+1)}{2^{2n+4}(n+2)(n+1)(2n+1)\pi},$$
$$c_{n,2} = \begin{cases} \frac{1}{3840\pi} & \text{if} \quad n=2 \\ \frac{(n+1)}{2^{2n+7}(2n+3)(2n+5)\pi} & \text{if} \quad n>2 \end{cases},$$
$$c_{n,0} = \begin{cases} \frac{1}{1280\pi} & \text{if} \quad n=2 \\ \frac{(n/2+1)(n+1)}{2^{2n+7}(2n+1)(2n+3)(2n+5)\pi} & \text{if} \quad n>2 \end{cases}$$

Scalars

$$c_{n,4} = \frac{(5n^2+5n+2)}{2^{2n+5}n(n+2)(n+1)(2n+1)\pi},$$
$$c_{n,2} = \begin{cases} -\frac{1}{3840\pi} & \text{if} \quad n=2 \\ \frac{(n+1)}{2^{2n+8}(n+2)(2n+3)(2n+5)\pi} & \text{if} \quad n>2 \end{cases}$$
$$c_{n,0} = \begin{cases} \frac{1}{1920\pi} & \text{if} \quad n=2 \\ \frac{(n+1)}{2^{2n+9}(2n+1)(2n+3)(2n+5)\pi} & \text{if} \quad n>2 \end{cases}$$

Fermions

We can now analyze the sign of Det[K_n]

$$K_n = \begin{pmatrix} g_2 & g_4 & g_6 & \cdots \\ g_4 & g_6 & g_8 & \cdots \\ g_6 & g_8 & g_{10} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & g_n \end{pmatrix}$$

