2D-CFT's

The blossoming of the 80's

Jean-Bernard Zuber (LPTHE, Sorbonne Université)

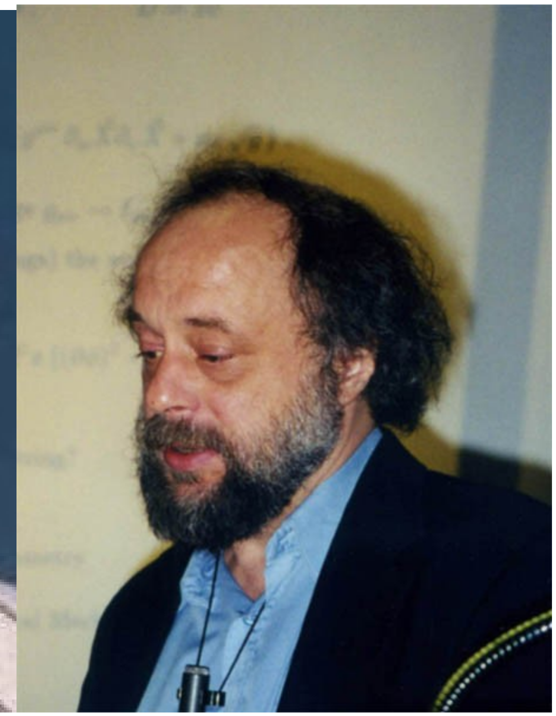
Saclay, 5 December 2018

Thirty five years after...

Thirty five years after. . . BPZ 1983 papers.

Alexander Belavin

Alexander Polyakov



Alexander Zamolodchikov

Journal of Statistical Physics, Vol. 34, Nos. 5/6, 1984

Infinite Conformal Symmetry of Critical Fluctuations in Two Dimensions

A. A. Belavin,[1] A. M. Polyakov,[1] and A. B. Zamolodchikov[1,2]

Received October 27, 1983

We study the massless quantum field theories describing the critical points in two dimensional statistical systems. These theories are invariant with respect to the infinite dimensional group of conformal (analytic) transformations. It is shown that the local fields forming the operator algebra can be classified according to the irreducible representations of the Virasoro algebra. Exactly solvable theories associated with degenerate representations are analized. In these theories the anomalous dimensions are known exactly and the correlation functions satisfy the system of linear differential equations.

KEY WORDS: Second order phase transitions; two-dimensional systems; operator algebra; conformal symmetry; Vivasoro algebra; Kac formula.

According to the scaling hypothesis, fluctuations of order parameters right at the point of a second-order phase transition possess invariance under the scaling transformations

$$\xi^a \to \lambda \xi^a \tag{1}$$

where ξ^a are the coordinates; $a = 1, 2, \ldots, \mathscr{D}$. In quantum field theory, taken as a mathematical tool for the theory of second-order phase transitions, the invariance is equivalent to the vanishing of the trace of the

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INFINITE CONFORMAL SYMMETRY IN TWO-DIMENSIONAL QUANTUM FIELD THEORY

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We present an investigation of the massless, two-dimensional, interacting field theories Their basic property is their invariance under an infinite-dimensional group of conformal (analytic) transformations It is shown that the local fields forming the operator algebra can be classified according to the irreducible representations of Virasoro algebra, and that the correlation functions are built up of the "conformal blocks" which are completely determined by the conformal invariance Exactly solvable conformal theories associated with the degenerate representations are analyzed In these theories the anomalous dimensions are known exactly and the correlation functions satisfy the systems of linear differential equations

1. Introduction

Conformal symmetry was introduced into quantum field theory about twelve years ago due to the scaling ideas in the second-order phase transition theory (see [1] and references therein). According to the scaling hypothesis, the interaction of the fields of the order parameters in the critical point is invariant with respect to the scale transformations

$$\xi^a \to \lambda \xi^a, \tag{1.1}$$

where ξ^a are the coordinates, $a = 1, 2, \dots, D$. In the quantum field theory the scale symmetry (1.1) takes place provided the stress-energy tensor is traceless

Thirty five years after... BPZ 1983 papers.

CFT didn't start with those papers ...

2D-Conformal Field Theories (CFT): Quantum Field Theories covariant under conformal transfos. In 2d: analytical changes of the variable $z = x_1 + \mathrm{i}\, x_2$, enforced by action of Virasoro algebra (or some "extended chiral algebra" $\mathcal{A} \supset$ Vir)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)$$

(a quantum realization of $\ell_z = -z^{n+1}\frac{\partial}{\partial z}$), $c =$ "central charge".

The old ingredients:

A. Polyakov's 1970 paper: Euclidean + scale invariance imply local scale, *i.e.,conformal* invariance

$\Rightarrow$ d-dimensional CFT : see the lectures by G. Mack and I. Todorov

Dual models: G. Veneziano 1968
Conformal invariance from reparametrization of the world sheet.
The Virasoro algebra: [M. Virasoro 1969 (+ J. H. Weis)],

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)$$

(unknowing of Gelfand–Fuchs' 1968 paper on the central extensions of the Witt algebra)

Many concepts to be rediscovered in the 80's already found then:

Vertex operators $V(z) =: \exp \mathrm{i}\, Q\, \phi(z) :$, [S.Fubini–G. Veneziano 1969]

Superconformal algebra: [P. Ramond; A. Neveu–J. Schwarz, 1971],

Conformal covariant fields ("primary"): [J.-L.Gervais & B. Sakita, 1971]

2D-current algebras $[J_{a,m}, J_{b,n}] = f^c_{ab} J_{c,m+n} + k\delta_{ab}\delta_{m+n,0}$, Sugawara construction $L_n \propto \sum : J_{a,m} J_{a,n-m} :$, coset construction. . . [K.Bardacki–M.Halpern, 1971]

Conformal Ward identities, OPE, etc, [S.Ferrara– A.Grillo–R.Gatto–G.Parisi '74]
Modularity and Counting of states:
[L.Brink–H.B.Nielsen 1973, W. Nahm 1974-76; C. Thorn 1980]

Unitarity constraint on the central charge
$c \geq \frac{1}{2}$ M. Lüscher and G. Mack, 1976 . . . unpublished !

