Virasoro symmetry and Coulomb gas integrals in higher dimensions

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Paolo Furlan, VP : arxiv:1806.03270,
(and Mod.Phys.Lett. A (1989), hep-th/0409213 (2004))

Outline

- Briefly on earlier work on Virasoro symmetry in higher dimensional CFT
- Liouville theory in $d = 2h$ space time dimensions
- 3-point function of vertex operators - Coulomb gas integrals
- 3-point function - extension to three arbitrary charges for different regions of parameters (analogs of real $c>25$ and $c<1$ regions)
- Open questions

Observation: [P. Furlan,VP, 1989]

The free field realization of the $c < 1$ minimal models in 2d CFT - Coulomb gas with a background charge - admits a generalization to arbitrary dimension d =2h of space time: based on a subcanonical logarithmic field

$$\frac{2}{(4\pi)^h\Gamma(h)}(-\Box\,)^h < \phi(x)\phi(0) >= \delta^{2h}(x)$$

Integrals interpreted as correlators of **vertex operators** $V_\alpha(x) = e^{2\alpha i\phi(x)}$

Fusion rules analogous to the 2d case, e.g.,

$$V_\alpha\; V_{\alpha_f} \to V_{\alpha+\alpha_f} + V_{\alpha-\alpha_f} \qquad \triangle\,(\alpha_f) = 2h\,\triangle_{2,1}$$

Further confirmed in the example of 2 integral 4-point function - two screening charges of different or the same type, analyzed in Mellin representation. “2h minimal model” , singularities. **Non-unitary theory.**

Hidden **Virasoro algebra** responsible, V_α - primary fields [Furlan,VP, 2004]

Two realizations, one with central charge and eigenvalue of $\mathbf{L}_0$ independent of h

Applied to 3-point functions, the оператор related to the level 2 singular vector

$$< \alpha \,|\, V_{\alpha_1}(x)\Big(t\,\mathbf{L}_{-1}^2(x) - \mathbf{L}_{-2}(x) \Big) \,|\, \triangle\,(\alpha_{2,1}) > \; = 0$$

reproduces the two term fundamental fusion rule; for the 4-point

$$< \alpha \,|\, V_{\alpha_2}(x_2) V_{\alpha_1}(x_1)\Big(t\,\mathbf{L}_{-1}^2(x) - \mathbf{L}_{-2}(x) \Big) \,|\, \triangle\,(\alpha_{2,1}) > \; = 0$$

with $x = x_1$, or $x = x_2$ - leads to the system of two **Appell - Kampe de Feriet** diff eqs for the double series hypergeometric function of type $\mathbf{F}_4(\alpha, \beta, \gamma, \gamma'; x, y)$ (in 2d reduces to two hypergeometric eqs.)

paper rejected: "the model has no obvious physical motivation/application, e.g. for a N=4 supersymmetric theory…"

The model was recently reconsidered in the noncompact region (c>25) in a broader context [Levy, Oz, 2018] "Liouville CFT in higher dimensions", motivated by applications to the **Theory of fluid turbulence**

The Liouville action and its dual in $d = 2h$ space time dimension

$$S^w = \frac{1}{(4\pi)^h \Gamma(h)} \int d^{2h}x \sqrt{g} \left(\phi(-\Box)^h \phi + 2Q_h(w)\phi\, \mathcal{G} \right) + \mu_w \int d^{2h}x \sqrt{g}\, e^{2w\phi}$$

$$w = b, \mu_b = \mu,\ or,\ w = \frac{h}{b},\ \mu_{h/b} = \tilde{\mu}$$

$$Q_h(b) = b + \frac{h}{b} = \sqrt{h}(\bar{b} + \frac{1}{\bar{b}}) = \sqrt{h}Q(\bar{b})$$

On the 2h sphere S^{2h}

$$\frac{1}{(4\pi)^h \Gamma(h)} \int d^{2h} \sqrt{g}\, \mathcal{G}(x) = 1$$

reference metric locally flat with the only singularity of the curvature related factor $\mathcal{G}(x)$ localized at a point at infinity; equivalent to the insertion of vertex operator $V_{-Q_h} = e^{-2Q_h\phi}$

$$\int d^2x\, V_b(x)\,, \quad \int d^2x\, V_{\frac{h}{b}}(x)$$

vertex operators of dim 2h

$$V_\beta = e^{2\beta\phi}\,,\ 2 \triangle_h(\beta) = 2\beta(Q_h - \beta) = 2h \triangle(\bar{\beta}) = 2h\bar{\beta}(Q(\bar{b}) - \bar{\beta})$$