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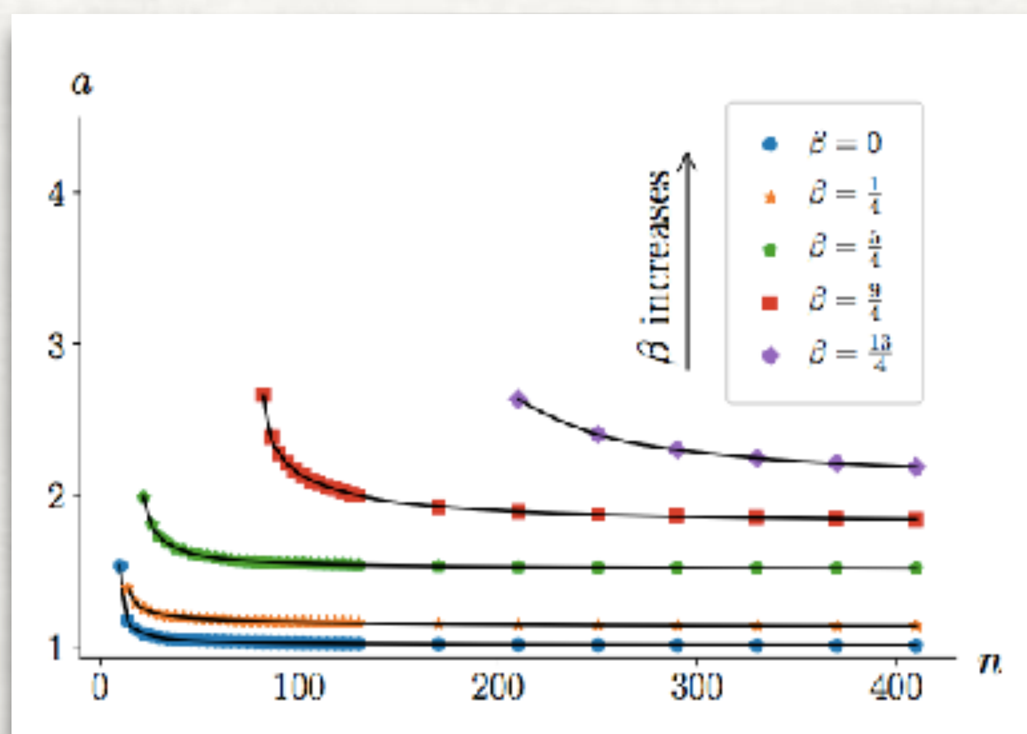
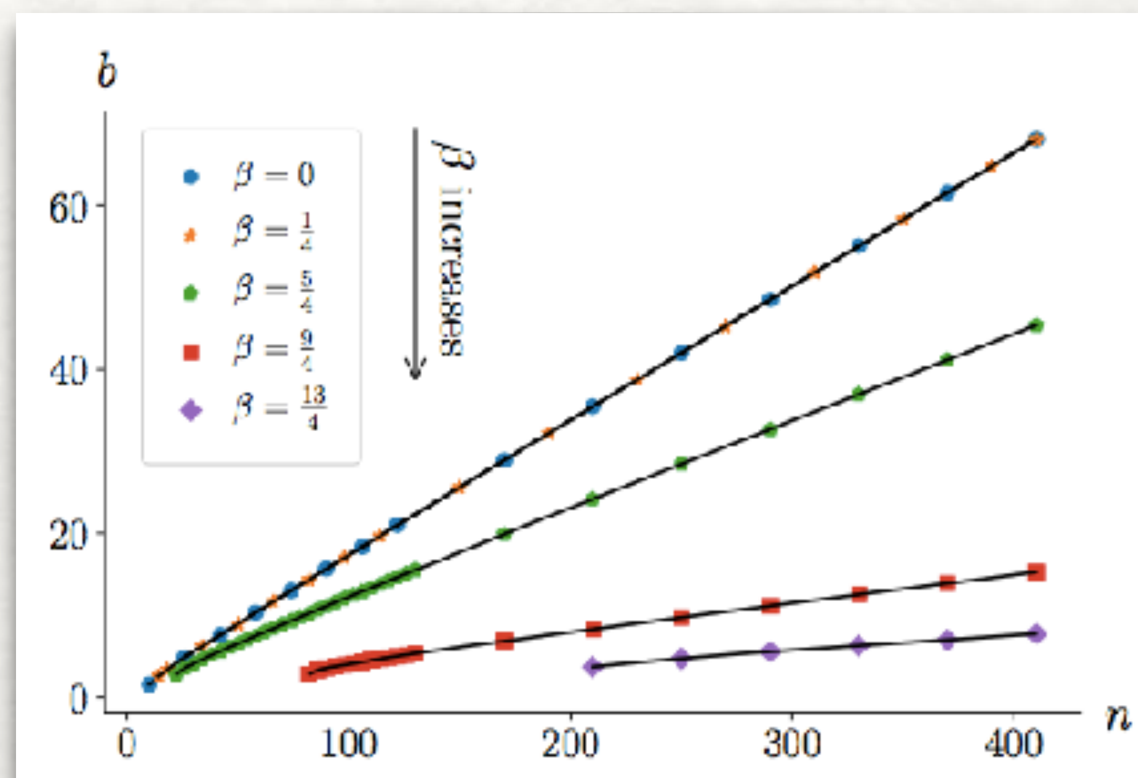
$$K_n = \begin{pmatrix} g_2 & g_4 & g_6 & \cdots \\ g_4 & g_6 & g_8 & \cdots \\ g_6 & g_8 & g_{10} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & g_n \end{pmatrix}$$

In general Det[K_n]>0, imposes 0<z<a and b<z. For example the positivity of $\det[K_{206}]$

