Fluctuations far from equilibrium

K. Mallick

Institut de Physique Théorique Saclay (France)

CEA, October 2, 2018

Outline

Introduction

1. Current and density fluctuations

2. Tagged Particle dynamics

3. Hydrodynamic limit

Conclusion

Introduction

The canonical Law of Equilibrium Statistical Mechanics

The statistical mechanics of a system at thermal equilibrium is encoded in the Boltzmann-Gibbs canonical law:

$$P_{\mathrm{eq}}(\mathcal{C}) = \frac{\mathrm{e}^{-E(\mathcal{C})/kT}}{Z}$$

the Partition Function Z being related to the Thermodynamic Free Energy F:

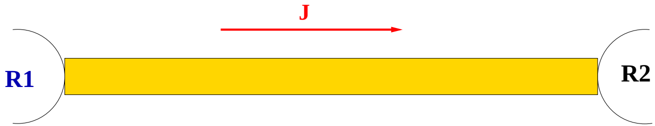
$$F = -kTLog\, Z$$

This provides us with a well-defined prescription to analyze systems *at equilibrium*:
(i) Observables are mean values w.r.t. the canonical measure.
(ii) Statistical Mechanics predicts fluctuations (typically Gaussian) that are out of reach of Classical Thermodynamics.

Equilibrium is a dynamical concept: Thermodynamic observables are nothing but average values of fluctuating, probabilistic, microscopic quantities.

Systems near equilibrium

Consider a Stationary Driven System in contact with two reservoirs at temperatures T_1 and T_2 (or chemical, or electric, potentials μ_1, μ_2).

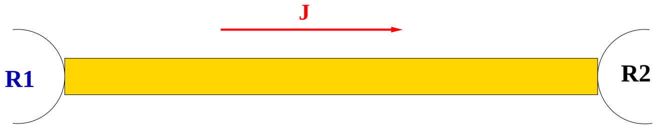


- If $T_1 = T_2$: Equilibrium Statistical Mechanics. The state of the system, characterized by very few parameters, is determined by optimizing the relevant thermodynamic potential and leads to an equation of state.

  This allows us to study phase transitions, universality classes, statistical fluctuations (generically Gaussian).

Systems *near* equilibrium

Consider a Stationary Driven System in contact with two reservoirs at temperatures T_1 and T_2 (or chemical, or electric, potentials μ_1, μ_2).



- When $|T_1 - T_2| \ll T_1$: A stationary current, breaking time reversal invariance, sets in, proportional to the temperature gradient.

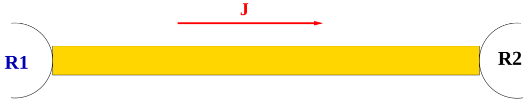
  This flow of the current implies that entropy is continuously generated and keeps on increasing with time.

  Conductivity determined by quadratic correlations *at equilibrium* (Einstein-Kubo linear response theory): mobility = diffusivity/kT

  Minimal Entropy Production Rate (Prigogine): an elegant way to reformulate linear response theory.

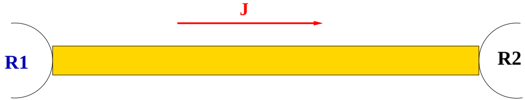
Systems far from equilibrium

Consider now a Stationary Driven System in contact with reservoirs at different potentials: no microscopic theory is yet available.



Systems far from equilibrium

Consider now a Stationary Driven System in contact with reservoirs at different potentials: no microscopic theory is yet available.



- What are the relevant macroscopic parameters?
- Which functions describe the state of a system?
- Do Universal Laws exist? Can one define Universality Classes?
- Can one postulate a general form for the microscopic measure?
- What do the fluctuations look like ('non-gaussianity')?

In the steady state, a non-vanishing macroscopic current J flows.

What can we say about the non-equilibrium properties of observables (e.g., current) from the point of view of Statistical Physics?

Density Fluctuations

Consider a gas in a room, at thermal equilibrium. The probability of observing a density profile $\rho(x)$ takes the form:

$$\Pr\{\rho(x)\} \sim \mathrm{e}^{-\beta V\,\mathcal{F}(\{\rho(x)\}}$$

What is $\mathcal{F}(\{\rho(x)\}$?

Density Fluctuations

Consider a gas in a room, at thermal equilibrium. The probability of observing a density profile $\rho(x)$ takes the form:

$$\Pr\{\rho(x)\} \sim \mathrm{e}^{-\beta V \mathcal{F}(\{\rho(x)\}}$$

What is $\mathcal{F}(\{\rho(x)\}$?

