Basso-Dixon Correlators in Two-Dimensional Fishnet CFT

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Abstract

We compute explicitly the two-dimensional version of Basso-Dixon type integrals for the planar 4-point correlation functions given by conformal "fishnet" Feynman graphs. These diagrams are represented by a fragment of a regular square lattice of power-like propagators, arising in the recently proposed integrable bi-scalar fishnet CFT. The formula is derived from first principles, using the formalism of separated variables in integrable $SL(2,\mathbb{C})$ spin chain. It is generalized to anisotropic fishnet, with different powers for propagators in two directions of the lattice.

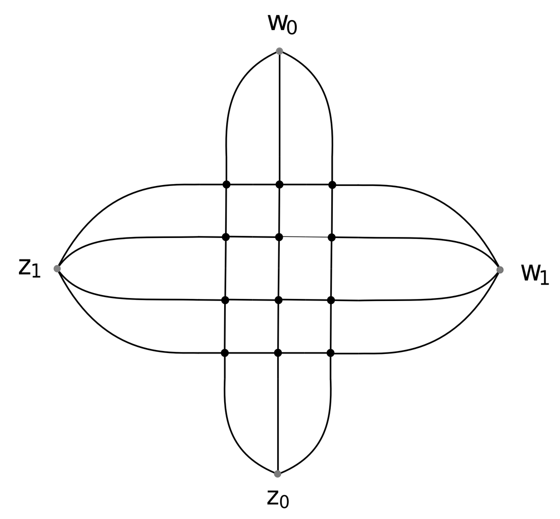
Introduction

Action of the $2D$ bi-scalar fishnet model [Kazakov, E.O. '18], with coupling ξ^2 and deformation parameter $-1/2 < \omega < 1/2$:

$$\mathcal{S} = N_c \int d^2x \; \mathrm{tr}\left[X^\dagger(-\partial^\mu\partial_\mu)^{1/2+\omega}X + Z^\dagger(-\partial^\mu\partial_\mu)^{1/2-\omega}Z + (4\pi)^2\xi^2 X^\dagger Z^\dagger X Z\right] , \qquad (1)$$

Some specific four-point correlator gets a single contribution in the planar limit, with the topology of square lattice fishnet.

$$I^{BD}_{L,N}(z_0,z_1,w_0,w_1) = \left\langle \mathrm{tr}\left(X^L(z_0)Z^N(z_1)X^{\dagger L}(w_0)Z^{\dagger N}(w_1)\right)\right\rangle . \qquad (2)$$



This class of fishnet graphs can be computed explicitly as a function of coordinates for any numbers of external fields N and L.

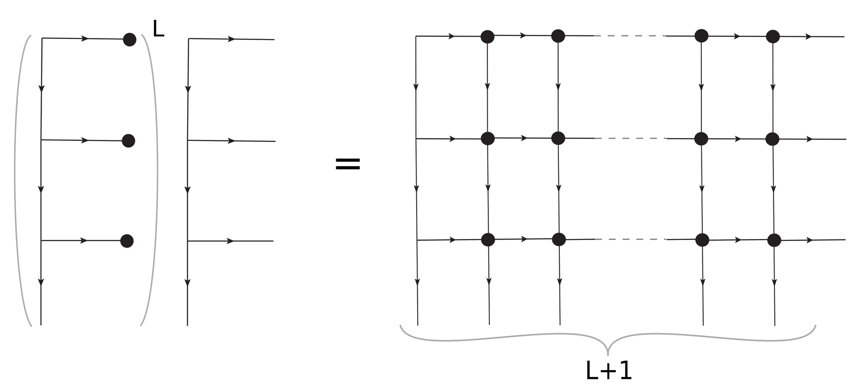
$$I^{BD}_{L,N}(z_0,z_1,w_0,w_1) = \frac{|z_1-z_0|^{(1-2\omega)N}[w_1-w_0]^{(1-2\omega)N}}{[z_0-w_0]^{(1-2\omega)(N-L)}}\,|\eta|^{(\frac{1}{2}-\omega)N}B^{(\omega)}_{L,N}(\eta) \qquad (3)$$