$$0 < |z| < 1.02, \text{ or } |z| > 34.82 .$$

The asymptotic behavior of (a,b) is very different





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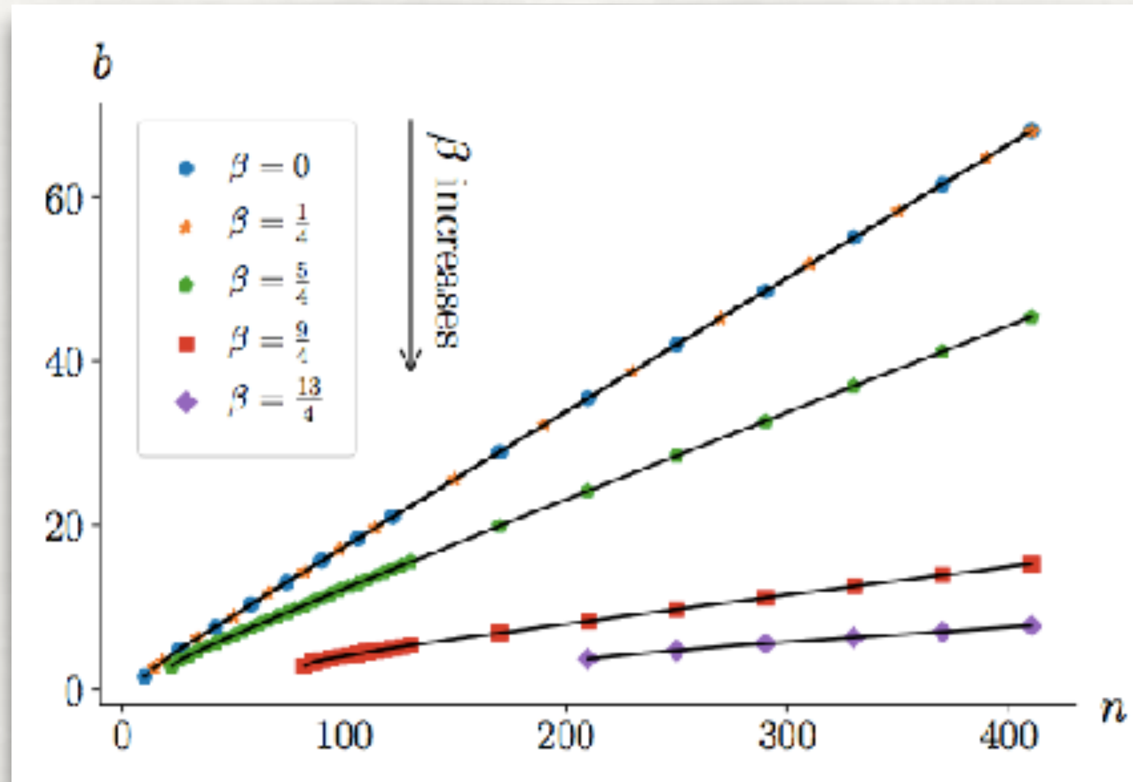

If we assume the linear rise tend to infinity, then we conclude that unitarization of gravity forbids an isolated low mass state with z>1!

But for a standard model like spectrum we will have the electron with z~10^22!

However, in three-dimensions, we have the special feature that the Wilson coefficients are dominated by the largest z and the smallest m!

$$c_i\left(\frac{qg}{m}, M_{pl}\right) = \frac{1}{mM_{pl}^2}\left(\alpha_i\, z^0 + \beta_i\, z^2 + \gamma_i\, z^4\right)\frac{1}{m^{2i}}$$