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 \left(f(\rho(x), T) - f(\bar{\rho}, T)\right) d^3x$$

Equilibrium Free Energy can be seen as a Large Deviation Function.

Density Fluctuations

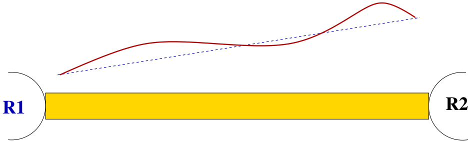
Consider a gas in a room, at thermal equilibrium. The probability of observing a density profile $\rho(x)$ takes the form:

$$\Pr\{\rho(x)\} \sim \mathrm{e}^{-\beta V\,\mathcal{F}(\{\rho(x)\})}$$

What is $\mathcal{F}(\{\rho(x)\})$?

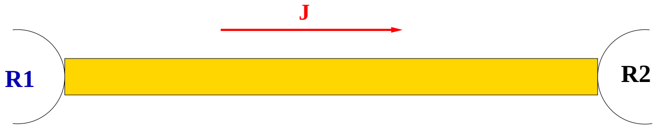
$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 \left(f(\rho(x), T) - f(\bar{\rho}, T)\right) d^3x$$

Equilibrium Free Energy can be seen as a Large Deviation Function.



What is the probability of observing an atypical density profile in the steady state? What does the functional $\mathcal{F}(\{\rho(x)\})$ look like for such a non-equilibrium system?

Large Deviations of the Total Current



Let Q_t be the total charge transported through the system (integrated total current) between time 0 and time t.

In the stationary state: a non-vanishing mean-current $\frac{Q_t}{t} \to J$

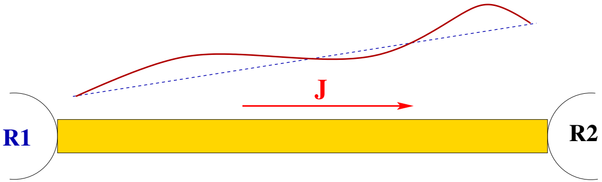
The fluctuations of Q_t obey a Large Deviation Principle:

$$P\left(\frac{Q_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

$\Phi(j)$ being the *large deviation function* of the total current.

Note that $\Phi(j)$ is positive, vanishes at $j = J$ and is convex (in general).

The General Large Deviations Problem



The Probability to observe an atypical current $j(x, t)$ and the density profile $\rho(x, t)$ during $0 \leq s \leq L^2\, T$ (L being the size of the system) assumes a Large Deviation behaviour

$$\mathrm{Pr}\{j(x,t), \rho(x,t)\} \sim \mathrm{e}^{-L\,\mathcal{I}(j,\rho)}$$

Knowing $\mathcal{I}(j, \rho)$, one could deduce the large deviations of the current and of the density profile. For instance, $\Phi(j) = \min_\rho\{\mathcal{I}(j, \rho)\}$.

Is there a Principle which gives this large deviation functional for systems out of equilibrium?

The importance of Large Deviations

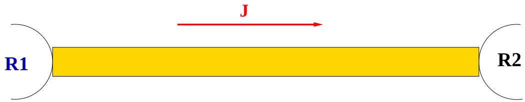
- Equilibrium Thermodynamic potentials (Entropy, Free Energy) can be defined as large deviation functions.
- Large deviations are well defined far from equilibrium: they are good candidates for being non-equilibrium potentials.
- Large deviation functions obey remarkable identities, valid far from equilibrium (Gallavotti-Cohen Fluctuation Theorem; Jarzynski and Crooks Relations).
- These identities imply, in the vicinity of equilibrium, the fluctuation dissipation relation (Einstein), Onsager's relations and linear response theory (Kubo).

Solutions of specific models (Ising, SAW) played a key role in Equilibrium Statistical Mechanics as benchmarks of analytical, numerical (Monte-Carlo) perturbative (Diagrammatics, RG) methods.

Models are also very useful far from equilibrium.

