On multiplicities of irreducibles in large tensor product of representations of simple Lie algebras

Olga Postnova[1], Nicolai Reshetikhin[2]

[1]Laboratory of Mathematical Problems of Physics, St. Petersburg Department of Steklov Mathematical Institute,191023, Fontanka 27, St. Petersburg, Russia;

[2]Department of Mathematics, University of California, Berkeley, CA 94720, USA & Physics Department, St. Petersburg University, Russia &KdV Institute for Mathematics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam

Asymptotic representation theory of S_N

The study of statistics in "large" natural representations goes back to works of Vershik-Kerov [1],[2] and Logan-Shepp[3]. They studied statistics of irreducible components for Plancherel measure for the symmetric group S_N as $N \to \infty$.

Representations of symmetric group are parametrized by Young diagrams $\lambda = (\lambda_1, \lambda_2, \dots \lambda_n)$.

The dimension of irreducible representation of symmetric group is given by hook-length formula:

$$\dim \tau_\lambda = \frac{N!}{\prod h_{(i,j)}}, \tag{1}$$

where the product is taken over cells (i,j) of λ. The Plancherel measure:

$$P^{(N)}(\lambda) = \frac{(\dim \tau_\lambda)^2}{N!}. \tag{2}$$

To each λ it is possible to associate a broken line function $\lambda(u)$ by switching from coordinates (i,j) to coordinates (u,v):

$$u = \frac{j-i}{\sqrt{N}} \qquad v = \frac{j+i}{\sqrt{N}}. \tag{3}$$

Then the border of the diagram λ can be represented as the broken line function $\lambda(u)$.

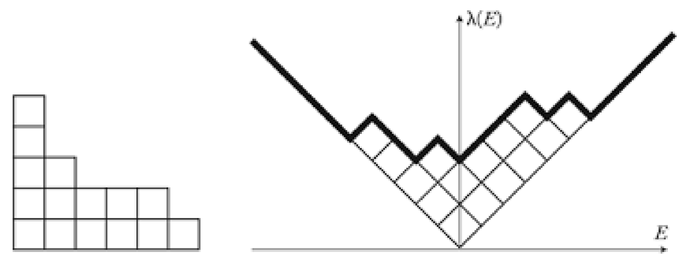

Figure 1: Young diagram

Vershik and Kerov [1],[2] have shown that as $N \to \infty$ the broken line $\lambda(u)$ converges in probability to the limit continuous function $\Omega(u)$:

$$\Omega(u) = \frac{2}{\pi}\left(x \arcsin \frac{u}{2} + \sqrt{4-u^2}\right), \; |u| \le 2, \quad \Omega(u) = |u|, \; |u| \ge 2 \tag{4}$$

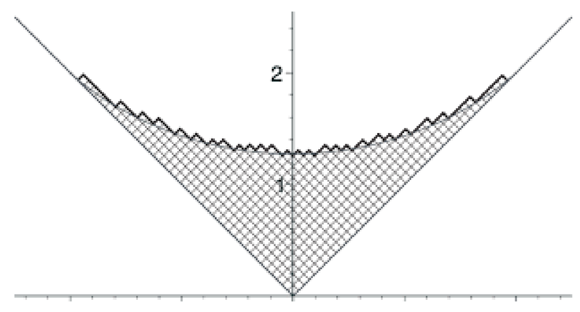

Figure 2: Large Young diagram

Probability measure on the tensor products of modules for Lie algebra A_n

Consider the space of tensors $V_{N,n} = \otimes_{k=1}^N C^n$. In this space the symmetric group S_N acts by permutation of components and the Lie group $GL(n)$ acts on each component. Due to Schur-Weyl duality

$$V_{N,n} = \bigoplus_{\lambda \in Y_{N,n}} V_\lambda^{S_N} \otimes V_\lambda^{GL(n)}, \tag{5}$$

where $Y_{N,n}$ is the space of Young diagrams with N boxes and no more than n rows. On this space the Plancherel-type measure can be introduced:

$$\mu^{(N)}(\lambda) = \frac{\dim \pi_\lambda \dim \tau_\lambda}{n^N}, \tag{6}$$

where $\dim \tau_\lambda$ is the dimension of the irreducible representation of S_N, and $\dim \pi_\lambda$ is the dimension of the irreducible representation of $GL(n)$. Asymptotic of this measure was studied in [4],[5].

