Harmonic analysis and light-ray operators

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Outline

1. Knapp-Stein intertwiners and CFT's
2. Analyticity in spin and light-ray operators
3. OPE of averaged null energy operators

Based on arXiv:1805.00098 with D. Simmons-Duffin and work in progress with D. Simmons-Duffin, A. Zhiboedov, M. Koloğlu.

Shadow transform

Consider an object $\phi(x)$ which transforms under Euclidean conformal transformation as a scalar primary of dimension Δ and consider the following expression

$$\mathbf{S}_\Delta[\phi](x) = \int d^d y \frac{1}{|x-y|^{2\widetilde{\Delta}}} \phi(y), \qquad \widetilde{\Delta} = d - \Delta.$$

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It is convenient to rewrite it using the following notation

$$\mathbf{S}_\Delta[\phi](x) = \int d^d y \, \langle \widetilde{\phi}(x) \widetilde{\phi}(y) \rangle \, \phi(y).$$

Since the dimensions at point y add to 0, the integral is conformally-invariant.
The result transforms as a scalar operator of dimension $\widetilde{\Delta}$.

Shadow transform

This "shadow transform" can be readily generalized to arbitrary spin

$$\mathbf{S}_\Delta[\mathcal{O}]^a(x) = \int d^d y \, \langle \widetilde{\mathcal{O}}^a(x) \widetilde{\mathcal{O}}_b^\dagger(y) \rangle \, \mathcal{O}^b(y)$$

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$$(\Delta, J, \lambda) \mapsto (d - \Delta, J, \lambda^R)$$

Conformal invariance of the transform implies that this should preserve the Casimir eigenvalues, e.g. $(\lambda = 0)$

$$C_2 = \Delta(\Delta - d) + J(J + d - 2),$$
$$C_4 = J(J + d - 2)(\Delta - 1)(\Delta - d - 1).$$

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(\Delta, J, \lambda) &\mapsto (\Delta, 2-d-J, \lambda^R), \\
(\Delta, J, \lambda) &\mapsto (1-J, 1-\Delta, \lambda), \\
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including those which also mix in λ with J and Δ.

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However, we would like the weights to makes sense for $\mathrm{SO}(d+1,1)$, which means that we want (J, λ) to be dominant for $\mathrm{SO}(d)$, i.e. $J \in \mathbb{Z}_{\geq 0}$ etc. For generic Δ only shadow transform preserves this condition.

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(Sometimes when Δ is (half-)integer these reflections are fine. This situation was discussed in Vladimir's talk.)

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The requirement $J \in \mathbb{Z}_{\geq 0}$ comes from compactness of $\mathrm{SO}(d)$. In Lorentzian signature $\mathrm{SO}(d-1,1)$ is non-compact and we can make sense of $J \in \mathbb{C}$, just like $\Delta \in \mathbb{C}$ makes sense for $\mathrm{SO}(d+1,1)$.

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This adds new reflections to the restricted Weyl group, and makes it $W' = \mathrm{D}_8$. Knapp&Stein then give us 6 new integral transforms.

Continuous-spin operators

Given a traceless-symmetric operator $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$ we can encode it by a function

$$\mathcal{O}(x,z) \equiv z_{\mu_1}\cdots z_{\mu_J}\mathcal{O}^{\mu_1\cdots\mu_J}(x).$$

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It suffices to take z to be null and future-pointing.

For example, in $d=4$ setting ($z^{\pm} = z^0 \pm z^1$)

$$z^+ = 1, \quad z^- = \mathrm{z}\bar{\mathrm{z}}, \quad z^2 = \frac{\mathrm{z}+\bar{\mathrm{z}}}{2}, \quad z^3 = \frac{\mathrm{z}-\bar{\mathrm{z}}}{2i}$$

one essentially recovers the parametrization in Vladimir's talk.

Continuous-spin operators

By construction $\mathcal{O}(x, z)$ is

- a function on the forward null cone $z^2 = 0$
- homogeneous in z with degree J
- is a restriction to null cone of a polynomial in z (equivalently, it satisfies a particular differential equation)

From any such function a traceless-symmetric tensor can be recovered using Todorov operator.

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We generalize to $J \in \mathbb{C}$ by dropping the last requirement. Continuous-spin operator $\mathcal{O}(x,z)$ is

- a function on the forward null cone $z^2 = 0$
- homogeneous in z with degree $J \in \mathbb{C}$

Knapp-Stein operators in Lorentzian signature

Consider the following integral transform ("light transform")

$$\mathbf{L}[\mathcal{O}](x,z) \equiv \int_{-\infty}^{+\infty} d\alpha (-\alpha)^{-\Delta-J} \mathcal{O}(x - z/\alpha, z)$$

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Moreover, $\mathbf{L}$ converges for $\Delta+J>1$. Therefore, $\mathbf{L}$ immediately produces lots of continuous-spin operators in a unitary CFT.