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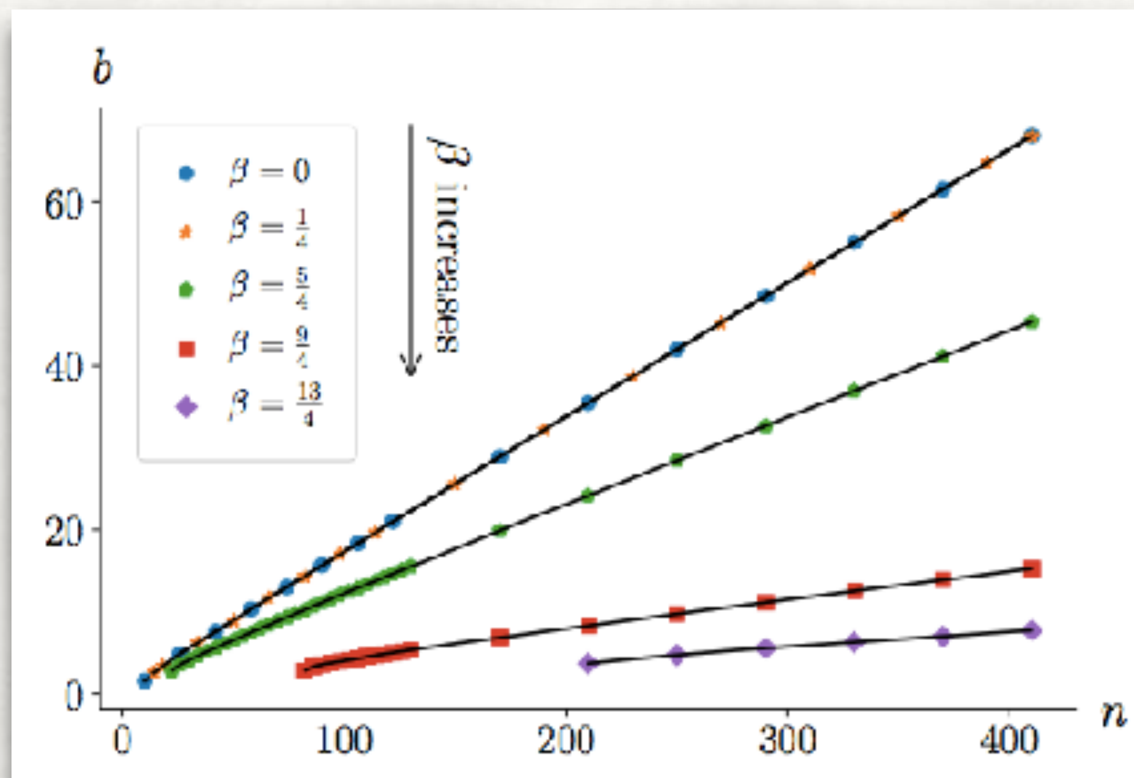
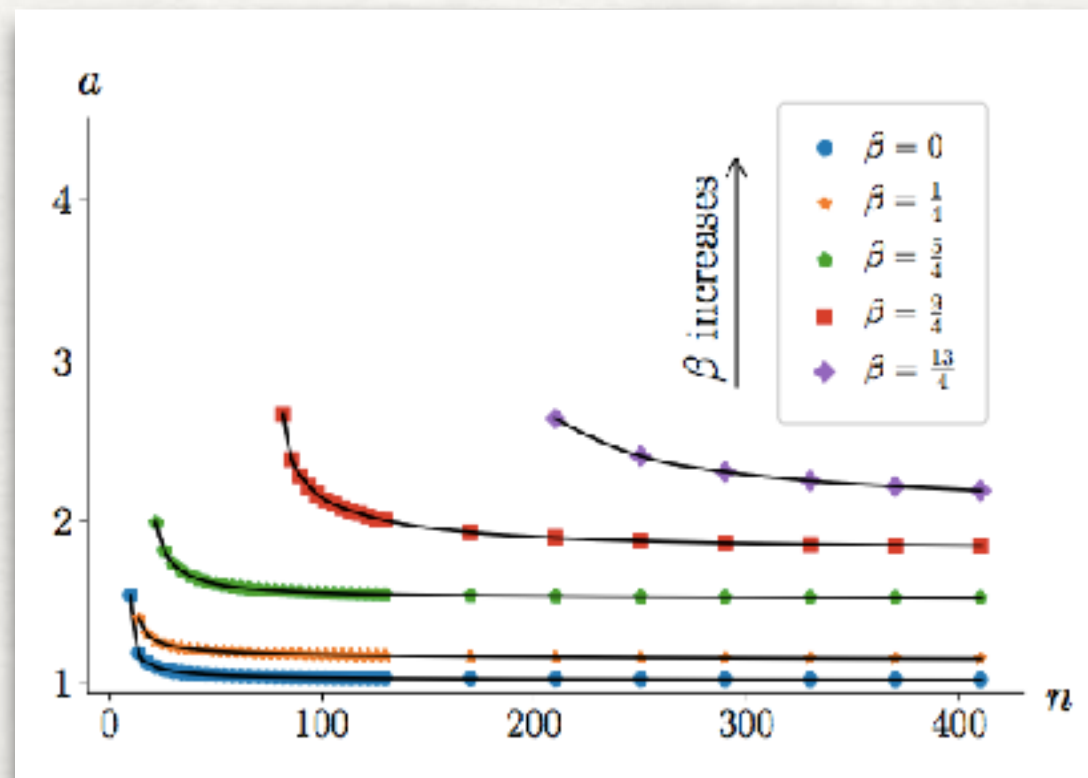