The dependance over cross-ratios takes the form of a determinant of matrix whose elements are derivatives of ladder diagrams, a structure of the same type as the 4D result by Basso and Dixon [Basso, Dixon '17].

$$B^{(\omega)}_{L,N}(\eta,\bar\eta) = (2\pi)^{-N}\pi^{-N^2}N!\,\underset{1\le j,k\le N}{\mathrm{Det}}\left[(\eta\partial_\eta)^{i-1}(\bar\eta\partial_{\bar\eta})^{k-1}I^{(\omega)}_{N+L}(\eta,\bar\eta)\right], \qquad \eta = \frac{z_0-w_1}{w_1-w_0}\frac{z_1-w_0}{z_0-z_1} \qquad (4)$$

Diagonalization of the graph - Separated Variables

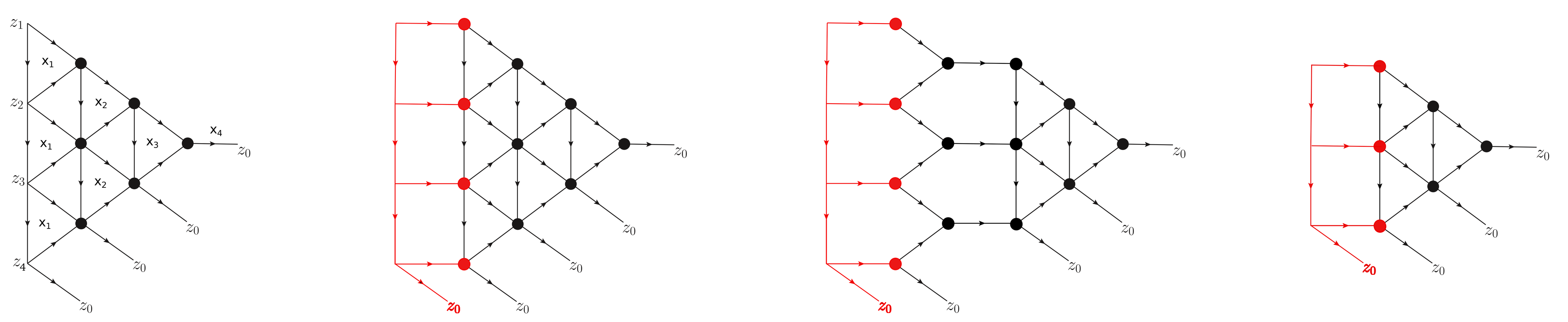
The graph can be built up starting from the iterated action of an integral kernel.



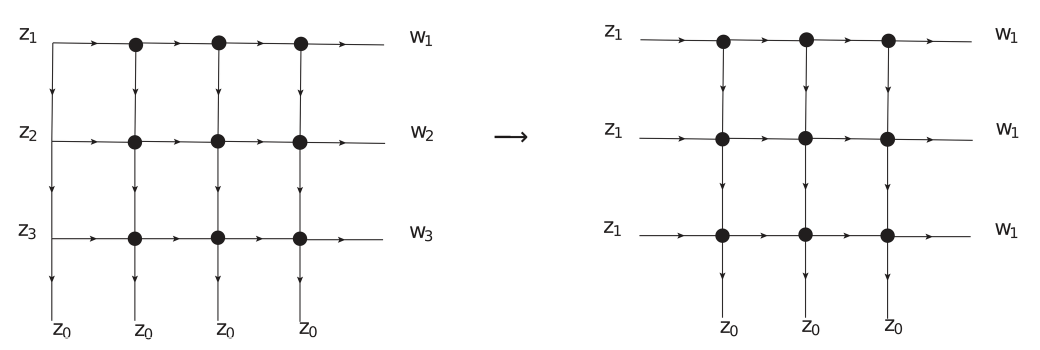
This kernel can be regarded as the transfer matrix of an open spin chain $SL(2,\mathbb{C})$ of N sites, carrying in each site a scalar irrep of the complementary series. As an integral operator it commutes with the operator $A(u)$ in the monodromy matrix of a closed $SL(2,\mathbb{C})$ chain. The construction of its eigenfunctions mimics step-by-step that for $B(u)$ operator (i.e. eigenfunctions of quantum separated variables), with the obvious replacement of the role of translations ∂_z with scalings $z\partial z$ [Derkachov,Manashov'15]. The eigenfunctions are labelled by parameters having the same spectrum of separated variables