A familiar example

In fact we are very much familiar with one example, $\mathbf{L}[T]$

$$\mathbf{L}[T](-\infty z, z) = \int_{-\infty}^{+\infty} d\alpha T(\alpha z, z) = \int_{-\infty}^{+\infty} dx^- T_{--}(x).$$

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Conformally-invariant statement of averaged null energy condition is then

$$\mathbf{L}[T](x, z) \geq 0.$$

For example, "conformal collider" setup of Hofman&Maldacena is $\mathbf{L}[T](\infty, z)$.

Other transforms

Define additionally

$$\mathbf{S}_J[\mathcal{O}](x,z) = \int D^{d-2}z'(-2z\cdot z')^{2-d-J}\mathcal{O}(x,z)$$

then all the transforms are

w	order	Δ'	J'	λ'
1	1	Δ	J	λ
$\mathrm{S}_\Delta \quad = \mathrm{L}\mathrm{S}_J\mathrm{L}$	2	$d-\Delta$	J	λ^R
S_J	2	Δ	$2-d-J$	λ^R
$\mathrm{S} \quad = (\mathrm{S}_J\mathrm{L})^2$	2	$d-\Delta$	$2-d-J$	λ
L	2	$1-J$	$1-\Delta$	λ
$\mathrm{F} \quad = \mathrm{S}_J\mathrm{L}\mathrm{S}_J$	2	$J+d-1$	$\Delta-d+1$	λ
$\mathrm{R} \quad = \mathrm{S}_J\mathrm{L}$	4	$1-J$	$\Delta-d+1$	λ^R
$\overline{\mathrm{R}} \quad = \mathrm{L}\mathrm{S}_J$	4	$J+d-1$	$1-\Delta$	λ^R

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w	order	Δ'	J'	λ'
1	1	d	2	0
$\mathrm{S}_\Delta = \mathrm{L}\mathrm{S}_J\mathrm{L}$	2	0	2	0
S_J	2	d	$-d$	0
$\mathrm{S} = (\mathrm{S}_J\mathrm{L})^2$	2	0	$-d$	0
L	2	-1	$1-d$	0
$\mathrm{F} = \mathrm{S}_J\mathrm{L}\mathrm{S}_J$	2	$d+1$	1	0
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$\mathbf{R}$-transform of T

We find that $w(x,z) \equiv \mathbf{R}[T](x,z)$ is a dimension -1 spin-1 operator. It is easy to check that

1. properties of $\mathbf{R}$ ensure $w(x,z) = z_\mu w^\mu(x)$
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In fact one can prove the operator identity

$$\mathbf{R}[T](x,z) = w_{AB}(x,z)L^{AB}$$

where w_{AB} is a basis of solutions to CKE and L^{AB} are the conformal generators.

A bit more on light transform

Claim: If $\Delta + J > 1$ then $\mathbf{L}[\mathcal{O}]$ kills the vacuum

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Proof: Any Wightman function $\langle 0|\cdots \mathbf{L}[\mathcal{O}](x,z)|0\rangle$ can be shown to vanish by contour deformation + an OPE argument for dropping the arc at infinity.

This can be explicitly checked in CFT three-pt functions,

$$\langle 0|\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}]|0\rangle = \langle 0|\mathbf{L}[\mathcal{O}]\mathcal{O}_1\mathcal{O}_2|0\rangle = 0,$$

while

$$\langle 0|\mathcal{O}_1\mathbf{L}[\mathcal{O}]\mathcal{O}_2|0\rangle \neq 0$$

and in general is some interesting expression.

Analyticity in spin

Four-point functions have conformal partial wave expansions

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{J=0}^{\infty}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}c(\Delta,J)\Psi_{\Delta,J}(x_1,x_2,x_3,x_4),$$

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- Singularities of $c(\Delta, J)$ for $J \in \mathbb{Z}_{\geq 0}$ are related to local primary operators of spin J.

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Caron-Huot '17 derives a formula for $c(\Delta, J)$

$$c(\Delta, J) \propto$$
$$\int d^d x_1 \ldots d^d x_4 \langle 0|[\phi(x_4), \phi(x_1)][\phi(x_2), \phi(x_3)]|0\rangle \widetilde{G}_{J+d-1,\Delta-d+1}(x_i)$$

This converges for $\Delta \in \frac{d}{2} + i\mathbb{R}$, $J > 1$ and agrees for $J \in \mathbb{Z}_{\geq 2}$ with $c(\Delta, J)$ in Euclidean CPW expansion.