See "The Birth of String Theory" [A. Cappelli, E. Castellani, F. Colomo, P. Di Vecchia] for a historical survey.

What made BPZ's 1983 papers possible?

Renewed interest in string theory (Polyakov 1981 papers on Liouville, [M. Green–J. Schwarz 1982-84]), and in anomalies [B.Zumino, R. Stora,'83], WZWN model, bosonization. . . [S. Novikov '82, A.Polyakov–P.Wiegmann '83, E.Witten '83]

Renewed interest in critical 2D statistical models [B. Nienhuis 1983]

New ingredient: Mathematics:

representation theory of Vir: [V. Kac 1979, B. Feigin–D. Fuchs 1982]

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V. Kac, B. Feigin–D. Fuchs: highest weight representation of Vir:

$$\mathcal{M}_{(c,h)} = \mathrm{Span}\{\underbrace{L_{-1}^{\alpha_1} L_{-2}^{\alpha_2} \cdots L_{-m}^{\alpha_m}|h\rangle}_{\text{“level” } n=\sum j\alpha_j} \;\Big|\; L_0|h\rangle = h|h\rangle \;;\, L_{n>0}|h\rangle = 0\}$$

is reducible iff $c = 1 - \frac{6}{x(x+1)}, \quad h = h_{r,s} := \frac{(r(x+1)-sx)^2-1}{4x(x+1)}, \quad x \in \mathbb{C},\ r,s \in \mathbb{N}.$
(roots of Kac' determinant)

Different patterns of reducibility and of the corresponding irreducible rep $\mathcal{V}_{(c,h)} = \mathcal{M}_{(c,h)}/\cdots$ depending whether $x \notin \mathbb{Q}$ or $x = \frac{p'}{p-p'} \in \mathbb{Q}\ldots$

Ensuing character formulae [A. Rocha-Caridi 1984; V. Dobrev 1985]

$$\chi_h(q) := \mathrm{tr}_{\mathcal{V}_{(c,h)}} q^{L_0 - \frac{c}{24}} = q^{h-\frac{c}{24}} \sum_{n=0}^{\infty} \#_n\, q^n = \frac{q^{h-\frac{c}{24}} - \cdots}{\prod(1-q^n)}$$

BPZ:

basic formalism of CFT:

– complex coordinates $z = x_1 + \mathrm{i}\, x_2$ and $\bar{z} = x_1 - \mathrm{i}\, x_2$ “decouple” : 2 copies of Vir

– energy–momentum tensor $T(z), \bar{T}(\bar{z})$ and its anomalous conformal transformations

– conformal Ward identities; Virasoro action on fields = differential operators

– “primary” (aka “ancestor”) fields ϕ and their covariant transformations: $\tilde{\phi}(z,\bar{z}) = \left(\frac{dz}{d\zeta}\right)^{h} \left(\frac{d\bar{z}}{d\bar{\zeta}}\right)^{\bar{h}} \phi(\zeta,\bar{\zeta})$

– primary field correlators determine those of their descendent fields $L_{-1}^{\alpha_1} L_{-2}^{\alpha_2} \cdots \phi$

– OPE and conformal bootstrap

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OPE and conformal bootstrap

New:

– $\exists$ "Degenerate" (reducible) representations of Vir: quotienting out the "null fields" leads to (partial) differential equations satisfied by the correlators

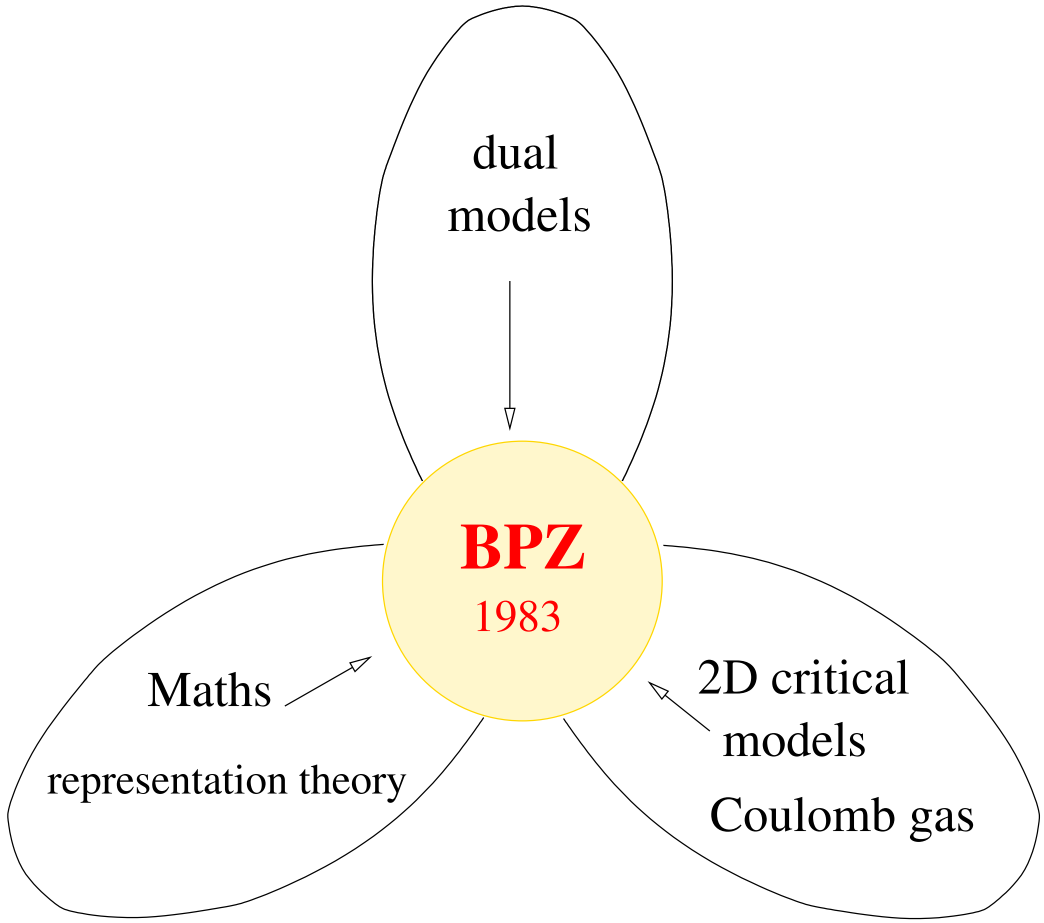
– For $c < 1$, $\exists$ "minimal models" $\mathcal{M}(p,p')$ with a *finite* number of primaries:

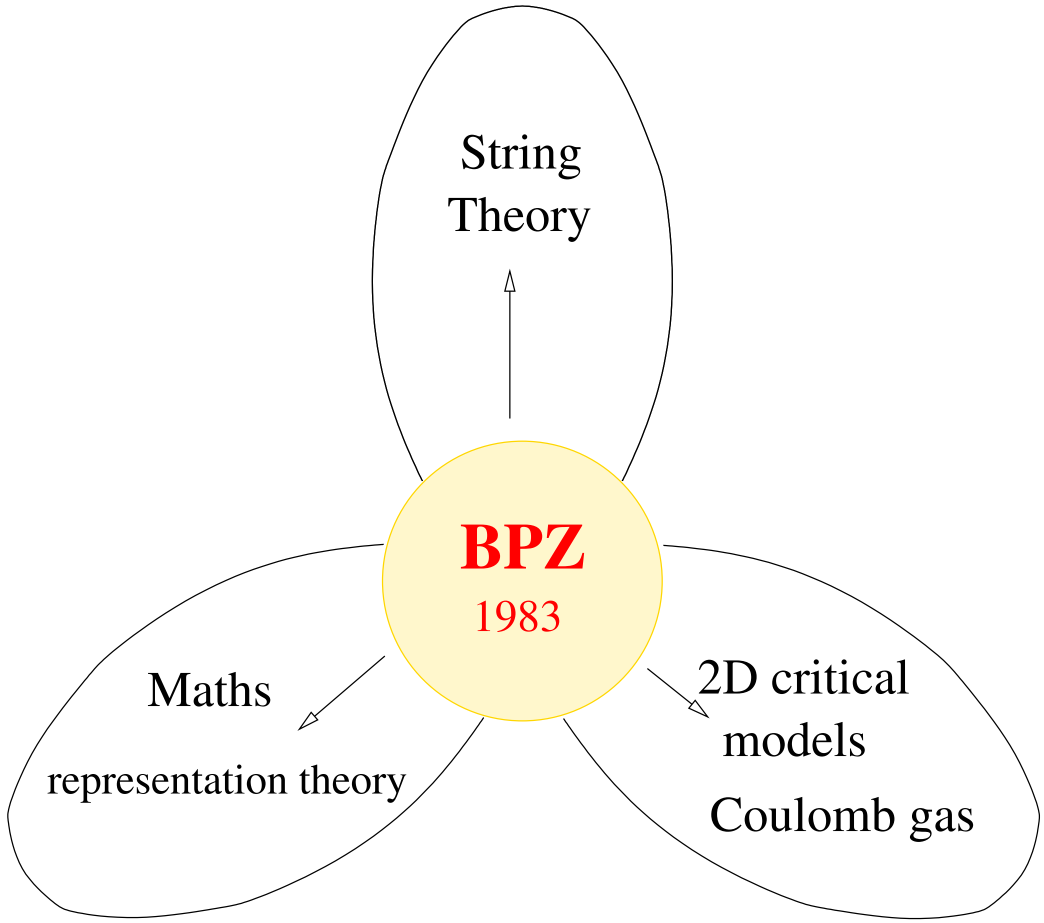
$c = 1 - \frac{6(p-p')^2}{p\,p'}$, $\qquad h_{r,s} = \frac{(r\alpha_+ + s\alpha_-)^2 - (\alpha_+ + \alpha_-)^2}{4}$, $1 \le r \le p'-1,\ 1 \le s \le p-1$

$\alpha_\pm = \pm(p/p')^{\pm\frac{1}{2}}$ $\qquad\qquad = \frac{(rp-sp')^2-(p-p')^2}{4pp'}$

– OPA closes on this finite number of fields

– Ising correlators $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\epsilon\epsilon\sigma\sigma\rangle$, etc satisfy hypergeometric differential equations


dual
models
BPZ
1983
Maths
representation theory
2D critical
models
Coulomb gas


String
Theory
BPZ
1983
Maths
representation theory
2D critical
models
Coulomb gas

Why is BPZ's 1983 paper regarded as a new start ?

In the aftermath of BPZ: explosion of activity (and # papers!) in

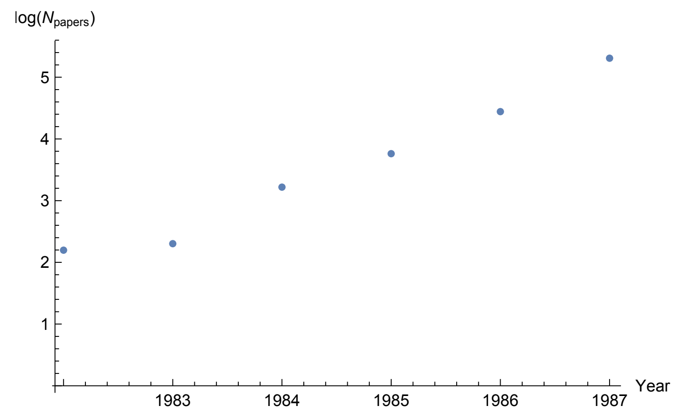
– CFT and string theory

– 2D critical phenomena and applications to spin chains

– extensions of BPZ

– mathematics of infinite dim algebras and related topics

– related integrable systems



John Cardy

New directions, new results : 1984-86

Minimal models, including Ising, 3-state Potts, . . .