- **Biane** [5] studied asymptotic at $N, n \to \infty$ with $\frac{\sqrt{N}}{n} = c$ - fixed constant. He found that the shape of Young diagrams converge in probability to the continuous limit curve $\Omega(c,u)$ which depends on the value of c.

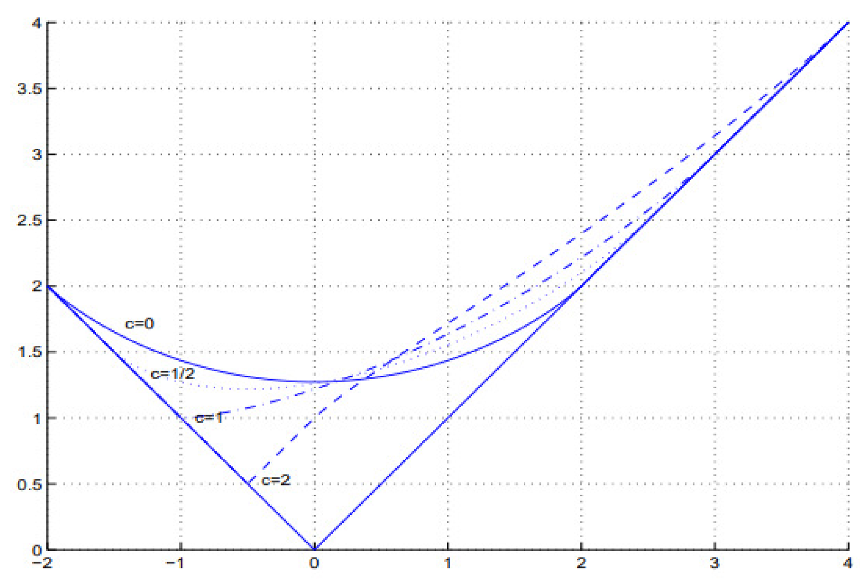

Figure 3: Limit shapes of Young diagrams for different values of c

- **Kerov** [4] studied asymptotic at $N \to \infty$ with n - fixed. He found that the measures $\mu^{(N)}(\lambda)$ converge weakly to the continuous measure

$$\phi_n(x) = \frac{n^{\frac{n(n-1)}{2}}}{1!2!\dots(n-1)!}\left(\frac{n}{2\pi}\right)^{\frac{(n-1)}{2}} \prod_{i \le j}(x_i - x_j)^2 e^{-\frac{n}{2}\sum_k x_k^2}, \tag{7}$$

where $x_k = \frac{\lambda_k - \frac{N}{n}}{\sqrt{N}}, k = 1, 2, \dots n$ are the specific values of row length of diagram λ. The vector $x = (x_1, \dots_n)$ lies in the hyper plane $H_n = \{x | \sum_{k=1}^n x_k = 0\}$

Probability measure on the tensor products of modules for Lie algebra B_n

- **Nazarov, Postnova**[6] studied the asymptotic $N \to \infty$ with n-fixed for the Plancherel-type measure for modules of Lie algebra B_n.

The decomposition of tensor powers of the spinor fundamental module V_{ω_n} of Lie algebra B_n:

$$V_{\omega_n}^{\otimes N} \cong \bigoplus_\lambda W_\lambda(N) \otimes L^\lambda, \tag{8}$$

here L_λ is the irreducible representation with the highest weight λ and $W_\lambda(N) \simeq \mathrm{Hom}_{\mathfrak{g}}(L^\lambda, V_{\omega_n}^{\otimes N})$.

The dimension $M_\lambda(N) = \dim W_\lambda(N)$ is the multiplicity of L^λ in the tensor product decomposition.

The Plancherel-type probability measure $\mu_\lambda^{(N)}$ on the space of the dominant integral weights λ:

$$\mu_\lambda^{(N)} = \frac{M_\lambda(N) \dim L^\lambda}{\dim V_{\omega_n}^N}. \tag{9}$$

In the limit $N \to \infty$ with fixed n these measures converge weakly to the continuous measure with the probability density function:

$$\phi(\{x_i\}) = \prod_{i<j}(x_i^2 - x_j^2)^2 \prod_{l=1}^n x_l^2 \exp\left(-\frac{1}{2}\sum_k x_k^2\right) \cdot \frac{2^{2n} n!}{(2n)!(2n-2)!\dots 2!}, \tag{10}$$

where $x_i = \frac{1}{\sqrt{N}} a_i$ and $a_i = \lambda_i + \rho_i$ are the shifted Euclidean coordinates on weight space of B_n, where ρ is the Weyl vector.

