Semiclassics
&
multi-leg amplitudes

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with G. Badel, G. Cuomo, A. Monin, in preparation

Quantum regimes nicely classified via the path integral

- Weak coupling: loop expansion around leading trajectory γ

$$e^{-W} = e^{-[S_0+S_1+S_2+\dots]}$$

can further distinguish semi-classical and quantum observables

$$\langle O\rangle = O_c(\gamma) + \delta_q O$$

$\delta_q O \ll O_c(\gamma)$ semi-class — Ex: every day life

$\delta_q O \gtrsim O_c(\gamma)$ quantum — Ex: $\langle\phi\phi\phi\phi\rangle$ around $\phi(\gamma)=0$

- Strong coupling: PI cannot be described by leading trajectory

Common practice: few legs in weakly coupled QFT

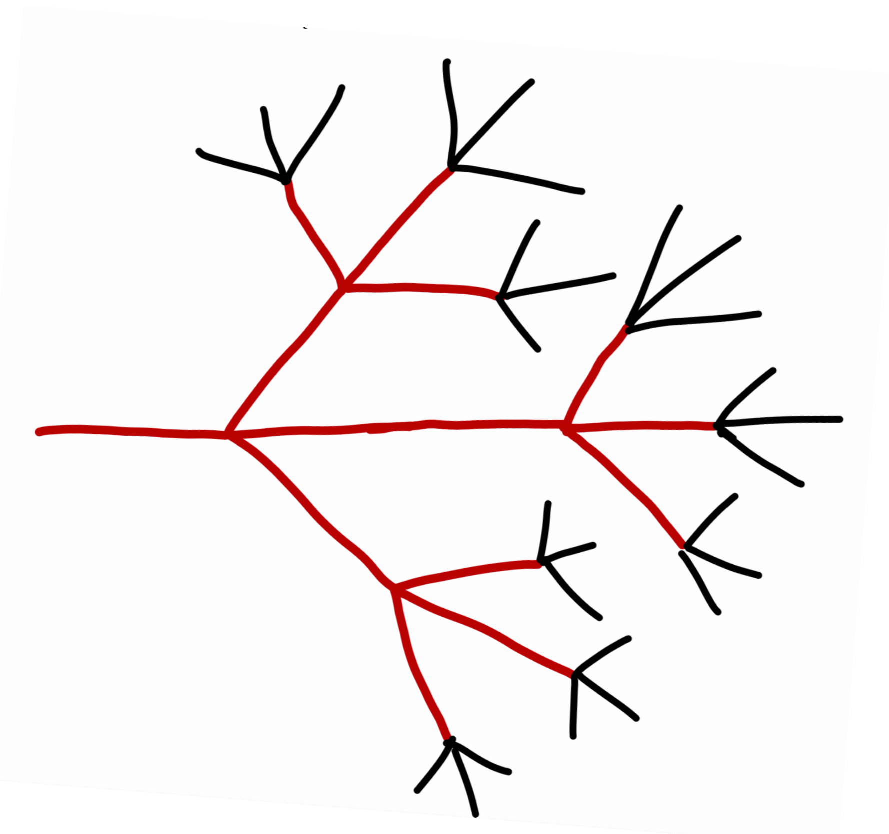
= small fluctuations around trivial trajectory

However when the number of legs grows expansion breaks down

How do we describe physics in this regime?

n-legs in ϕ^4

see old review by Rubakov, arXiv:9511236, 1995



$$A_{1\to n} \propto n! \left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}$$

$$\sigma_{1\to n} \propto n! \left(\frac{\lambda}{8m^2}\right)^{n-1} \epsilon^{\frac{3n}{2}} \qquad \epsilon = \frac{E-nm}{n}$$

$$A_{loop} = A_{tree}\left(1 + B\lambda n^2 + C\lambda^2 n^3 + \dots\right)$$

a mess ?!