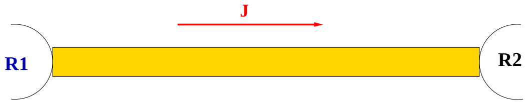
Classical Transport in 1d

A picture of a non-equilibrium system

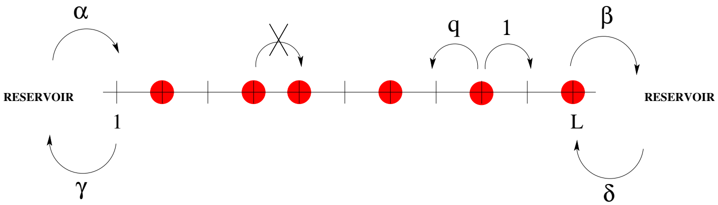


Classical Transport in 1d

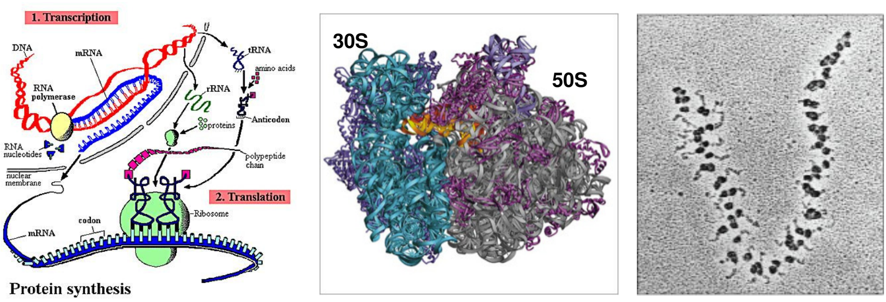
A picture of a non-equilibrium system



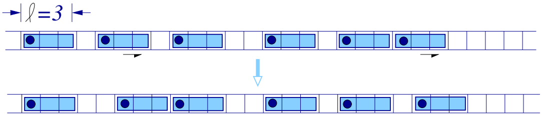
A paradigm: the asymmetric exclusion model with open boundaries



A building block in many realistic models of 1d transport and studied extensively in probability, combinatorics, condensed matter physics...
Thousands of articles devoted to this model in the last 20 years.



(a) (b) (c)



C. T. MacDonald, J. H. Gibbs and A.C. Pipkin, Kinetics of biopolymerization on nucleic acid templates, *Biopolymers* (1968).

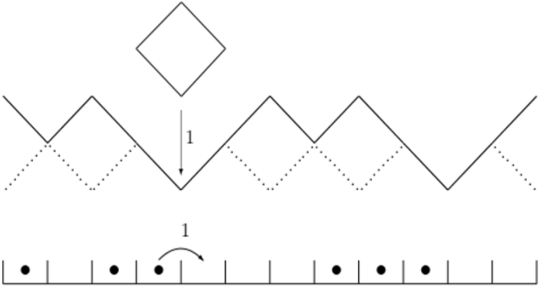
ORIGINS

- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.
- Interface dynamics. KPZ equation

APPLICATIONS

- Traffic flow.
- Sequence matching.
- Molecular motors.

The Kardar-Parisi-Zhang equation in 1d



The height of an interface $h(x,t)$ satisfies the generic KPZ equation

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \xi(x,t)$$

The ASEP is a discrete version of the KPZ equation in one-dimension.

INTEGRABILITY

Interacting particle processes are complex enough to exhibit a rich phenomenology that *captures* the physics involved.

On the other hand, some of these models have intricate mathematical properties that allows us to solve them exactly: they are **integrable.**

Exact solutions are benchmarks for testing general theories or more versatile approximation methods.

Some techniques involved:

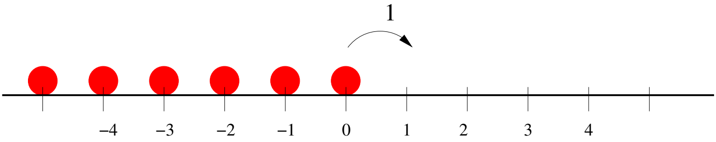
- Coordinate and Algebraic Bethe Ansatz (cf H. Bethe's solution of the Heisenberg spin chain).
- Integrable probabilities and determinantal processes.
- Continuous limits: Stochastics hydrodynamics and Macroscopic Fluctuation Theory.

NEXT: Some examples of exact results.

Current/Density Fluctuations

Current fluctuations on the Infinite Line

Consider the Totally Asymmetric case with a step initial condition:
What is the statistics of the total current Q_t that has flown through the (0,1) bond in the exclusion process during time t?



The Probability that $(Q_t \geq N)$ is equal to the Probability that the Mth particle has jumped at least N steps (to the right).
Here, the summation over the Green function can be done explicitly:

$$\mathrm{Prob}(Q_t \geq N) = \frac{1}{\mathcal{Z}_N} \int_{[0,t]^N} \mathrm{d}^N x \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{i=1}^{N} \mathrm{e}^{-\mathrm{x_i}}$$

This integral has a meaning in random matrix theory: this is the distribution of the Largest Eigenvalue in the Laguerre ensemble.