Problem : Compute the 3-point function

$$C(\alpha_1, \alpha_2, \alpha_3) = < \alpha_3 \,|\, V_{\alpha_2}(e) \,|\, \alpha_1 >$$

[Levy-Oz]: correlator in the light charge semiclassical limit :

[2d: ZZ (Zamolodchikov^2), 1996]

$$b \to 0, \quad \alpha_a = b\sigma_a, \quad \sigma_a = \text{finite}$$

Coulomb gas computation - charges restricted - charge conservation condition

$$\boxed{\sum_a \alpha_a + sb = Q_h(b) = \frac{h}{b} + b\,, \quad \text{s screening charges} \quad \int d^2x V_b(x)}$$

$\Rightarrow$ a **class of conformally invariant integrals in d=2h**

generalization of the Dotsenko-Fateev 2d volume integral formula

$$I_s(p_1, p_2, p_3)(x) = \int d\mu_s(t)\, D_s^{-2b^2}(t) \prod_{i=1}^{s} |\, t_i \,|^{2p_1} \,|\, t_i - x \,|^{2p_2}\,, \quad p_a = -\, 2\alpha_a b$$

$$d\mu_s(t) = \frac{1}{\pi^{hs} s\,!} \prod_{i=1}^{s} d^{2h}(t_i)\,, \quad D_s(t) = \prod_{1 \le i < j \le s} |\, t_i - t_j \,|^2$$

Derivation based on a formula generalising the BF formula

$2h$=2: [Baseilhac, Fateev, 1998][Fateev, Litvinov, 2007]….

$$\int d\mu_n(y) D_n^h(y) \prod_{i=1}^{n} \prod_{j=1}^{n+m+1} |y_i - t_j|^{2p_j} = \prod_{j=1}^{n+m+2} \frac{1}{\gamma_h(-p_j)} \frac{1}{\gamma_h(h(n+1) + \sum_j p_j)} \times \qquad (*)$$

$$\prod_{i<j}^{n+m+1} |t_{ij}|^{2h+2p_i+2p_j} \int d\mu_m(u) D_m^h(u) \prod_{i=1}^{m} \prod_{j=1}^{n+m+1} |u_i - t_j|^{-2h-2p_j}$$

$$\boxed{\gamma_h(\delta) := \frac{\Gamma(h-\delta)}{\Gamma(\delta)}}$$

Two equivalent free field realizations of the *N*-point correlator of vertex operators related by $V_{\alpha_j}(t_j) = r(\alpha_j) V_{Q_h - \alpha_j}(t_j)$

with **integer number** of screening charges ***m*** and ***n*** **-** only possible for $N = m + n + 2$ and fixed value of the parameter $w \to 2w^2 = -h$, i.e. $b^2 = -\frac{h}{2}$, or, $b^2 = -2h$

In this case the reflection factor is computed from N=3, m=0, n=1

Take m=0, n=s-1, all $p_i = -h - b^2$, getting $D_s^{-h-2b^2}(t)$ from the factor in the r.h.s., represented by the integral in the l.h.s

Allows to derive and solve a recursion relation for the 3-point Coulomb gas integrals

$$I_s(p_1, p_2, p_3)(x) = \text{const}\ (x^2)^{h+p_1+p_2} I_{s-1}(p_1 - b, p_2 - b, p_3)(x)$$

$$\mathbf{C}_s(\beta_1, \beta_2, \beta_3) = \frac{1}{\gamma_h^s(-b^2)} \prod_{k=0}^{s-1} \frac{\gamma_h((k-s)b^2)}{\prod_{a=1}^{3} \gamma_h(2\beta_a b + kb^2)}, \qquad \sum_a \beta_a - Q_h = -sb$$

Similarly for the dual correlator

$$\tilde{\mathbf{C}}_s(\beta_1, \beta_2, \beta_3)\,, \text{ with } \quad \sum \beta_a + s\frac{h}{b} = Q_h\,;\ b \to \frac{h}{b}$$

Next step - analytic continuation, getting rid of the restriction on the 3 charges, generalization of the 2d DOZZ formula for the Liouville 3-point constant