Indeed .. but not completely

all large effects can be proven to resum into

$$\sigma_{1\to n} \sim e^{nF(\lambda n,\epsilon)}$$

Libanov, Rubakov, Son, Troitsky 1994

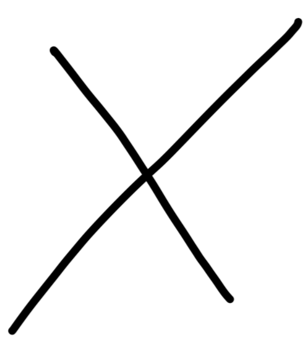
Exponential form suggest existence of non-trvial semiclassical trajectory describing the process

Something indeed proven by Son,
back in the 90's...still a reasonable mess to work out quantitatively

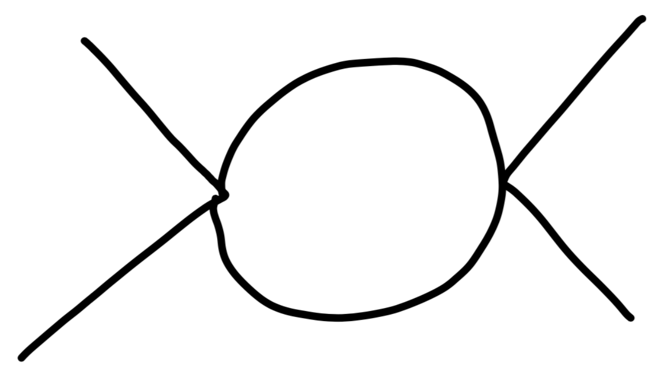
Charged ϕ^4

$$\mathcal{L} = \partial_\mu \overline{\phi}\, \partial^\mu \phi + \frac{\lambda}{4} (\overline{\phi}\phi)^2$$

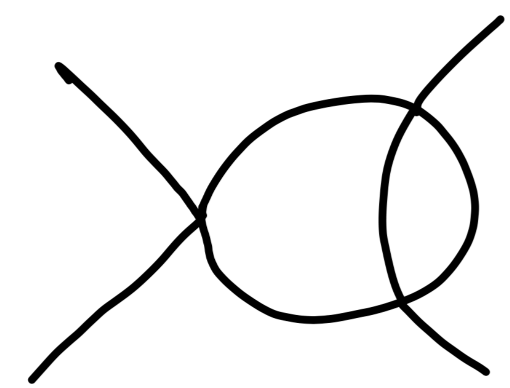
Few Legs



λ



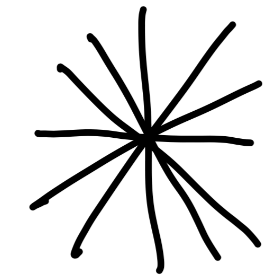
$$\frac{\lambda^2}{16\pi^2}$$



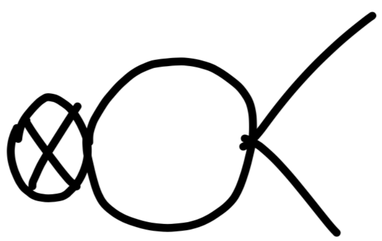
$$\frac{\lambda^3}{(16\pi^2)^2}$$

Many Legs

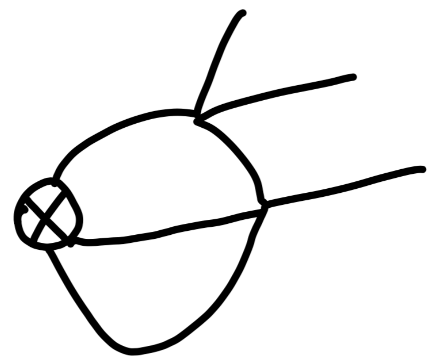
Ex. anomalous dimension of ϕ^n $\quad n \gg 1$