The Tracy-Widom Law

The statistics of Q_t can be extracted from the random matrix integral:

$$Q_t = \frac{t}{4} + \frac{t^{1/3}}{2^{4/3}}\,\xi_{TW}$$

The random variable ξ_{TW} follows the Tracy-Widom distribution F_2 (K. Johansson, 2000), which is the cumulative distribution of the maximal eigenvalue λ_{max} in a GUE:

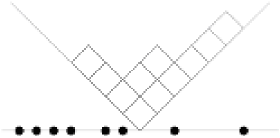
$$\text{Prob}\,(\xi_{TW} \leq s) = 1 - F_2(-s)$$

$$\text{with} \quad F_2(s) = \exp\left(-\int_s^\infty (x-s)\,u(x)^2 dx\right)$$

$u(x)$ being the solution of Painlevé II equation $u'' = xu + 2u^3$, matching the Airy function at infinity.

TASEP and Corner Growth

The exclusion process is equivalent to a dynamically growing Young Tableau.



In particular, the position of the rightmost particle is equal to the length of the first line of the Young Tableau.

Furthermore, the length of the first line of the Young Tableau is equal to the length of the largest increasing subsequence in a permutation.

Ulam's Problem: Choose a random permutation σ, e.g.,

$$2 \quad 5 \quad 1 \quad 3 \quad 7 \quad 4 \quad 6$$

extract a largest increasing subsequence and call $l(\sigma)$ its length.

What are the statistical properties of $l(\sigma)$?

Ulam's problem and Patience Sorting

It has been proved (Baik-Deift-Johansson, circa 2000) that

$$l(\sigma) = 2\sqrt{n} + n^{1/6}\,\xi_{TW}$$

HOW?

• A random permutation can be mapped to a (two) Young Tableau(x), via the RSK (Robinson-Knuth-Schensted) correspondence.

• A Young Tableau is a corner growth and therefore a configuration of TASEP.

• TASEP with step initial condition can be studied by "Bethe Ansatz"

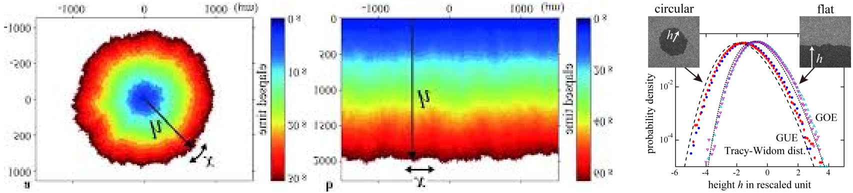
This can be related to sorting algorithms: Aldous and Diaconis (1999).

Solution of KPZ and universality classes

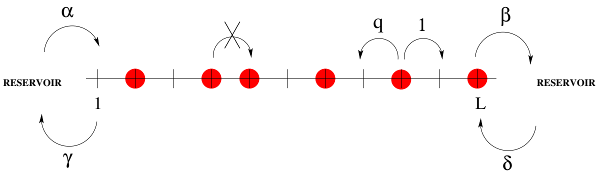
Exact solution to the KPZ equation in one dimension: Sasamoto-Spohn, Corwin-Amir-Quastel, Dotsenko, Le Doussal-Calabrese-Rosso...

Generalizations to different settings and processes are possible: Various distributions appear, related to the Tracy-Widom Law, which has a strong universality in mathematics and in physics. This is an extremely active field of mathematics: **Integrable Probability** (Borodin, Corwin, Ferrari, Sasamoto...).

The Tracy-Widom Law has been precisely measured in liquid crystals experiments of Takeuchi and Sano, 2010.



Large deviations in presence of reservoirs



The stationary probability of a configuration $\mathcal{C}$ is given by (Derrida et al., 1993)

$$P(\mathcal{C}) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} \left(\tau_i D + (1 - \tau_i) E \right) | V \rangle$$

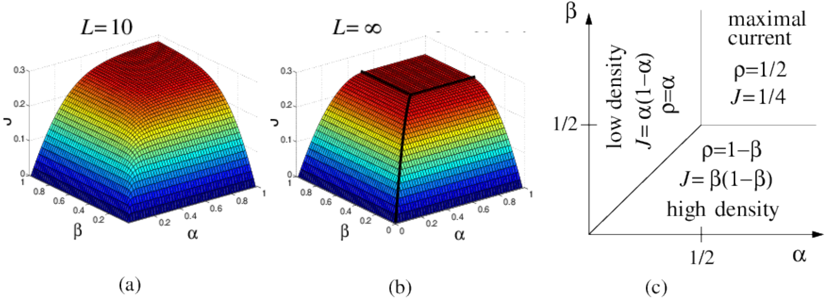
where $\tau_i = 1$ (or 0) if the site i is occupied (or empty). The operators D and E, the vectors $\langle W|$ and $|V\rangle$ satisfy a quadratic algebra

$$\begin{aligned} D\,E - qE\,D &= (1-q)\,(D + E) \\ (\beta\,D - \delta\,E)\,|V\rangle &= |V\rangle \\ \langle W|(\alpha\,E - \gamma\,D) &= \langle W| \end{aligned}$$

What is the statistics of the density profile and the current?