Barnes double Gamma function $\Gamma_b(x) = \Gamma_{\frac{1}{b}}(x)$ poles at $x = -nb - m/b\,,\ n, m \in \mathbb{Z}_{\geq 0};$

$$\frac{\Gamma_b(x+b^\epsilon)}{\Gamma_b(x)} = \sqrt{2\pi}\frac{b^{\epsilon(b^\epsilon x-\frac{1}{2})}}{\Gamma(b^\epsilon x)}\,, \qquad \epsilon = \pm 1$$

$$\Upsilon_b^{(h)}(x) := \frac{1}{\Gamma_b(x)\Gamma_b(Q_h - x)} = \Upsilon_b^{(h)}(Q_h - x)$$

$\Rightarrow$ functional relations

$$\boxed{\begin{aligned} \frac{\Upsilon_b^{(h)}(x+b)}{\Upsilon_b^{(h)}(x)} &= b^{h-2bx}\gamma_h(xb) \\ \frac{\Upsilon_{\frac{h}{b}}^{(h)}(x+\frac{h}{b})}{\Upsilon_{\frac{h}{b}}^{(h)}(x)} &= (\frac{h}{b})^{h-2\frac{h}{b}x}\,\gamma_h(x\frac{h}{b}) \end{aligned}}$$

$\dfrac{\Upsilon_b^{(h)}(x+\frac{h}{b})}{\Upsilon_b^{(h)}(x)}$ is computed applying h times the second functional relation for ε =1

For h=1 these two functions $\Upsilon_b^{(h)}(x)\,,\ \Upsilon_{\frac{h}{b}}^{(h)}(x)$ coincide

We get $\beta_{123} = \beta_1 + \beta_2 + \beta_3$

$$C(\beta_1,\beta_2,\beta_3) = \frac{\Gamma(h)}{b^{h-1}}\left(\frac{-\pi^h \mu b^{2h-2b^2}}{b^{4h}\gamma_h(-b^2)}\right)^{\frac{Q_h-\beta_{123}}{b}} \prod_{k=1}^{3} \frac{\Upsilon_b^{(h)}(2\beta_k)}{\Upsilon_b^{(h)}(\beta_{123}-2\beta_k)} \frac{\Upsilon_b^{(h)}(b)}{\Upsilon_b^{(h)}(\beta_{123}-Q_h)}$$

$$\text{s.t. } Res_{\beta_{123}-Q_h=-sb}\, C(\beta_1,\beta_2,\beta_3) = (-\pi^h\mu)^s\, C_s(\beta_1,\beta_2,\beta_3)$$

while starting from the dual 3-point Coulomb gas correlator

$$\tilde{C}(\beta_1,\beta_2,\beta_3) = \frac{\Gamma(h)}{(\frac{h}{b})^{h-1}}\left(\frac{-\pi^h \tilde{\mu}(\frac{h}{b})^{2h-2(\frac{h}{b})^2}}{(\frac{h}{b})^{4h}\gamma_h(-(\frac{h}{b})^2)}\right)^{\frac{b(Q_h-\beta_{123})}{h}} \prod_{k=1}^{3} \frac{\Upsilon_{\frac{h}{b}}^{(h)}(2\beta_k)}{\Upsilon_{\frac{h}{b}}^{(h)}(\beta_{123}-2\beta_k)} \Upsilon_{\frac{h}{b}}^{(h)} \frac{\Upsilon_{\frac{h}{b}}^{(h)}(\frac{h}{b})}{(\beta_{123}-Q_h)}$$

$$Res_{\beta_{123}-Q_h=-s\frac{h}{b}}\, \tilde{C}(\beta_1,\beta_2,\beta_3) = (-\pi^h\tilde{\mu})^s\, \tilde{C}_s(\beta_1,\beta_2,\beta_3)$$

Starting from the two Coulomb gas 3-point expressions we get different unrelated analytic continuations expressed by the two different functions $\Upsilon_b^{(h)}, \Upsilon_{\frac{h}{b}}^{(h)}$

The two Liouville theories with interaction terms defined by the two dual screening charges produce different correlators for h>1, related by $b, \mu \to \frac{h}{b}, \tilde{\mu}$

The theory is **not self-dual for *h*>1** - even if we impose some relation between the two coupling constants as in the h=1.