Singular vectors lead to (partial) differential equations for correlators (conformal blocks): example of $c = \frac{4}{5}$, $\mathcal{M}(5,6)$: 3-state Potts model [V.Dotsenko 1984]

Integrable (Coulomb gas) representations [V. Dotsenko–V. Fateev 1984]

More applications to stat mech [J. Cardy]

– surface critical behavior [J.Cardy 1984]

– c measures a finite size (Casimir) effect $E_0 = fL - \frac{\pi c}{6L}$

[H.Blöte–J.Cardy–M.Nightingale ; I.Affleck,1986]

– boundaries and finite size scaling [J.Cardy 1984-86]

and condensed matter: quantum spin chains [I. Affleck1985, . . .]

Unitarity constraints [D.Friedan–Z.Qiu–S.Shenker, 1984]

Unitarity: $L_n^\dagger = L_{-n}$ and $c < 1$ only consistent if $c = 1 - \frac{6}{m(m+1)}$

and $h = h_{rs} = \frac{(r(m+1)-sm)^2-1}{4m(m+1)}$

Extensions to higher symmetries [A. Zamolodchikov]

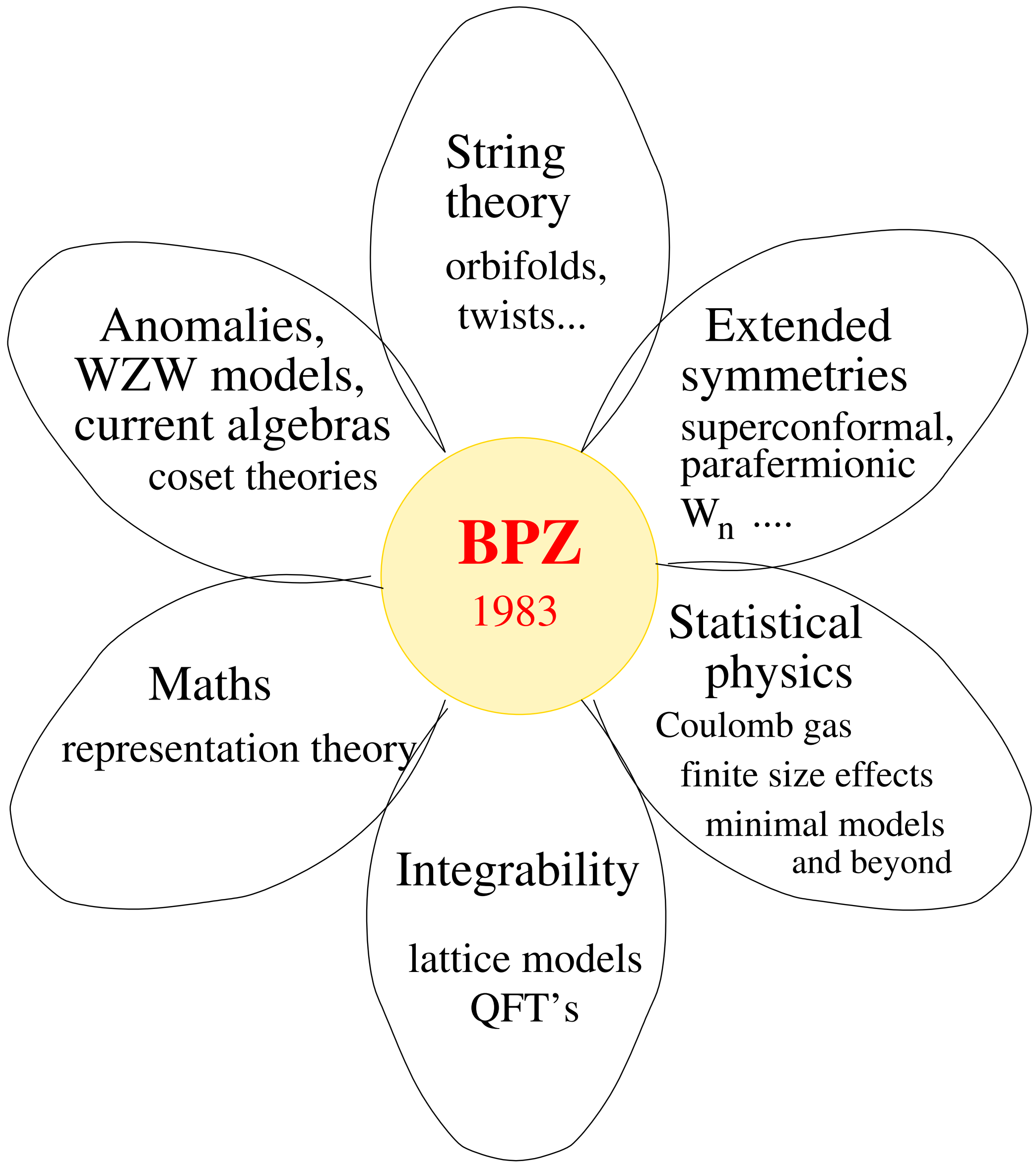
– 2-d current (aka affine Kac–Moody) algebras $\widehat{\mathfrak{g}}_k$, $k \in \mathbb{N}$, [V.Knizhnik–A.Z. '84],
– superconformal algebra [D.Friedan–Z.Qiu–S.Shenker 1984] ("Neveu–Schwarz and Ramond sectors");
– W_n algebra, parafermionic theories [A.Zamolodchikov '85, –V.Fateev, '87, F.Bais–P.Bouwknegt–M.Surridge–K.Schoutens '87]

Coset theories [P.Goddard, A.Kent, D.Olive, 1985] $\mathfrak{g}/\mathfrak{h}$

– A most useful tool to manufacture new CFT's !
– $\widehat{\mathfrak{s}u}(2)_k \times \widehat{\mathfrak{s}u}(2)_1/\widehat{\mathfrak{s}u}(2)_{k+1}$ has $c = 1 - \frac{6}{(k+2)(k+3)}$: FQS condition is sufficient, and all $c < 1$ unitary theories exist !