The Phase Diagram of the open ASEP

The Matrix Ansatz leads to Stationary State Properties (currents, correlations, fluctuations) and to the Phase Diagram in the infinite size limit.



Finite size corrections (a) to the infinite-size limit (b) are analytically accessible.

A very large body of knowledge has been developed thanks to the Matrix Ansatz (Review of R. Blythe and M. R. Evans, 2007).

Large Deviations of the Density Profile

- In the equilibrium case, $\rho_1 = \rho_2 = \bar{\rho}$, we have

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left\{ (1 - \rho(x)) \log \frac{1 - \rho(x)}{1 - \bar{\rho}} + \rho(x) \log \frac{\rho(x)}{\bar{\rho}} \right\}$$

- When $\rho_1 \neq \rho_2$, the Large Deviation Functional of the profile, $\mathrm{Pr}\{\rho(x)\} \sim \mathrm{e}^{-L\,\mathcal{F}(\{\rho(x)\}}$, is given by (for $q = 0$)

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left(B(\rho(x), F(x)) + \log \frac{F'(x)}{\rho_2 - \rho_1} \right)$$

where $B(u, v) = (1 - u) \log \frac{1-u}{1-v} + u \log \frac{u}{v}$ and $F(x)$ satisfies

$$F\left(F'^2 + (1 - F)F''\right) = F'^2 \rho \quad \text{with} \quad F(0) = \rho_1 \text{ and } F(1) = \rho_2 \,.$$

This functional is non-local and is not given by local equilibrium (B. Derrida, J. Lebowitz E. Speer, 2002).

Current Fluctuations

The large deviation function $\Phi(j)$ of the total current is, as above,

$$P\left(\frac{Y_t}{t}=j\right)\sim e^{-t\Phi(j)}$$

.

- Low Density (and High Density) Phases: writing $j=(1-q)r(1-r)$

$$\boxed{\Phi(j)=(1-q)\left\{\rho_a-r+r(1-r)\ln\left(\frac{1-\rho_a}{\rho_a}\frac{r}{1-r}\right)\right\}}$$

- Maximal Current Phase: The Legendre transform $\mathcal{E}$ of $\Phi(j)$ is

$$\mu = -\frac{L^{-1/2}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(2k)!}{k!k^{(k+3/2)}}B^k$$

$$\mathcal{E}-\frac{1-q}{4}\mu = -\frac{(1-q)L^{-3/2}}{16\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(2k)!}{k!k^{(k+5/2)}}B^k$$

These exact results were obtained using a Generalized Matrix Product (S. Prolhac, A. Lazarescu, K. M.).

A special case

In the totally asymmetric case with $\rho_L = 1, \rho_R = 0$, a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]!\,[k(L+2)]!} \frac{B^k}{2k} ,$$

$$E = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)-2]!}{[k(L+1)-1]!\,[k(L+2)-1]!} \frac{B^k}{2k} .$$

First cumulants of the current

- Mean Value : $J = \frac{L+2}{2(2L+1)}$
- Variance : $\Delta = \frac{3}{2} \frac{(4L+1)![L!(L+2)!]^2}{[(2L+1)!]^3(2L+3)!}$
- Skewness :
  $E_3 = 12 \frac{[(L+1)!]^2[(L+2)!]^4}{(2L+1)[(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)![(2L+2)!]^2[(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$
  For large systems: $E_3 \to \frac{2187-1280\sqrt{3}}{10368} \pi \sim -0.0090978...$

Tagged particle dynamics

Motion of a tagged particle

We now focus the Symmetric Exclusion Process, $(p = q = 1)$, on an *infinite one-dimensional line* with a finite density ρ of particles. This model was defined by F. Spitzer in 1970.

Suppose that we tag and observe a particle that was initially located at site 0 and monitor its position X_t with time.

On the average $\langle X_t \rangle = 0$ but how large are its fluctuations?

- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.

Motion of a tagged particle

We now focus the Symmetric Exclusion Process, $(p = q = 1)$, on an *infinite one-dimensional line* with a finite density ρ of particles. This model was defined by F. Spitzer in 1970.