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- This formula defines $c(\Delta, J)$ for $J \notin \mathbb{Z}_{\geq 0}$.
- Singularities of $c(\Delta, J)$ for $J \in \mathbb{Z}_{\geq 0}$ are related to local primary operators of spin J.
- What is the meaning of these singularities for $J \notin \mathbb{Z}_{\geq 0}$?

Analyticity in spin

Can we have non-integer spin operators $\mathbb{O}_J(x,z)$ which reduce to local operators $\mathbb{O}_J(x,z) = \mathcal{O}_J(x,z)$ at $J \in \mathbb{Z}_{\geq 0}$?

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1. We can compute conformal Casimirs of this state. Generically they will not be in the spectrum of Casimirs on the physical Hilbert space, or even in the spectrum of positive-energy irreps.
2. We can check two- and three-point functions. The three-pt functions

   $$\langle 0|\mathcal{O}_1\mathcal{O}_2\mathbb{O}_J|0\rangle \text{ and } \langle 0|\mathbb{O}_J\mathcal{O}_1\mathcal{O}_2|0\rangle$$

   fail Wightman analyticity properties, but

   $$\langle 0|\mathcal{O}_1\mathbb{O}_J\mathcal{O}_2|0\rangle$$

   is fine.

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Proposal: there exist analytic families of non-local primary operators $\mathbb{O}_{i,J}(x)$, such that

$$\mathbb{O}_{i,J}(x) = \mathbf{L}[\mathcal{O}_{i,J}](x,z) \qquad \text{for } J \in \mathbb{Z}_{\geq 0}.$$

(Note that J is not the spin of $\mathbb{O}_{i,J}$ anymore.)

(Generalized) free theories

In free theory we have a family of operators

$$\mathcal{O}_J(u,v) = :\phi(u,v)\partial_u^J\phi(u,v): + \partial_u(\cdots)$$

$$\mathbf{L}[\mathcal{O}_J](-\infty,0) = \int_{-\infty}^{+\infty} du \, :\phi(u,0)\partial_u^J\phi(u,0):$$

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Define the primary

$$\mathbb{O}_J(-\infty,0) \propto \int du ds \frac{1}{(s+i\epsilon)^{1+J}} :\phi(u+s,0)\phi(u-s,0):$$

Using that for $J \in 2\mathbb{Z}_{\geq 0}$

$$\frac{1}{(s+i\epsilon)^{1+J}} + \frac{1}{(-s+i\epsilon)^{1+J}} \propto \partial^J\delta(s)$$

We find that indeed

$$\mathbb{O}_J = \mathbf{L}[\mathcal{O}_J] \qquad \text{for } J \in 2\mathbb{Z}_{\geq 0}.$$

General CFTs

In a general CFT we define an object $\mathbb{O}_{\Delta,J}$ by

$$\mathbb{O}_{\Delta,J}(x,z) = \int d^dx_1 d^dx_2 K(x_1,x_2;x,z)\phi(x_1)\phi(x_2)$$

and the families $\mathbb{O}_{i,J}$ are obtained by

$$\mathbb{O}_{\Delta,J} \sim \frac{\mathbb{O}_{i,J}}{\Delta - \Delta_i(J)}.$$

By construction, matrix elements of $\mathbb{O}_{i,J}$ agree with matrix elements of $\mathbf{L}[\mathcal{O}_{i,J}]$ at integer spin.

General CFTs

Computing matrix elements

$$\langle\phi(x_4)\mathbb{O}_{\Delta,J}(x,z)\phi(x_3)\rangle =$$
$$\int d^dx_1 d^dx_2 K(x_1,x_2;x,z)\langle\phi(x_4)\phi(x_1)\phi(x_2)\phi(x_3)\rangle$$

we find

$$\langle\phi(x_4)\mathbb{O}_{\Delta,J}(x,z)\phi(x_3)\rangle = c(\Delta,J)\langle\phi(x_4)\mathbb{O}_{\Delta,J}(x,z)\phi(x_3)\rangle^{(0)}$$

with $c(\Delta, J)$ given by Lorentzian inversion formula.