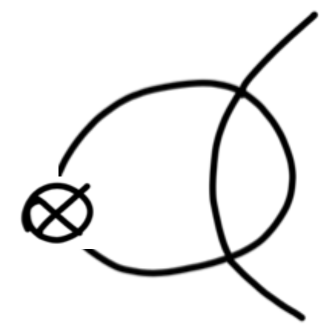
$\equiv \otimes$



$\lambda n(n-1)$



$\lambda^2 n(n-1)(n-2)$



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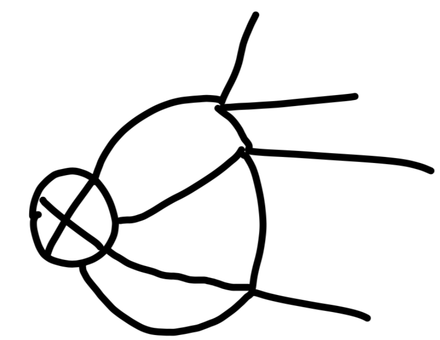

$\lambda^3 n(n-1)(n-2)(n-3)$

leading $\quad = \; n\left[\, \lambda n + (\lambda n)^2 + (\lambda n)^3 + \ldots \right]$

perturbation theory breaks down at $\quad \dfrac{\lambda n}{16\pi^2} \gtrsim 1$

series can be organized as a double expansion

$$\frac{\gamma_n}{n} = P_0(\lambda n) + \lambda P_1(\lambda n) + \lambda^2 P_2(\lambda n) + \ldots$$

similar to RG $\quad F_0(\lambda\, Log) + \lambda F_1(\lambda\, Log) + \ldots$

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Yes

by expanding path integral around suitable trajectory

Sketch of idea

$$\int D\phi\ \bar{\phi}^n(x_f)\,\phi^n(x_i)\ e^{-\frac{S}{\lambda}} \;=\; \int D\phi\ e^{-\frac{1}{\lambda}(S+n\lambda\ln\phi_f+n\lambda\ln\phi_i)}$$

saddle point $\qquad \phi_{cl} \equiv \phi_{cl}(\lambda n, x_f - x_i)$

$$\langle\,\bar{\phi}^n(x_f)\,\phi^n(x_i)\,\rangle \;=\; e^{-\frac{1}{\lambda}S_{cl}(\lambda n, x_f - x_i) - S_1 - \lambda S_2 + \dots}$$

$$\frac{\gamma_n}{n} = P_0(\lambda n) + \lambda P_1(\lambda n) + \dots$$

semiclassical expansion valid for all λn as long as $\lambda \ll 1$

Must reproduce diagrammatic expansion at $\lambda n \ll 1$!