Large tensor products of representations of simple Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra, $V_i, \quad i = 1 \dots m$ be its finite dimensional representations and $N_k \ge 0$ be integers. Any finite dimensional representation of a simple Lie algebra is completely reducible and therefore:

$$\bigotimes_{k=1}^m V_k^{\otimes N_k} \cong \bigoplus_\lambda W_\lambda(\{V_k\}, \{N_k\}) \otimes V_\lambda. \tag{11}$$

The sum is taken over irreducible components of the tensor product, V_λ is the irreducible $\mathfrak{g}$-module with the highest weight λ and $W_\lambda(\{V_k\}, \{N_k\})$ is the "space of multiplicities":

$$W_\lambda(\{V_k\}, \{N_k\}) \simeq \mathrm{Hom}_{\mathfrak{g}}(\bigotimes_{k=1}^m V_k^{\otimes N_k}, V_\lambda). \tag{12}$$

Its dimension $m_\lambda(\{V_k\}, \{N_k\})$ is the **multiplicity** of V_λ in the tensor product. We assume $V_k = V_{\nu_k}$, otherwise $V_k \simeq \bigoplus V_\nu^{\oplus m_{\nu,k}}$.

Choose a Borel subalgebra $\in \mathfrak{g}$ and let Δ_+ be corresponding positive roots, $\mathfrak{h}$ be the corresponding Cartan subalgebra and $\alpha_1, \dots, \alpha_r$ be enumerated fundamental roots. Here $r = rank(\mathfrak{g}) = dim(\mathfrak{h})$ is the rank of the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{R}}$ be the split real form of $\mathfrak{g}$ and $\mathfrak{h}$ be its Cartan subalgebra.

We study the asymptotic behavior of multiplicities m_λ in the limit when $N_k \to \infty$ and $\lambda \to \infty$ such that $N_k = \tau_k/\epsilon$ and $\lambda = \xi/\epsilon, \quad \epsilon \to 0$, where $\tau_k \in_{\ge 0}$ and $\xi \in \mathfrak{h}^*_{\ge 0}$.

Asymptotic of the multiplicity function

Theorem

If $\xi = \epsilon\lambda$ remain regular as $\epsilon \to 0$ the asymptotic of the multiplicity has the following form

$$m_\lambda(\{V_k\}, \{N_k\}) = \epsilon^{\frac{r}{2}} \frac{\sqrt{detK}}{(2\pi)^{\frac{r}{2}}} \Delta(x) e^{-(\rho,x)} e^{\frac{1}{\epsilon}S(\tau,\xi)}(1 + \mathcal{O}(\epsilon)) \tag{13}$$

Here $x \in \mathfrak{h}$ is the Legendre image of $\xi \in \mathfrak{h}^*$, $\Delta(x)$ is the denominator in the Weyl formula for characters:

$$\Delta(x) = \prod_{\alpha \in \Delta_+} (e^{\frac{(x,\alpha)}{2}} - e^{-\frac{(x,\alpha)}{2}})$$

The function $S(\tau, \xi)$ is the Legendre transform of the function

$$f(\tau, x) = \sum_k \tau_k \ln(\chi_{\nu_k}(e^x)). \tag{14}$$

$$S(\tau, \xi) = \min_y (f(\tau, y) - (y, \xi)) = f(\tau, x) - (x, \xi), \tag{15}$$

where $(y, \xi) = \sum_{ab} y_a B_{ab} \xi_b$ and x is the critical point where the minimum is achieved. It is the unique solution to the equation:

$$\frac{\partial}{\partial x_a} f(\tau, x) = \sum_b B_{ab}\xi_b. \tag{16}$$

The matrix K is defined as

$$K_{ac} = \sum_{b,d} B_{ad}(D^{-1})_{db} B_{bc} \tag{17}$$

$$D_{ab} = \frac{\partial^2}{\partial y_a \partial y_b} f(\tau, y)|_{y=x} \tag{18}$$

where x is as above.