Other applications to stat. mech., in minimal models and beyond

– [G.Andrews–R.Baxter–P.Forrester, D.Huse, 1984]: integrable lattice (height, or RSOS) models; one of their critical regimes is described by unitary minimal models.
– Percolation, polymers [B.Duplantier, H.Saleur 1986], ...


String theory
orbifolds, twists...
Anomalies, WZW models, current algebras
coset theories
Extended symmetries
superconformal, parafermionic
W_n ....
BPZ
1983
Maths
representation theory
Statistical physics
Coulomb gas
finite size effects
minimal models and beyond
Integrability
lattice models
QFT's

Newer directions, more results : 1986-89

While the previous topics continue to flourish,
– orbifold CFT's, covariant quantization of the string [D.Friedan–E.Martinec–S.Shenker'86, L.Dixon–F–M–S '87], N=2 SCFT models of superstrings [Gepner'87] ; Coulomb gas revisited, etc

new directions develop:
– modularity
– integrability: integrable lattice models and integrable QFT's
– perturbed CFT's, Zamolodchikov c-theorem
– newer maths, quantum groups, Verlinde algebra and algebraic geometry. . .

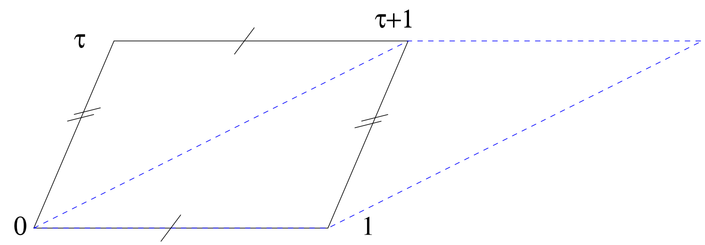
Modularity

Compute the partition function Z on a *torus*, *i.e.*, with doubly periodic boundary conditions [J. Cardy '86]

– Z must be modular invariant [A.Ferdinand–M.Fisher 1969, C.Thorn 1980]

– If Hilbert space $\mathcal{H} = \oplus_{i,\bar{i}} N_{i\bar{i}} \mathcal{V}_i \otimes \mathcal{V}_{\bar{i}}$, a finite sum in a minimal (or "rational") CFT, then

$$Z = \sum_{i,\bar{i}} N_{i\bar{i}}\, \chi_i(q) \chi_{\bar{i}}(q^*)$$



where $\chi_i(q) = \mathrm{tr}_{\mathcal{V}_i}\, q^{L_0 - \frac{c}{24}}$ but now $q = e^{2\pi \mathrm{i}\, \tau}$, $\tau =$ torus modular ratio.

– Characters χ transform under a (unitary) finite representation of the modular group

$$\chi_i(\tau + 1) = e^{2\pi \mathrm{i}\,(h_i - c/24)} \chi_i(\tau) \qquad \chi_i(-1/\tau) = S_{ij} \chi_j(\tau)$$

In particular $Z = \sum_i \chi_i(q) \chi_i(q^*)$ is modular invariant. But $\exists$ other solutions. . .

Opens the way to

– classification of RCFT's, more below . . .

– Verlinde fusion formula $N_{ij}^{k} = \sum_\ell \frac{S_{i\ell} S_{j\ell} S^*_{k\ell}}{S_{1\ell}}$ [E.Verlinde 1988]

– BCFT, b.c. changing operators, "Cardy's consistency equation" [J.Cardy '89]

Integrability

–All minimal CFT's admit an integrable lattice realization [G.Andrews–R.Baxter–P.Forrester, D.Huse 1984, V.Pasquier 1986-88].

– Features of CFT (OPE, fusion algebra, boundary conditions) have a counterpart on lattice and vice versa [V.Pasquier '87]

– Quantum group [V.Drinfeld,M.Jimbo 1985] is another common feature. . . [J.Fröhlich '87, V.Pasquier–H.Saleur '88-89, E. Lusztig '88; G.Moore–N.Seiberg'89, C.Gomez–G.Sierra '90,G.Mack–V.Schomerus '90, P.Furlan–A.Ganchev–V.Petkova '91 . . .]

Modular tensor category.

Perturbing CFT's

Perturb a CFT: generically, get a massive theory and a RG flow toward another IR fixed point (another CFT)

– Zamolodchikov c-theorem: (in unitary theories) there is a function c interpolating between central charges in a monotonous decreasing way. [A.Zamolodchikov, 1986]

– for specific perturbations, $\exists$ conserved quantities, the massive theory remains integrable, S-matrix may be computed [A.&Al. Zamolodchikov 1979, A.Zamolodchikov 1990]

— Case of the Ising model in a magnetic field, E_8 theory, [A. Zamolodchikov, 1988] now observed in neutron scattering experiments !

$\Rightarrow$ A flurry of works on various models off criticality, their S-matrix and correlation functions, form factors, . . . Perturbative methods in the vicinity of the CFT. . .

[T.Hollowood–P.Mansfield, H.Braden–E.Corrigan–P.Dorey–R.Sasaki'89, P.Christe–G.Mussardo'89, . . .]

From 1989 to the 90's and 2000's

– Axiomatizing CFT [G.Moore–N.Seiberg'89, G.Segal'91]

– Perturbing CFT's

Integrable perturbations, S matrix. . . [. . .]

RG flows between CFT's [P.Dorey–F.Ravanini 1992, V.Fateev'93,. . .]

– Coupling CFT's to 2D-gravity $\leftrightarrow$ matrix models [V.Knizhnik–A. Polyakov–A.Zamolodchikov 1988, F. David, J.Distler–H.Kawai '89]; . . . Minimal models on "random lattices" [I. Kostov '92]

– $N = 2$ superCFT's and Topological Field Theories [E.Martinec, W.Lerche–C.Vafa–N.Warner, 1989, . . .]

– 2D CFT's and 3D Topological theories [J.Fuchs–I.Runkel–C.Schweigert 2002–]. Operator algebra approach [A. Ocneanu, J.Böckenhauer–D.Evans–Y.Kawahigashi et al. '99, F.Xu '98. . .].

– Boundary CFT, defects . . .

– CFT's and finite groups [T.Gannon '99,. . .]