This feature of the h>1 theory is reflected also in the semiclassical light charge limit $b \to 0\,, \beta_a = b\,\sigma_a\,, \;\; \sigma_a - \text{finite}$

The fixed area correlator

$$C^A(b\sigma_1, b\sigma_2, b\sigma_3) = (\mu A)^{\frac{\sigma b - Q_h}{b}} \frac{C(b\sigma_1, b\sigma_2, b\sigma_3)}{\Gamma(\frac{\sigma b - Q_h}{b})}$$

$$\to c_h(b) \left(\frac{A}{\pi^h}\right)^{\frac{\sigma b - Q_h}{b}} \prod_{k=1}^{3} \frac{\Gamma(\sigma - 2\sigma_k)}{\Gamma(2\sigma_k)} \Gamma(\sigma - 1)\,, \quad \sigma = \sigma_{123}$$

For h = 1 reproduces the semiclassical fixed area correlator in [ZZ]

The (properly normalised) dual correlator $\tilde{C}(b\sigma_1, b\sigma_2, b\sigma_3)$ produces similar expression but the Gamma's depend on h :

$$\Gamma(\sigma - 2\sigma_a) \;\to\; \Gamma(h\,(\sigma - 2\sigma_a))\,,\ \text{etc}\,.$$

Comparison with [Levy,Oz] - qualitatively - the second formula with h-dependent arguments, if some relation for the two coupling constants is postulated, generalising the case h=1, but the coefficient is rather the one in the first formula.

Compact ("matter") region c<1

The Coulomb gas representation for the correlator of the vertex operators

$V_e^{(M)}(x) = e^{2e\, i\chi(x)}$ is given by changing

$$b^2 \to -b^2 \quad \text{while} \quad -2\beta_a b \to -2e_a b\,, \quad 2\triangle^{(M)}(e) = 2e(e - e_0^{(h)})\,, \quad e_0^{(h)} = \frac{h}{b} - b$$

The 3-point constant for three arbitrary charges is

$$C^{(M)}(e_1, e_2, e_3) = \left(-\pi^h \mu_{_M} \frac{b^{-2be_0^{(h)}}}{\gamma_h(b^2)}\right)^{\frac{e_{123}-e_0^{(h)}}{b}} \prod_{k=1}^{3} \frac{\Upsilon_b^{(h)}(e_{123} - 2e_k + b)}{\Upsilon_b^{(h)}(2e_k + b)} \frac{\Upsilon_b^{(h)}(e_{123} - e_0^{(h)} + b)}{\Upsilon_b^{(h)}(b)}$$

$$\text{s.t.}\ C^{(M)}(e_1, e_2, e_3)_{|e_{123}-e_0^{(h)}=sb} = C_s^{(M)}(e_1, e_2, e_3)\,, \quad C_0^{(M)}(e_1, e_2, e_3) = 1$$

i.e., reproduces the Coulomb gas expression; effectively one uses the special function as a shorthand notation and substitutes at the end the expression for the integer s; similarly for b-> h/b

Back to the **basic generalized BF formula** - can we check it ?

for m=0 simplifies

$$I_n = \int d\mu_n(y) D_n^h(y) \prod_{i=1}^{n} \prod_{j=1}^{n+1} |y_i - t_j|^{-2\delta_j} = \prod_{j=1}^{n+2} \frac{1}{\gamma_h(\delta_j)} \prod_{i<j}^{n+1} |t_{ij}|^{2h-2\delta_i-2\delta_j}$$

Ex. : **n=2** - in the l.h.s.: 2 integrals, 4-point function (with one of the coordinates taken to infinity)

$$\textit{factor} \;\; D_2^h(y) = (y_{12}^2)^h \text{ polynomial, expand } |(y_1 - t_1) - (y_2 - t_1)|^{2h}$$

and replace by derivatives with respect to $\mathbf{t_1}$ distributed to the two integrals

$$\Rightarrow I_2 = \sum D\, I_1(\delta_1 - h) D\, I_1(\delta_1 - h)$$

$$I_1(\delta_1 - h) = I_1(\delta_1 - h, \delta_2, \delta_3, \delta_4)\,(u,v)\,, \;\; \sum_a \delta_a - h = 2h$$

the simplest 1-integral conf. 4-point function

- The result so far is inconclusive - **coeff in the r.h.s. reproduced** in the leading order, but vanishing of the remaining terms still not obvious - relevant expansion of the blocks?

- Another check - analyze the crossing relation with two arbitrary coefficients - would lead to a functional equation for their ratio generalising the 2d case

Conclusions:

- The Liouville theory in higher dimensions shares many of the features of the 2d case, however the selfduality property is lost.
- The particular Coulomb gas integrals provide new examples of computable conformally invariant integrals.
- The basic generalized BF formula still needs more checks.
- The possible applications of this non-unitary theory may justify further studies

recently - supersymmetric extension [Levy, Oz, Raviv-Moshe, 2018]

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