General CFTs

Computing matrix elements

$$\langle \phi(x_4) \mathbb{O}_{\Delta,J}(x,z) \phi(x_3) \rangle = \int d^d x_1 d^d x_2 K(x_1, x_2; x, z) \langle \phi(x_4)\phi(x_1)\phi(x_2)\phi(x_3)\rangle$$

we find

$$\langle \phi(x_4) \mathbb{O}_{\Delta,J}(x,z) \phi(x_3) \rangle = c(\Delta, J) \langle \phi(x_4) \mathbb{O}_{\Delta,J}(x,z) \phi(x_3) \rangle^{(0)}$$

with $c(\Delta, J)$ given by Lorentzian inversion formula.

$$c(\Delta, J) = \frac{1}{2\pi i} \int d^d x_i \langle 0 | [\phi(x_4), \phi(x_1)][\phi(x_2), \phi(x_3)] | 0 \rangle \left[\frac{\langle \phi\phi \mathbf{L}[\mathcal{O}] \rangle^{-1} \langle \phi\phi \mathbf{L}[\mathcal{O}] \rangle^{-1}}{\langle \mathbf{L}[\mathcal{O}] \mathbf{L}[\mathcal{O}] \rangle^{-1}} \right]$$

An application: OPE of energy detectors

Consider the following expectation value

$$f(n_1, n_2) = \langle 0|\mathcal{O}^\dagger(-p)\mathbf{L}[T](\infty, z_1)\mathbf{L}[T](\infty, z_2)\mathcal{O}(p)|0\rangle$$

For $z_i = (1, n_i)$ this measures two-point function of energy flux in directions $n_i \in S^{d-2}$ in state $\mathcal{O}(p)|0\rangle$.

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We would like to understand the expansion of this correlator as $n_1 \to n_2$. Use CPW expansion on S^{d-2}

$$\begin{aligned} f(n_1, n_2) &= \int \frac{d\delta}{2\pi i} \widetilde{c}(\delta) g_\delta(n_1, n_2), \\ \widetilde{c}(\delta) &= \int d^{d-2}n_1 d^{d-2}n_2 f(n_1, n_2) g^*_\delta(n_1, n_2) \end{aligned}$$

An application: OPE of energy detectors

Consider the following expectation value

$$f(n_1, n_2) = \langle 0|\mathcal{O}^\dagger(-p)\mathbf{L}[T](\infty, z_1)\mathbf{L}[T](\infty, z_2)\mathcal{O}(p)|0\rangle$$

For $z_i = (1, n_i)$ this measures two-point function of energy flux in directions $n_i \in S^{d-2}$ in state $\mathcal{O}(p)|0\rangle$.

We would like to understand the expansion of this correlator as $n_1 \to n_2$. Use CPW expansion on S^{d-2}

$$\begin{aligned} f(n_1, n_2) &= \int \frac{d\delta}{2\pi i}\widetilde{c}(\delta)g_\delta(n_1, n_2), \\ \widetilde{c}(\delta) &= \int d^{d-2}n_1 d^{d-2}n_2 f(n_1, n_2) g^*_\delta(n_1, n_2) \end{aligned}$$

Comparing this inversion integral with generalized Lorentzian inversion formula we find

$$\widetilde{c}(\delta) = c(\Delta = \delta + 1, J = 3)$$

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Using $\widetilde{c}(\delta) = c(\Delta = \delta + 1, J = 3)$ and

$$f(n_1, n_2) = \int \frac{d\delta}{2\pi i} \widetilde{c}(\delta) g_\delta(n_1, n_2)$$

we can deform the contour and write OPE expansion for $f(n_1, n_2)$, which is equivalent to

$$\mathbf{L}[T](\infty, z_1)\mathbf{L}[T](\infty, z_2) = \sum_i \mathcal{B}(z_1, z_2, \partial_{z_2})\mathbb{O}_{i,3}(\infty, z_2).$$

Summary

- Tools of harmonic analysis on Lorentzian conformal group are essential for understanding Lorentzian inversion formulas and analyticity in spin
- Using these tools we have found a plausible story for spin-analytic families of non-local light-ray operators
- These operators control things such as Regge limit and OPE of average null energy operators

Summary

- Tools of harmonic analysis on Lorentzian conformal group are essential for understanding Lorentzian inversion formulas and analyticity in spin
- Using these tools we have found a plausible story for spin-analytic families of non-local light-ray operators
- These operators control things such as Regge limit and OPE of average null energy operators

- What is the general analytic structure of $c(\Delta, J)$ for $J \notin \mathbb{Z}_{\geq 0}$?
- Is our construction rigorously well-defined in a general non-perturbative CFT?
- Can we put this theory to a test in a non-perturbative CFT like 3d Ising?

Fin

Thank you!