γ_n is scheme dependent away from fixed point

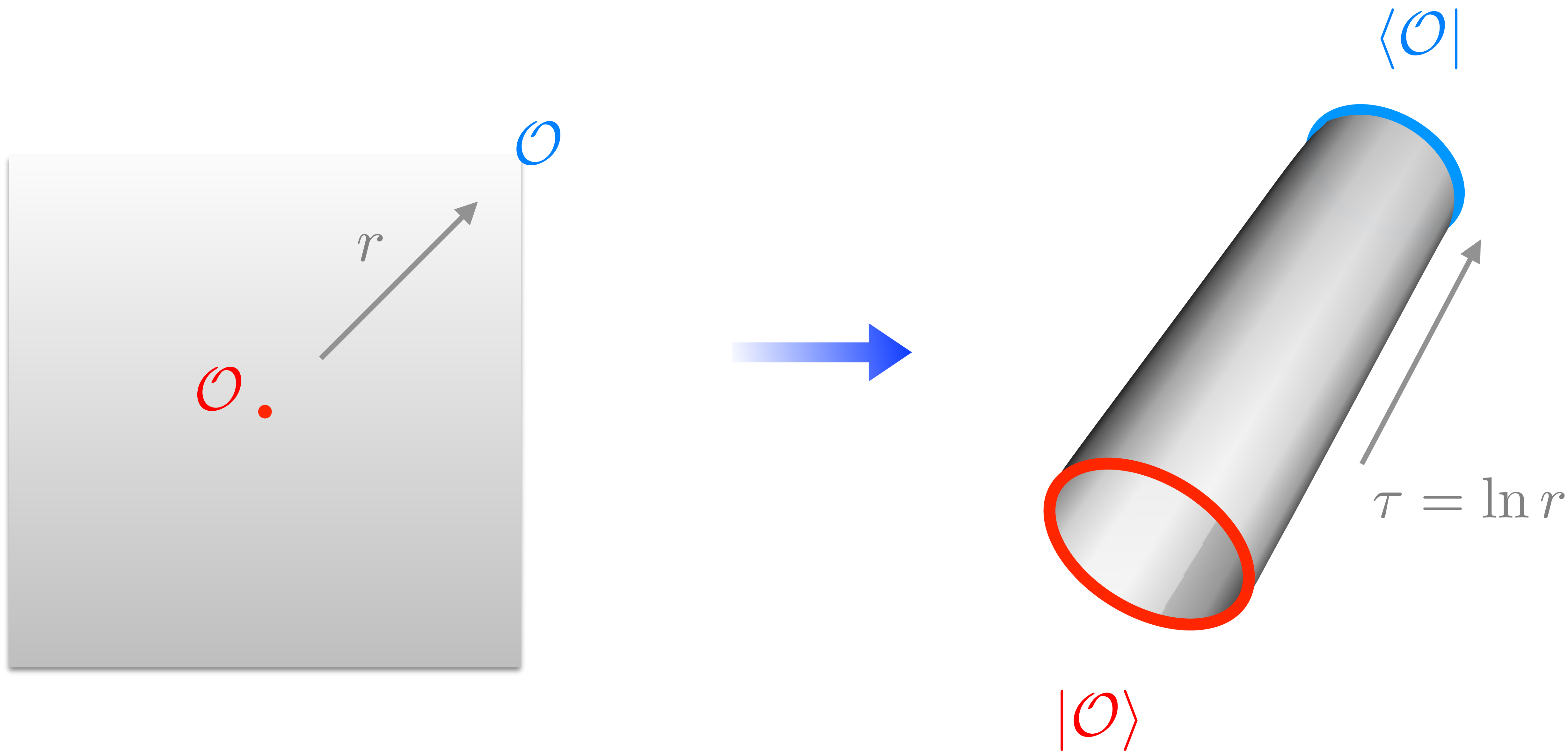
focus on Wilson-Fischer fixed point in $d = 4 - \epsilon$

$$\beta_\lambda = \lambda \left[-\epsilon + 5 \frac{\lambda}{(4\pi)^2} - 15 \frac{\lambda^2}{(4\pi)^4} + O(\lambda^3) \right]$$

$$\beta(\lambda_*) = 0 \qquad \frac{\lambda_*}{(4\pi)^2} = \frac{\epsilon}{5} + \frac{3\epsilon^2}{25} + O(\epsilon^3)$$

Compute by mapping theory to cylinder $\mathbb{R} \times S_{d-1}$

Mapping to the cylinder & operator state correspondence



$$\langle\, \mathcal{O}(r)\, \mathcal{O}(0)\, \rangle \;=\; \frac{1}{r^{2\Delta}} \quad \longrightarrow \quad \langle \mathcal{O} | e^{-H\tau} | \mathcal{O} \rangle \;=\; e^{-\Delta\tau}$$

$$\langle \phi^n | e^{-H(\tau_f - \tau_i)} | \phi^n \rangle \; = \; \int D\phi \, e^{-S_{cyl}}$$

$$S_{cyl} \; = \; \int d\tau d\Omega_{d-1} \left(|\partial\phi|^2 + \xi_d \mathcal{R} |\phi|^2 + \frac{\lambda}{4} |\phi|^4 - in\chi_f + in\chi_i \right)$$

$$\xi_d \mathcal{R} = \left(\frac{d-2}{2} \right)^2 \equiv m_d^2$$ ‘conformal mass’ on sphere

$$\phi \; \equiv \; \rho \, e^{i\chi}$$

$$e^{in\chi} \propto e^{in \ln \phi}$$ projects on state of charge n

solution of lowest action: homogeneous superfluid

$$\chi = -i\mu\tau \qquad \rho = \text{const}$$

boundary eom
$$2\rho^2\mu = \frac{n}{S_{d-1}}$$

$$S_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

bulk eom
$$-\mu^2 + m_d^2 + \frac{\lambda}{2}\rho^2 = 0$$

$$\boxed{\begin{aligned} \mu(\mu^2 - m_d^2) &= \frac{\lambda n}{4S_{d-1}} \\ \Delta_n &= \frac{n}{4\mu}(3\mu^2 + m_d^2) \end{aligned}} \Rightarrow \boxed{\Delta_n = n\,P_0(\lambda n, d)}$$