– Probability (percolation, self-avoiding walks) [J. Cardy '92, . . .], combinatorics (for ex. meander problems [P.Di Francesco–O.Golinelli–E.Guitter]) and SLE

[G.Lawler-O.Schramm–W.Werner, S.Smirnov,'99-'01 . . . J.Cardy, M.Bauer–D.Bernard '02]

– Log CFT's: [V.Gurarie '93, F.Rohsiepe'96, M.Flohr'97,. . . , H.Saleur-et-al. . .],

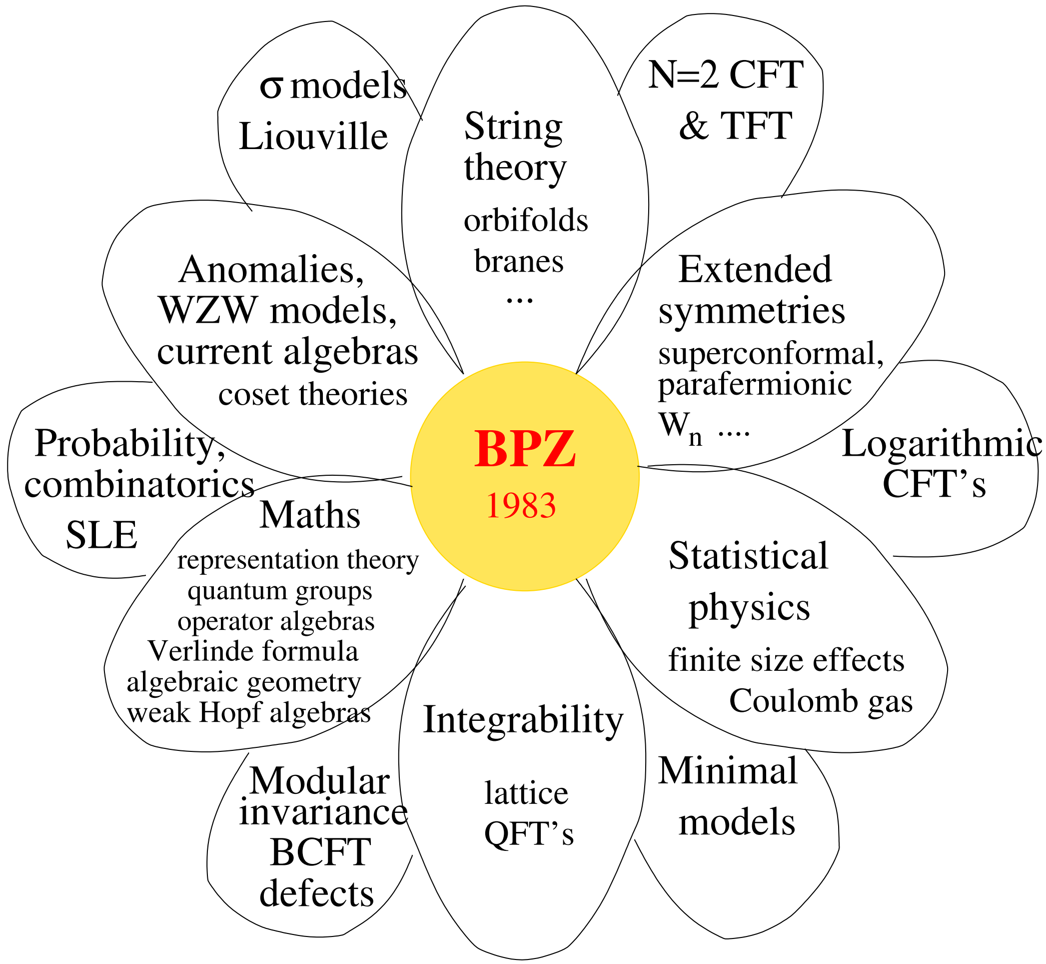
– Non rational CFT's, σ-models, Liouville theory [H.Dorn–H.-J.Otto'92, A.&Al.Zamolodchikov'96, J.Teschner, V.Schomerus, 2003, . . .]

– 2D CFT's as a laboratory: computation of entanglement entropy [J.Cardy–P.Calabrese '04,. . .]

$\vdots$

etc, etc

$\vdots$


BPZ
1983
σ models
Liouville
String theory
orbifolds
branes
...
N=2 CFT & TFT
Anomalies, WZW models, current algebras
coset theories
Extended symmetries
superconformal, parafermionic
W_n ....
Probability, combinatorics
SLE
Logarithmic CFT's
Maths
representation theory
quantum groups
operator algebras
Verlinde formula
algebraic geometry
weak Hopf algebras
Statistical physics
finite size effects
Coulomb gas
Integrability
lattice
QFT's
Modular invariance
BCFT
defects
Minimal models

The offspring of BPZ ?

The lure of Classification...

2D-CFT tools permit to envisage some classification programs

– classification of “good” representations [Feigin–Fuchs, BPZ]

– classification of unitary $c < 1$ theories [FQS]

– classification of OPA [P.Christe–R.Flume '87,... M.Caselle–G.Ponzano–F.Ravanini'91]

– classification of RCFT's: are all obtained from WZW by cosets, orbifolds and/or twists ?

– classification of modular invariant partition functions after Cardy:

ADE classification of minimal theories and su(2) affine theories

[A.Cappelli, C.Itzykson, JBZ 1986-88, D.Gepner–Z.Qiu '87, A.Kato '88]

$\leftrightarrow$ Pasquier ADE lattice models [V.Pasquier 1986]

level	Z	diagram
$k \geq 0$	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	A_{k+1}
$k = 4\rho \geq 4$	$\sum_{\lambda\ \text{odd}\ =1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$D_{2\rho+2}$
$k = 4\rho - 2 \geq 6$	$\sum_{\lambda\ \text{odd}\ =1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda\ \text{even}=2}^{2\rho-2} \left(\chi_\lambda \bar{\chi}_{4\rho-\lambda} + \text{c. c.}\right)$	$D_{2\rho+1}$
$k = 10$	$\lvert\chi_1 + \chi_7\rvert^2 + \lvert\chi_4 + \chi_8\rvert^2 + \lvert\chi_5 + \chi_{11}\rvert^2$	E_6
$k = 16$	$\lvert\chi_1 + \chi_{17}\rvert^2 + \lvert\chi_5 + \chi_{13}\rvert^2 + \lvert\chi_7 + \chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + \left[(\chi_3 + \chi_{15})\bar{\chi}_9 + \text{c. c.}\right]$	E_7
$k = 28$	$\lvert\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}\rvert^2 + \lvert\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}\rvert^2$	E_8

Table 1: List of modular invariant partition functions of $\widehat{sl}(2)$ RCFTs

χ_λ

are characters of representations of the affine algebra at level k . The last column shows the associated ADE Dynkin diagram.