$$x_k = \frac{n_k}{2} + i\nu_k \;\; , \;\; \bar x_k = -\frac{n_k}{2} + i\nu_k$$

where $n_k \in \mathbb{Z}$ and $\nu_k \in \mathbb{R}$, and enter as weights of propagators in the graphical picture of the eigenfunctions. The diagonalization works as an iterative procedure:



The original topology of the graph can be restored by some amputation and reduction:



and the final expression of the diagonalized graph is a sum over the spectrum of separated variables:

$$B^{(\omega)}_{L,N}(\eta) = (2\pi)^{-N}\pi^{-N^2}\int \mathcal{D}_N\boldsymbol{x}\,\prod_{k=1}^{N}[\eta]^{-x_k}\lambda^{N+L}_{\omega}(x_k)\prod_{k<j}[x_k-x_j]\,; \qquad \prod_{k<j}[x_k-x_j] = \Delta(\underline{x})^2,$$

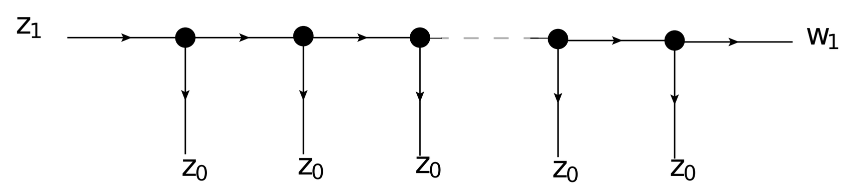
where

$$\lambda_\omega(x_k) = \pi\frac{\Gamma\left(\frac{1-2\omega}{4}-\bar x_k\right)\Gamma\left(\frac{1-2\omega}{4}+\bar x_k\right)\Gamma\left(\frac{1}{2}+\omega\right)}{\Gamma\left(\frac{3+2\omega}{4}-x_k\right)\Gamma\left(\frac{3+2\omega}{4}+x_k\right)\Gamma\left(\frac{1}{2}-\omega\right)}.$$

The square Vandermonde in the spectral measure, together with the factorization of the eigenvalues respect to separated variables, allows to rewrite the result in the form of determinant (4).

Ladders in 2D

The computation of an explicit answer for Basso-Dixon diagrams in 2D, needs the knowledge of the ladder graphs at any lenght M. Here we present the result for an isotropic fishnet $\omega = 0$.



It can be diagonalized as a the simple case $N=1$ of the more general fishnet, and written as a sum over one separated variable spectrum.

$$I_M = \int \mathcal{D}x\,[\eta]^{-x}\lambda^M(x) \propto \sum_{n\in Z}\int_{-\infty}^{+\infty} d\nu\,\frac{\Gamma^M(\frac{1}{2}-\frac{n}{2}-i\nu)\Gamma^M(\frac{n}{2}+i\nu)}{\Gamma^M(\frac{3}{2}-\frac{n}{2}+i\nu)\Gamma^M(\frac{n}{2}-i\nu)}[\eta]^{\frac{n}{2}-i\nu}$$

The integration can be explicitly carried on picking residues of the integrand and the result is expressible through ${}_MF_{M+1}$ hypergeometric functions of the cross-ratio.