Explicitly for $d \to 4$

$$\left[\begin{array}{l} \lambda = \lambda_* \propto \epsilon \to 0 \\ \lambda n \;=\; \text{fixed} \end{array}\right]$$

$\lambda n \gg 1$

$$\Delta_n = \frac{\pi^2}{\lambda}\left[\frac{3}{8}\left(\frac{\lambda n}{\pi^2}\right)^{4/3} + \left(\frac{\lambda n}{\pi^2}\right)^{2/3} - \frac{2}{3} + O\left(\left(\lambda n/\pi^2\right)^{-2/3}\right)\right]$$

Interpretation: $m_\rho^2 \sim \mu^2 \sim (\lambda n)^{2/3} \gg \dfrac{1}{R^2} = 1$

$\Rightarrow$ integrate out radial mode: 'pure' conformal superfluid EFT

$$\mathcal{L} \sim (\partial\chi)^4 + \mathcal{R}(\partial\chi)^2 + \mathcal{R}^2 + \ldots$$

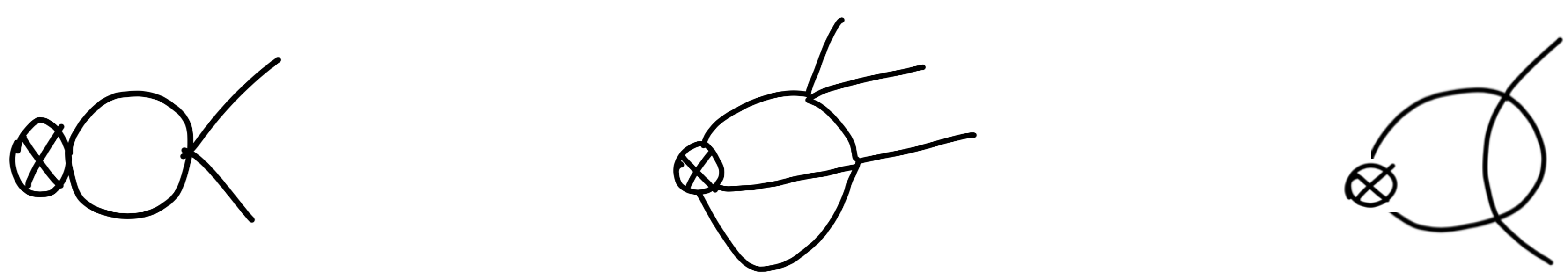
known large charge behaviour

Hellerman, Orlando, Reffert, Watanabe '15
Monin, Pirtskhalava, RR, Seibold '16
Jafferis, Mukhametzhanov, Zhiboedov, '17

$$\lambda n \ll 1 \qquad \Delta_n = n \sum a_p \left(\frac{\lambda n}{8\pi^2}\right)^p \qquad a_p = \frac{1}{p!}\partial_z^p \left.\frac{9z^4-1}{4z^{p+2}(1+z)^{p+1}}\right|_{z=1}$$

$$\Delta_n = n + \frac{\lambda n^2}{32\pi^2} - \frac{\lambda^2 n^3}{512\pi^4} + \frac{\lambda^3 n^4}{4096\pi^6} + O\left(\lambda^4 n^5\right)$$

which invites comparison with diagrams



$$\gamma_n = \frac{\lambda n(n-1)}{32\pi^2} - \frac{\lambda^2 n^2(n-1)}{512\pi^4} + \ldots$$

happily agreeing

Makes one want to check subleading terms

To do so fixed point condition in $d = 4 - \epsilon$ is essential

diagrammatic computation gives

$$\gamma_n \Big|_{\text{diag}} = \epsilon \frac{n(n-1)}{10} - \epsilon^2 \frac{n(n^2 - 4n)}{50} + O\left(\epsilon^3 n^4\right)$$