Suppose that we tag and observe a particle that was initially located at site 0 and monitor its position X_t with time.

On the average $\langle X_t \rangle = 0$ but how large are its fluctuations?

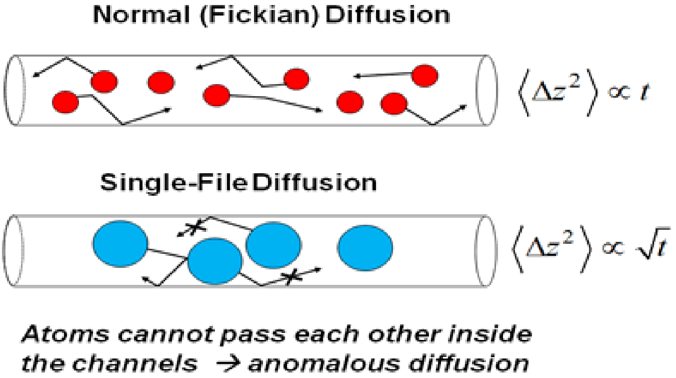
- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.
- Because of the exclusion condition, a particle displays an anomalous diffusive behaviour:

$$\langle X_t^2 \rangle = 2\frac{1-\rho}{\rho}\sqrt{\frac{Dt}{\pi}} \qquad \text{(Arratia, 1983)}$$

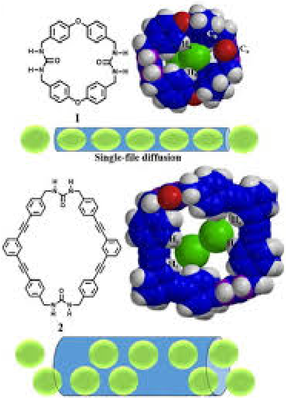
The full distribution of X_t has remained unknown for 35 years!

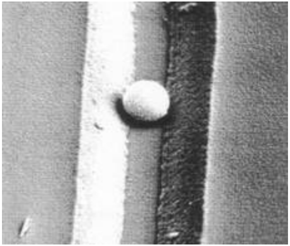
Single-file diffusion

SEP is a pristine model for single-file diffusion, an important phenomena soft-condensed matter (for example, transport in chanels through cell membranes).



Experimental realizations

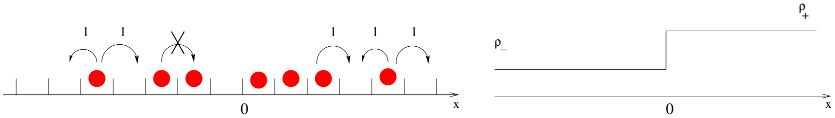




(C. Bechinger's group in Stuttgart)

SEP with step profile

We begin with a step-like profile (ρ_+, ρ_-) with the tagged particle is located at 0 and let the system evolve.



The goal is to calculate the large deviation function (LDF) $\phi(\xi)$ or, equivalently, the characteristic function of X_t, which behaves as

$$\langle e^{sX_t} \rangle \sim e^{-\sqrt{t}C(s)} \quad \text{when} \quad t \to \infty$$

and generates the cumulants of X_t.

This problem is solved using the newly developed methods of integrable probabilities

Exact finite time expression

There exists a formula for the distribution of the tracer *exact at any finite-time*, in terms of a Fredholm determinant:

$$\det(1+\omega K_{t,x})$$

where $\omega = \rho_+(e^{\lambda}-1)+\rho_-(e^{-\lambda}-1)+\rho_+\rho_-(e^{\lambda}-1)(e^{-\lambda}-1)$
$K_{t,x}$ is a compact operator

$$f \to \int K_{t,x}(\xi_1,\xi_2)f(\xi_2)d\xi_2$$

with kernel

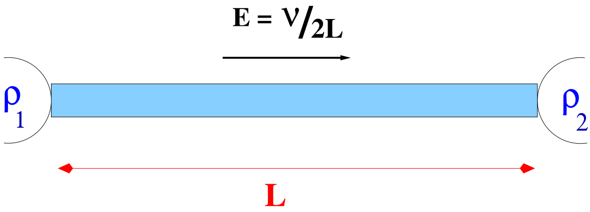
$$K_{t,x}(\xi_1,\xi_2)=\frac{\xi_1^x e^{\epsilon(\xi_1)t}}{\xi_1\xi_2+1-2\xi_2} \quad \text{with} \quad \epsilon(\xi)=\xi+\xi^{-1}-2$$

The arithmetical and combinatorial properties of such Kernels are systematized in the field of *Integrable probabilities*.