$$I_M = \frac{2\pi^{M+1}}{(M-1)!\,|\eta|^{\pm\frac{1}{2}}}\;\partial^{M-1}_{\varepsilon}\Big|_{\varepsilon=0}\frac{\Gamma^M\left(\frac{1}{2}-\varepsilon\right)\Gamma^M(1+\varepsilon)}{\Gamma^M\left(\frac{1}{2}+\varepsilon\right)\Gamma^M(1-\varepsilon)}[\eta]^{\mp\varepsilon}\,\mathcal{F}^{\frac{1}{2},\frac{1}{2}}_M(\eta^{\pm1},\bar\eta^{\pm1}|\varepsilon), \qquad (5)$$

$$\mathcal{F}^{\frac{1}{2},\frac{1}{2}}_M(\eta,\bar\eta|\varepsilon) = \left|{}_{M+1}F_M\left(\frac{1}{2}-\varepsilon\cdots\frac{1}{2}-\varepsilon,\;1;1-\varepsilon\cdots1-\varepsilon\,\middle|\,\eta\right)\right|^2 \quad \text{and} \quad |\eta|\lessgtr 1,$$

showing explicit $z_1 \longleftrightarrow w_1$ symmetry

$$I_M(\eta) = I_M\left(\frac{1}{\eta}\right) \qquad (6)$$

Duality $L \longleftrightarrow N$

The Basso-Dixon fishnet can be equally computed rotating the whole diagram by $\frac{\pi}{2}$ and repeating the procedure. This means that there is duality for $\eta \leftrightarrow \frac{1}{\eta}$, $L \leftrightarrow N$, $\omega \leftrightarrow -\omega$:

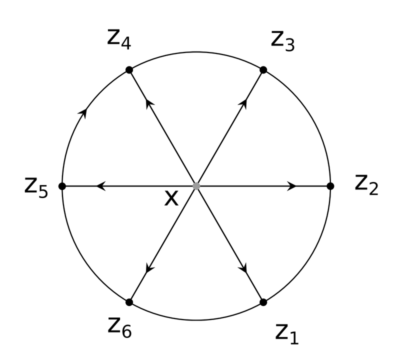
$$B^{(-\omega)}_{N,L}(1/\eta) = |\eta|^{(1/2-\omega)(N+L)-N}|1-\eta|^{-(N+L)(1-2\omega)+2N}\,B^{(\omega)}_{L,N}(\eta). \qquad (7)$$

Simple wheel graph in 2D

The computation of ladders allows to get the anomalous scaling dimension in the first perturbative order for operators $\mathrm{tr}(X^l)$ or $\mathrm{tr}(Z^l)$.

$$-\frac{l}{2}-\gamma(\xi^2) = \lim_{|x-y|\to\infty}\frac{\log\left\langle X^l(x)(X^\dagger)^l(y)\right\rangle}{\log(x-y)^2} \qquad (8)$$

This contribution correspond to the $\frac{1}{\varepsilon}$ divergence of a wheel graph with l spokes in dimensional regularization.



$$W_l(x) = \int\prod_{j=1}^{l}\frac{d^2z_j}{|x-z_{j,1}|^{1+2\omega}|z_j-z_{j+1}|^{1-2\omega}} \qquad (9)$$

The computation of the graph follows the lines of the Basso-Dixon one at $N=1$, but we take the trace of the iterative application of kernels, in order to realize the cyclic structure of wheels. The explicit answer reads ($\omega = 0$):

$$-\gamma(\xi^2)^{(1)} = \frac{2\pi^l\xi^{2l}}{(l-1)!}\,\frac{d^{l-1}}{d\varepsilon^{l-1}}\Big|_{\varepsilon=0}\frac{\Gamma^l(1+\varepsilon)\Gamma^l(1-\varepsilon)}{\Gamma^l(3/2+\varepsilon)\Gamma^l(-1/2-\varepsilon)}\left(\sum_{k=0}^{\infty}\frac{\Gamma^l(1/2+k-\varepsilon)}{\Gamma^l(1+k-\varepsilon)}\right)^2. \qquad (10)$$

What do to next

- Direct computation of 4D Basso-Dixon diagram by graph diagonalization and Separation of Variables techniques.
- Computation of Wheels and spiral graphs in $2D$ for any external states $\mathrm{tr}(X^LZ^N)$ and $\mathrm{tr}(Z^L(Z^\dagger)^N)$, in progress [Derkachov, Kazakov,E.O.].