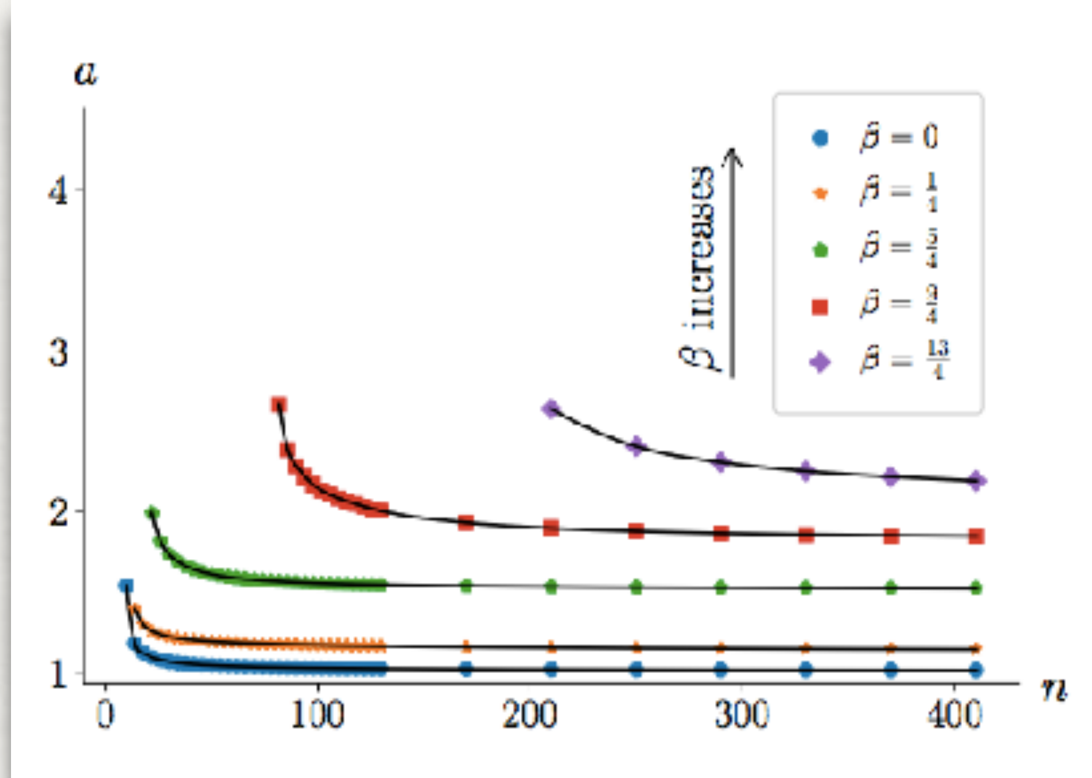
We can consider adding other neutral states, with β=m_e/m_0