Asymptotic of probability measure

When $g = e^t, \quad t \in \subset \mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ the characters $\chi_{\nu_k}(e^t)$ and $\chi_\lambda(e^t)$ are positive and as a consequence of the tensor product decomposition we have the identity

$$\prod_k \chi_{\nu_k}(e^t)^{N_k} = \sum_\lambda m_\lambda(\{V_k\}, \{N_k\}) \chi_\lambda(e^t) \tag{19}$$

Therefore

$$p_\lambda = \frac{m_\lambda(\{V_k\}, \{N_k\}) \chi_\lambda(e^t)}{\prod_k \chi_{\nu_k}(e^t)^{N_k}}. \tag{20}$$

is a natural probability measure on irreducible components of tensor product: $p_\lambda \ge 0$ and $\sum_\lambda p_\lambda = 1$.

The extreme non regular case is when $t = 0$. In this case the probability distribution is given by

$$p_\lambda = \frac{m_\lambda(\{V_k\}, \{N_k\}) dim(V_\lambda)}{\prod_k dim(V_{\nu_k})^{N_k}}. \tag{21}$$

By the analogy with the left regular representation of a compact group we will call it Plancherel measure.

Theorem

1. *If $\xi = \epsilon\lambda$ remain regular as $\epsilon \to 0$ and t is regular, the asymptotic of the the probability p_λ as $\epsilon \to 0$ is:*

$$p_\lambda = \epsilon^{\frac{r}{2}} \frac{\sqrt{detK}}{(2\pi)^{\frac{r}{2}}} \frac{\Delta(x)}{\Delta(t)} e^{-(\rho, x-t)} e^{\frac{1}{\epsilon}\widetilde{S}(\tau,\xi)}(1 + \mathcal{O}(\epsilon)) \tag{22}$$

where $\widetilde{S}(\tau,\xi) = S(\tau,\xi) - f(\tau,t) + (t,\xi)$. The exponent has maximum at η which the Legendre image of t and $\widetilde{S}(\tau,\eta) = 0$.

2. *If t is regular, the asymptotic probability distribution is localized at point η with a Gaussian distribution around this point. If we rescale random variable ξ near the critical point η as $\xi = \eta + \sqrt{\epsilon}a$, then in the limit $\epsilon \to 0$ the random variable $a \in \mathbb{R}^r$ is distributed as*

$$p(a) = \frac{\sqrt{detK}}{(2\pi)^{\frac{r}{2}}} e^{-\frac{1}{2}\sum_{bc} a_b K_{bc} a_c} \tag{23}$$

Theorem

As $\epsilon \to 0$ the Plancherel measure (21) weakly converges to

$$p(a) = \frac{\sqrt{det(B)}}{(2\pi)^{\frac{r}{2}}} \prod_{\alpha > 0} \frac{(a,\alpha)^2}{(\rho,\alpha)} e^{-\frac{1}{2}\sum_{b,c} a_b B_{bc} a_c} da$$

Example

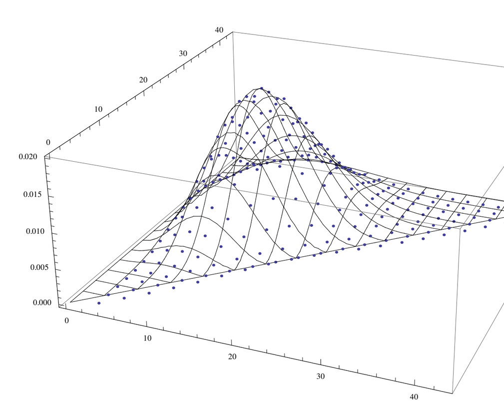

Figure 4: Probability density function for B_2

References

[1] A. Vershik and S. Kerov, Asymptotics of Plancherel measure of symmetrical group and limit form of young tables, Doklady Akademii Nauk SSSR, 233, 6, pp 1024–1027, 1977.

[2] A. Vershik and S. Kerov, Asymptotic of the largest and the typical dimensions of irreducible representations of a symmetric group, Functional analysis and its applications, 19, 1, pp. 21–31, 1985

[3] B. Logan, and L. Shepp, A variational problem for random Young tableaux, Advances in mathematics, 26, 2, pp. 206–222, 1977.

[4] S. Kerov, On asymptotic distribution of symmetry types of high rank tensors, Zapiski Nauchnykh Seminarov POMI, 155, pp. 181-186, 1986.

[5] Philippe Biane, Approximate factorization and concentration for characters of symmetric groups, arXiv: math/0006111.

[6] A. Nazarov, Postnova, The limit shape of a probability measure on a tensor product of modules on the B_n algebra, Zapiski Nauchnykh Seminarov POMI, 468, 82–97, 2018.

postnova.olga@gmail.com