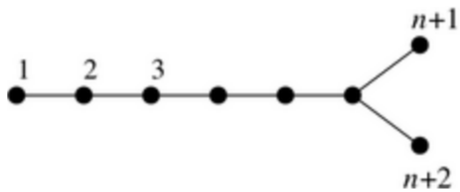
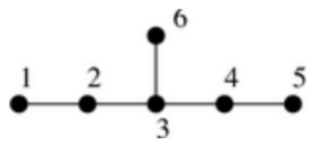
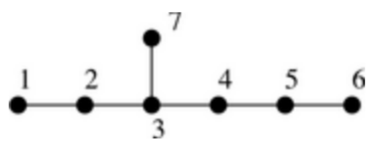
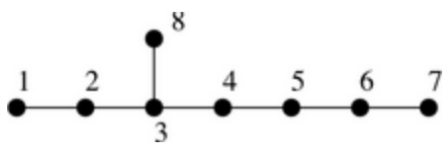
G	diagram	h	exponents ℓ_n
A_n		$n+1$	$1, 2, \cdots, n$
D_{n+2}		$2(n+1)$	$1, 3, \cdots, 2n+1, n+1$
E_6		12	$1, 4, 5, 7, 8, 11$
E_7		18	$1, 5, 7, 9, 11, 13, 17$
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$

Table 2: ADE Dynkin diagrams with Coxeter numbers h and exponents ℓ_n .

Eigenvalues of adjacency matrix of diagram $= 2\cos \pi\ell_n/h$

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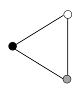
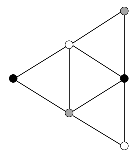
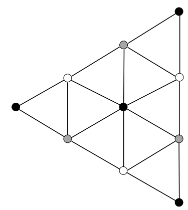
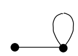
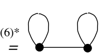
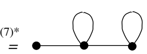
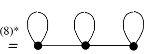
ADE classification of minimal theories and su(2) affine theories [A.Cappelli, C.Itzykson, JBZ 1986-88, D.Gepner–Z.Qiu '87, A.Kato '88]

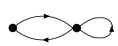
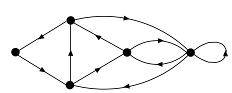
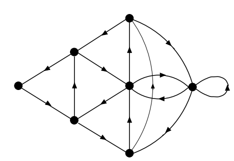
$\leftrightarrow$ Pasquier ADE lattice models [V.Pasquier 1986]

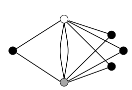
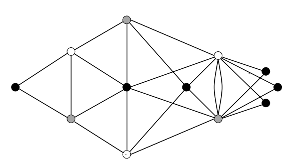
– Why ADE ? (several answers!) What about higher rank ? Graphs ?

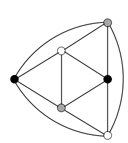
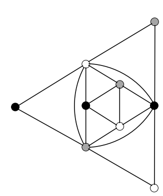
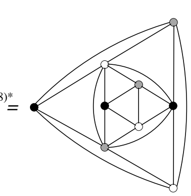
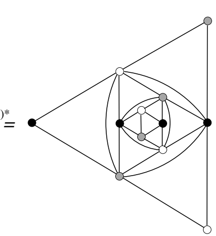
For su(3) ✓: modular invariants [D.Bernard 1987,..., T.Gannon 1994], graphs [I. Kostov 1988, P. Di Francesco–JBZ 1989,..., A. Ocneanu 2000]

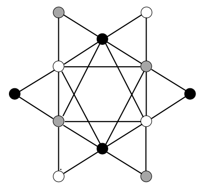
$A^{(4)}=$  $A^{(5)}=$  $A^{(6)}=$  . . .

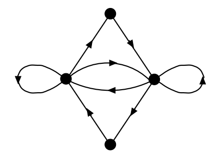
$A^{(4)*}=$  $A^{(5)*}=$  $A^{(6)*}=$  $A^{(7)*}=$  $A^{(8)*}=$  . . .

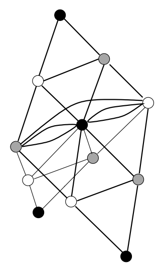
$D^{(5)}=$  $D^{(7)}=$  $D^{(8)}=$  . . .

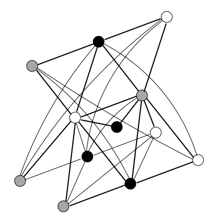
$D^{(6)}=$  $D^{(9)}=$  . . .

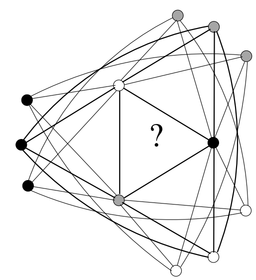
$D^{(6)*}=$  $D^{(7)*}=$  $D^{(8)*}=$  $D^{(9)*}=$  . . .