(T. Imamura, K. M. and T. Sasamoto, 2017)

Hydrodynamic limit

The Hydrodynamic Limit: deterministic case



Starting from the microscopic level, define local density $\rho(x,t)$ and current $j(x,t)$ with macroscopic space-time variables $x = i/L, t = s/L^2$ (diffusive scaling).

The average hydrodynamic evolution of the system is given by:

$$\partial_t \rho(x,t) = -\nabla J(x,t) \quad \text{with} \quad J = -D(\rho)\nabla\rho + \nu\sigma(\rho)$$

How can Fluctuations be taken into account?

Fluctuating Hydrodynamics

Let Y_t be the integrated current of particles transferred from the left reservoir to the right reservoir during time t.

- $\lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = D(\rho)\frac{\rho_1-\rho_2}{L} + \sigma(\rho)\frac{\nu}{L}$ for $(\rho_1 - \rho_2)$ small
- $\lim_{t\to\infty} \frac{\langle Y_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$ for $\rho_1 = \rho_2 = \rho$ and $\nu = 0$.

Then, the equation of motion is obtained as:

$$\partial_t \rho = -\partial_x j \quad \text{with} \quad j = -D(\rho)\nabla\rho + \nu\sigma(\rho) + \sqrt{\sigma(\rho)}\xi(x,t)$$

where $\xi(x,t)$ is a Gaussian white noise with variance

$$\langle \xi(x',t')\xi(x,t) \rangle = \frac{1}{L}\delta(x-x')\delta(t-t')$$

For the symmetric exclusion process, the 'phenomenological' coefficients are given by

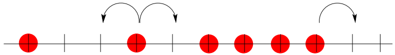
$$D(\rho) = 1 \quad \text{and} \quad \sigma(\rho) = 2\rho(1-\rho)$$

Values of Diffusivity and Conductivity

- Independent particles: $D = 1$, $\sigma = 2\rho$
- Simple Exclusion Process: $D_{\mathrm{SEP}} = 1$, $\sigma_{\mathrm{SEP}} = 2\rho(1-\rho)$
- Kipnis-Marchioro-Presutti model: $D_{\mathrm{KMP}} = 1$, $\sigma_{\mathrm{KMP}} = 2\rho^2$
- Repulsion Process (P. Krapivsky, 2015): Hops increasing the number of nearest neighbourg pairs are forbidden:

$$D_{\mathrm{RP}} = \begin{cases} \frac{1}{(1-\rho)^2} & \text{if } 0 < \rho < \frac{1}{2} \\ \frac{1}{\rho^2} & \text{if } \frac{1}{2} < \rho < 1 \end{cases} \quad \sigma_{\mathrm{RP}} = \begin{cases} \frac{2\rho(1-2\rho)}{1-\rho} & \text{if } 0 < \rho < \frac{1}{2} \\ \frac{2(1-\rho)(2\rho-1)}{\rho} & \text{if } \frac{1}{2} < \rho < 1 \end{cases}$$

- Exclusion with Avalanches: $D_{\mathrm{EPA}} = \frac{1}{(1-2\rho)^3}$, $\sigma_{\mathrm{EPA}} = \frac{2\rho(1-\rho)}{(1-2\rho)^3}$



Katz-Lebowitz-Spohn model (Driven Ising Model)

The Katz-Lebowitz-Spohn model is a driven lattice gas where the hopping rates depend on the neighbouring sites:

$$0100 \underset{1+\delta}{\overset{1+\delta}{\leftrightarrows}} 0010 \quad 1101 \underset{1-\delta}{\overset{1-\delta}{\leftrightarrows}} 1011 \quad 1100 \underset{1+\epsilon}{\overset{1-\epsilon}{\leftrightarrows}} 1010 \quad 0101 \underset{1-\epsilon}{\overset{1+\epsilon}{\leftrightarrows}} 0011$$

$$\sigma_{\mathrm{KLS}} = 2\frac{\lambda(\rho)[1+\delta(1-2\rho)]-2\epsilon\sqrt{\rho(1-\rho)}}{\lambda(\rho)^3} \text{ with } \lambda(\rho) = \frac{1+\sqrt{1-8\epsilon\rho(1-\rho)/(1+\epsilon)}}{2\sqrt{\rho(1-\rho)}}$$