Semiclassically on cylinder

$$e^{-\Delta_n \tau} = \frac{\int D\phi\, e^{-S+in\chi_f - in\chi_i}}{\int D\phi\, e^{-S}} \overset{\text{1-loop}}{=} \frac{e^{-S_{cl}(n)-\frac{1}{2}\ln\det S_n^{(2)}}}{e^{-\frac{1}{2}\ln\det S_0^{(2)}}}$$

$$\frac{1}{\tau}\left(\frac{1}{2}\ln\det S_n^{(2)} - \frac{1}{2}\ln\det S_0^{(2)}\right) = \frac{1}{2}\sum_{\ell} n_{\ell,d}\left[\omega_+(\ell,d)+\omega_-(\ell,d)-2\omega_0(\ell,d)\right]$$

Casimir Energy on S_{d-1}, convergent for sufficiently low d

in terms of the renormalized coupling λ_R

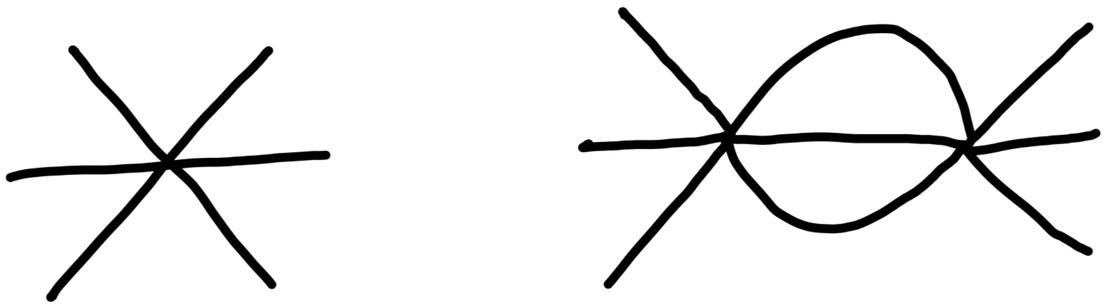
$$\gamma_n = \frac{\lambda_R n(n-1)}{32\pi^2} + \frac{\lambda_R \epsilon n^2}{64\pi^2} - \frac{\lambda_R^2(2n^3+3n^2)}{1024\pi^4} + \dots$$

fixed point

$$\epsilon\frac{n(n-1)}{10} - \epsilon^2\frac{n(n^2-4n)}{50} + O\left(\epsilon^3 n^4\right)$$

perfect match !

$$\frac{1}{\lambda}\left(|\partial\phi|^2 + |\phi|^6\right) \qquad \text{in} \qquad 3-\epsilon$$



$$\beta(\lambda) \;=\; \epsilon\lambda + a\lambda^3$$

in d=3 conformally invariant for any λ up to 1-loop

$$\Delta_n \;=\; \underbrace{\frac{1}{\lambda}f_0(\lambda n,\epsilon) + f_1(\lambda n,\epsilon)}_{\epsilon\to 0} + \lambda f_2(\lambda n,\epsilon) + \ldots$$

Can non trivially match universal quantum effects in genuine 3D CFT in large regime to diagrammatic computation ... and it works!

Summary

Wilson-Fisher fixed points: simple but rich playground to get structural insight on $\lambda n \gg 1$ regime in QFT

Loop expansion for γ_{ϕ^n} non-trivially and systematically encapsulated by semiclassical supefluid configuration

Properties of nearby operators, ex $\phi^{n-2}\partial\phi\partial\phi$, described by hydrodynamic modes

$\langle\phi^{n_1}\dots\phi^{n_p}\rangle$ can be studied by extension of method, providing dynamical information, akin to amplitudes

...but it would be nice to get back to the SM... someone certainly will before the Fcc begins