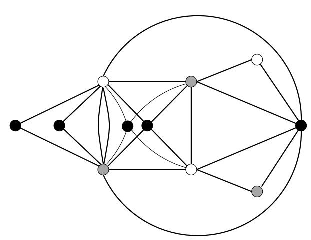
$E^{(8)} =$ 

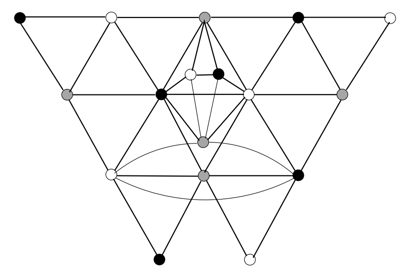
$E^{(8)*} =$ 

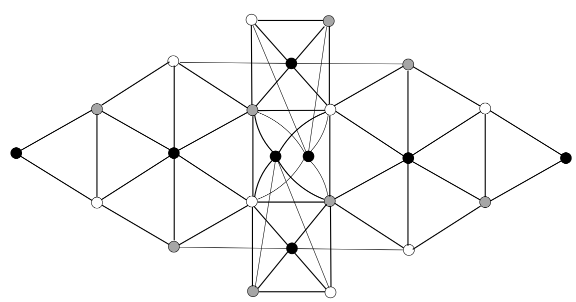
$E_1^{(12)} =$ 

$E_2^{(12)} =$ 

$E_3^{(12)} =$ 

$E_4^{(12)} =$ 

$E_5^{(12)} =$ 

$E^{(24)} =$ 

– Why graphs ? BCFT! [J. Cardy'88, M.Bauer–H.Saleur'89, P. Di Francesco-JBZ'89]. . . [I.Affleck–M.Oshikawa–H.Saleur'97, A.Sagnotti–Y.Stanev, J.Fuchs–C.Schweigert, A.Recknagel-V.Schomerus 1995-97, Runkel'98], [G.Watts, R. Behrend–P. Pearce–V. Petkova–JBZ 1998]:

adjacency matrices of graphs encode the boundary conditions and form a matrix representation of the fusion rules, so as to satisfy Cardy's equation.

Classify *non-negative integer matrix representations* ("nimreps") of the fusion algebra $n_i n_j = N_{ij}^k n_k$.

For $\widehat{\mathfrak{su}}(2)$ theories and minimal models, $n_1 \to$ ADE Dynkin diagrams!

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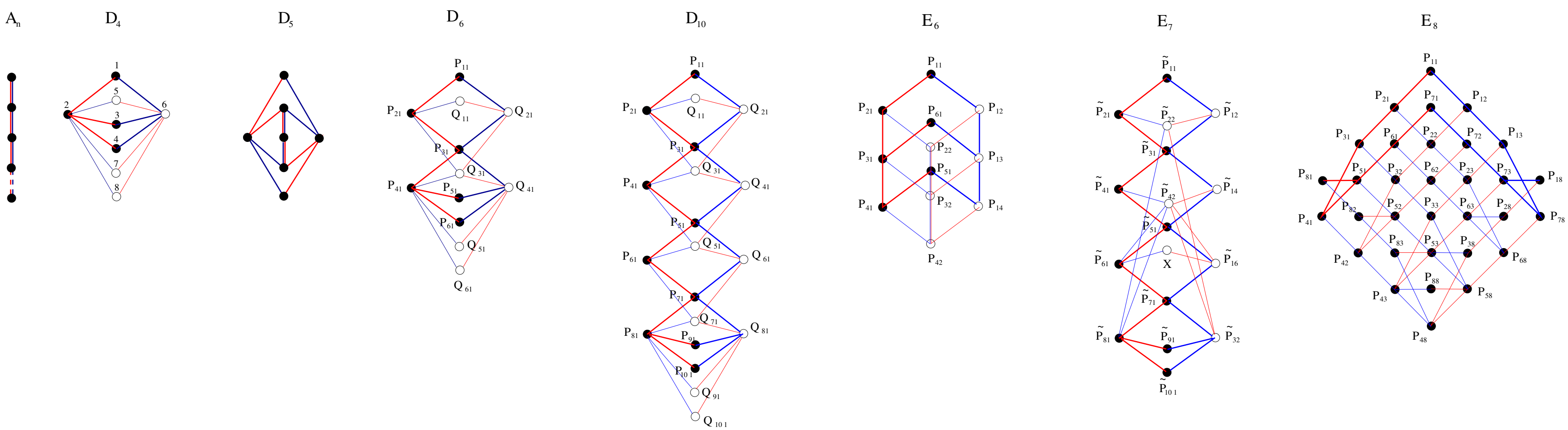

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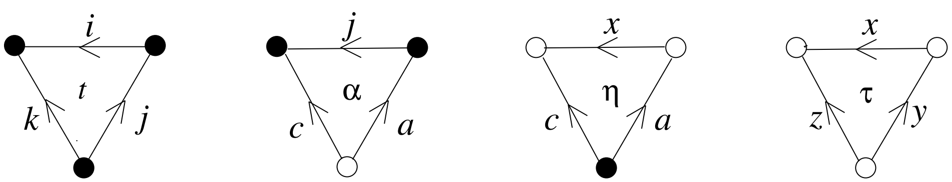

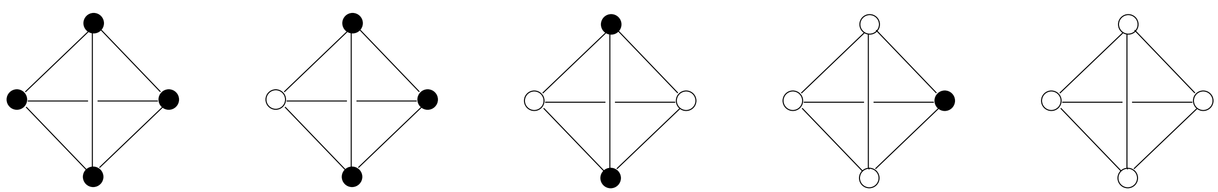

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The underlying structure is a "weak Hopf algebra" . . .

[G.Böhm-K.Szlachanyi'96,. . . P.Etingof–D.Nikshych–V.Ostrik'05]

– classification of weak Hopf algebras ??

To summarize

The BPZ paper of 1983 has opened a new chapter in the big book of QFT.

It had incredibly many ramifications and applications, from mathematics to string theory, stat. mech. and condensed matter.

The story is still going on, both in 2D and in higher dimensions.

$\star$

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for exciting times on an exciting subject. . .