The diffusivity is given by $D_{\mathrm{KLS}}(\rho) = \frac{1}{2}\chi(\rho)\,\sigma_{\mathrm{KLS}}(\rho)$, where $\chi(\rho)$ is obtained by eliminating the parameter h between the two equations:

$$\chi = \frac{1}{4}\,\frac{1+\epsilon}{1-\epsilon}\,\frac{\cosh h}{\left(\sinh^2 h + \frac{1+\epsilon}{1-\epsilon}\right)^{3/2}}$$

$$\rho = \frac{1}{2}\left(1 + \frac{\sinh h}{\sqrt{\sinh^2 h + \frac{1+\epsilon}{1-\epsilon}}}\right)$$

(Y. Kafri et al., 2013)

Towards a General Principle for Large Deviations

The probability to observe an atypical current $j(x,t)$ and the corresponding density profile $\rho(x,t)$ during a time $L^2 T$ (L being the size of the system) is given by

$$\mathrm{Pr}\{j(x,t),\rho(x,t)\} \sim \mathrm{e}^{-L\,\mathcal{I}(j,\rho)}$$

A general principle has been found (G. Jona-Lasinio et al.), to express this large deviation functional $\mathcal{I}(j,\rho)$ as an optimal path problem:

$$\mathcal{I}(j,\rho) = \min_{\rho,j}\left\{\int_0^T dt\int_0^1 dx\,\frac{\left(j-\nu\sigma(\rho)+D(\rho)\nabla\rho\right)^2}{2\sigma(\rho)}\right\}$$

with the constraint: $\partial_t\rho = -\nabla.j$ and suitable boundary conditions.

At present, most of the available results for this variational theory are given by exact solutions of solvable models (cf ASEP).

Macroscopic Fluctuation Theory

Mathematically, one has to solve the corresponding Euler-Lagrange equations. The Hamiltonian structure is expressed by a pair of conjugate variables (p, q).
After some transformations, one obtains a set of coupled PDE's (here, we take $\nu = 0$):

$$\begin{aligned} \partial_t q &= \partial_x[D(q)\partial_x q] - \partial_x[\sigma(q)\partial_x p] \\ \partial_t p &= -D(q)\partial_{xx} p - \frac{1}{2}\sigma'(q)(\partial_x p)^2 \end{aligned}$$

where $q(x, t)$ is the density-field and $p(x, t)$ is a conjugate field.
The transport coefficients $D(q)$ and $\sigma(q)$ contain the information of the microscopic dynamics relevant at the macroscopic scale.

Macroscopic Fluctuation Theory

Mathematically, one has to solve the corresponding Euler-Lagrange equations. The Hamiltonian structure is expressed by a pair of conjugate variables (p, q).
After some transformations, one obtains a set of coupled PDE's (here, we take $\nu = 0$):

$$\begin{aligned} \partial_t q &= \partial_x[D(q)\partial_x q] - \partial_x[\sigma(q)\partial_x p] \\ \partial_t p &= -D(q)\partial_{xx} p - \frac{1}{2}\sigma'(q)(\partial_x p)^2 \end{aligned}$$

where $q(x, t)$ is the density-field and $p(x, t)$ is a conjugate field.
The transport coefficients $D(q)$ and $\sigma(q)$ contain the information of the microscopic dynamics relevant at the macroscopic scale.

- A general framework but the MFT equations are very difficult to solve in general. By using them one can in principle calculate large deviation functions directly at the macroscopic level.
- The analysis of this new set of 'hydrodynamic equations' has just begun!

Conclusions

Non-Equilibrium Statistical Physics has undergone remarkable developments in the last two decades and a unified framework is emerging.

Large deviation functions (LDF) appear as a generalization of the thermodynamic potentials for non-equilibrium systems. They satisfy remarkable identities (Gallavotti-Cohen, Jarzynski-Crooks) valid far from equilibrium.

The LDF's are very likely to play a key-role in the future of non-equilibrium statistical mechanics.

Current fluctuations are a signature of non-equilibrium behaviour. The exact results (e.g. for the Exclusion Process) can be used to calibrate the more general framework of fluctuating hydrodynamics (MFT), which is currently being developed.

FT for a system with two heat reservoirs

For a system exchanging heat with two reservoirs, the heat current j breaks time reversal. The Fluctuation Theorem takes the form, when $t \to \infty$,

$$\frac{\mathrm{Prob}\left(\frac{Q_t}{t} = j\right)}{\mathrm{Prob}\left(\frac{Q_t}{t} = -j\right)} \simeq \mathrm{e}^{\left(\frac{1}{kT_2} - \frac{1}{kT_1}\right)jt}$$

Using the definition of the Large Deviation Function, this is equivalent to the Gallavotti-Cohen relation:

$$\boxed{\Phi(j) = \Phi(-j) - \left(\frac{1}{kT_2} - \frac{1}{kT_1}\right)j}$$