$$g_n = \frac{c_{n,4}z^4 + c_{n,2}z^2 + (1+\beta^{2n-3})c_{n,0}}{m_e^{2n-3}M_{pl}^2}$$





The presence of light neutral states alleviate the tension of having large z



The presence of light neutral states alleviate the tension of having large z

In general we would like to understand the analytic behavior of

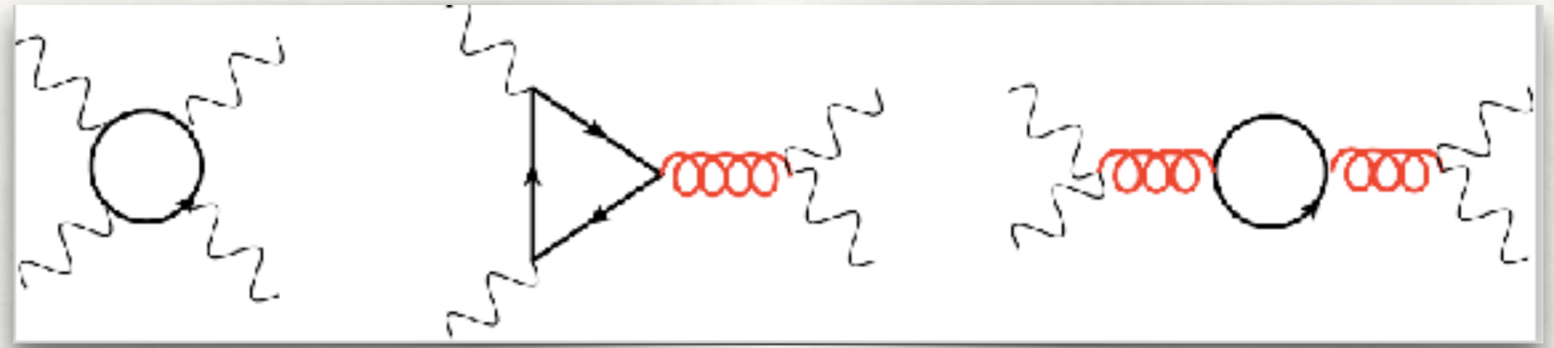
$$\boxed{0 < |z| \leq a_{asymp}(\beta)\,.}$$

$$a_{asymp}\left(0\right) = 1.01\,, \quad a_{asymp}\left(\tfrac{1}{4}\right) = 1.14$$
$$a_{asymp}\left(\tfrac{5}{4}\right) = 1.52\,, \quad a_{asymp}\left(\tfrac{9}{4}\right) = 1.84$$

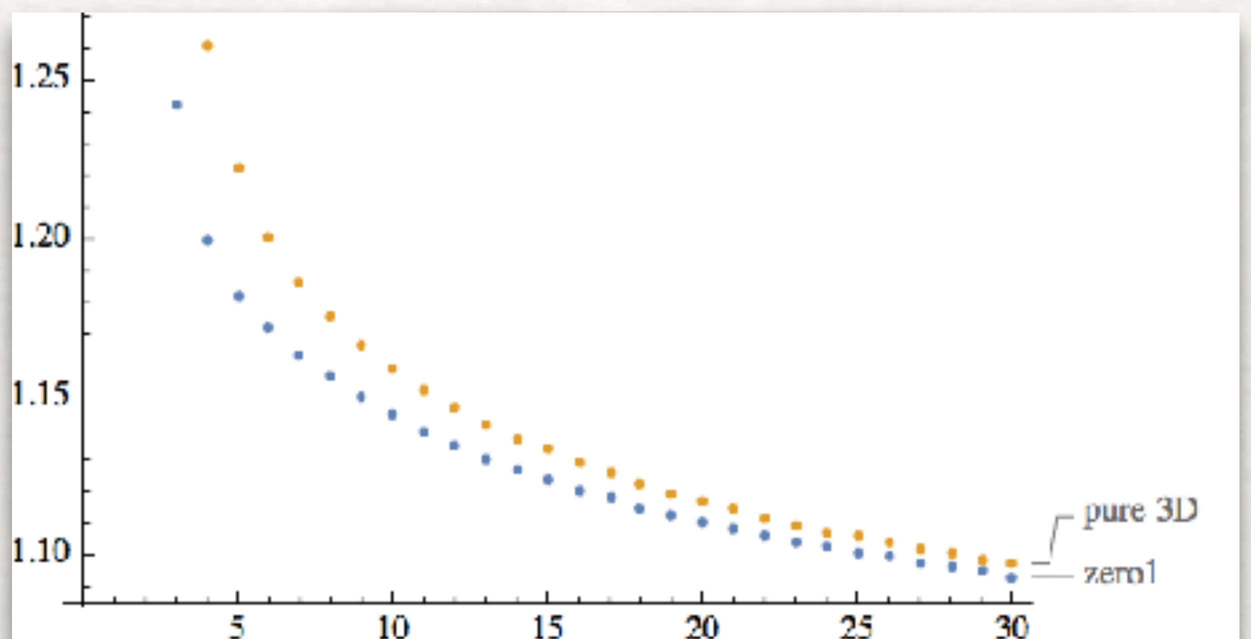
Implications for D=4

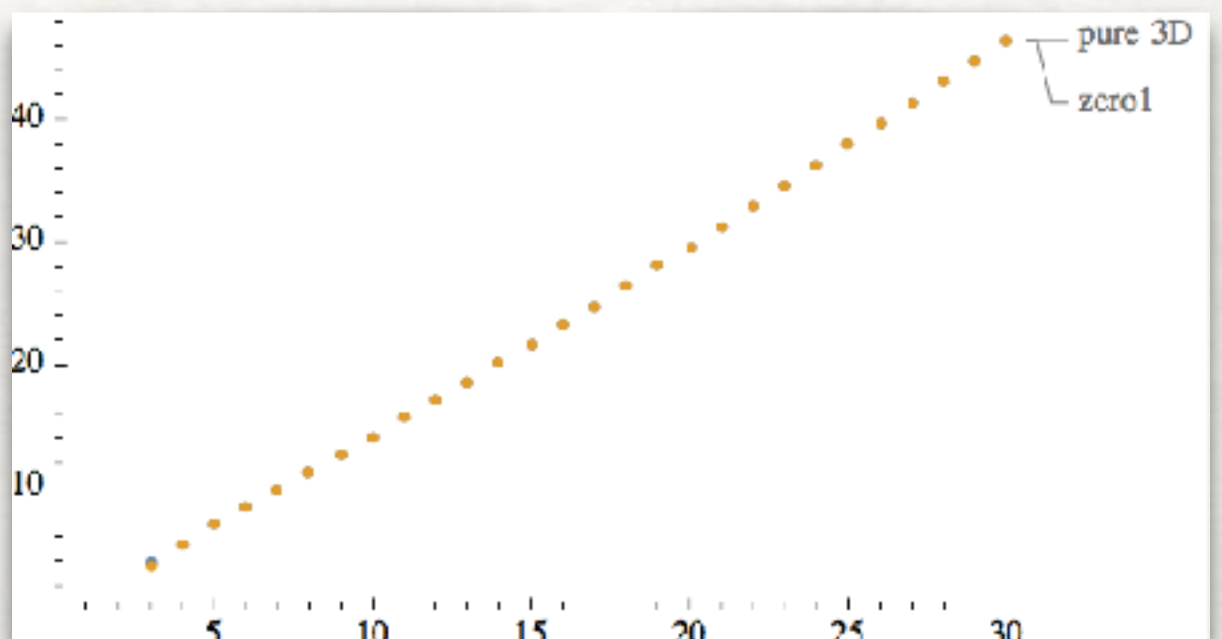
Let's consider our standard model (with z~10^22 electron), and compactified to D=3. and take the decoupling limit of the KK-modes. The massless spectrum will be modified,

Photons $\longrightarrow$ Photons, Dilatons, GraviPhotons



Assuming that the KK scale can be much higher than the string scale, such that asymptotic behavior satisfies <s^2 below the KK scale.





The constraint only becomes more stringent!

- The union of Lorentz invariance, Locality and Unitarity bounds the couplings of (dimension eight and beyond) operators to live in the space defined by the intersection of positive geometries. We are scratching the surface (no longer tip).

- Remarkably, this polytope has appeared in a wide range of problems in the past few years, CFT bootstrap, Modular bootstrap, amplitudes for gauge theories and scalar field theories, and Sphere packing problems