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"Conformal Invariance and
Harmonic Analysis"
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Invariant differential operators : from conformal symmetry to quantum groups

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Plan

Easy introduction via Maxwell equations

From $so(4,2)$ to $so(p,q)$

Other Examples :
$su(n,n)$, $sp(n,\mathbb{R})$, $so^*(2n)$, $E_{7(-25)}$, $E_{6(-14)}$,

Quantum groups

q-Minkowski space-time

q-Maxwell equations hierarchy

q-d'Alembert equations hierarchy

q-Weyl gravity equations hierarchy

Maxwell equations

It is well known that Maxwell equations may be written in several equivalent forms:

$$\partial^{\mu} F_{\mu\nu} \;=\; J_{\nu}\ , \qquad \partial^{\mu *} F_{\mu\nu} \;=\; 0 \tag{1}$$

or

$$\begin{aligned}
\partial_k E_k \;&=\; J_0\ (= 4\pi\rho), \\
\partial_0 E_k - \varepsilon_{k\ell m}\partial_\ell H_m \;&=\; J_k\ (= -4\pi j_k), \\
\partial_k H_k \;&=\; 0\ , \\
\partial_0 H_k + \varepsilon_{k\ell m}\partial_\ell E_m \;&=\; 0
\end{aligned} \tag{2}$$

where $E_k \;\equiv\; F_{k0},\;\; H_k \;\equiv\; (1/2)\varepsilon_{k\ell m}F_{\ell m}$,

or

$$\partial_k F_k^{\pm} \;=\; J_0\ , \qquad \partial_0 F_k^{\pm} \pm i\varepsilon_{k\ell m}\partial_\ell F_m^{\pm} \;=\; J_k \tag{3}$$

where

$$F_k^{\pm} \;\equiv\; E_k \pm iH_k \tag{4}$$

Not so well known is the fact that the eight equations in (3) can be rewritten as two conjugate scalar equations in the following way:

$$I^{+}\ F^{+}(z) \ = \ J(z,\bar{z})\ , \tag{5a}$$

$$I^{-}\ F^{-}(\bar{z}) \ = \ J(z,\bar{z}) \tag{5b}$$

where

$$\begin{aligned} I_0^{+} \quad &= \quad \bar{z}\partial_{+} + \partial_v \ - \qquad\qquad\qquad\qquad (6a) \\ &- \ \frac{1}{2}\Big(\bar{z}z\partial_{+} + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_{-}\Big)\partial_z\ , \\ I_0^{-} \quad &= \quad z\partial_{+} + \partial_{\bar{v}} \ - \qquad\qquad\qquad\qquad (6b) \\ &- \ \frac{1}{2}\Big(\bar{z}z\partial_{+} + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_{-}\Big)\partial_{\bar{z}} \end{aligned}$$

$$x_{\pm} \equiv x_0 \pm x_3, \quad v \equiv x_1 - ix_2, \quad \bar{v} \equiv x_1 + ix_2,$$
$$\partial_{\pm} \equiv \partial/\partial x_{\pm}, \quad \partial_v \equiv \partial/\partial v, \quad \partial_{\bar{v}} \equiv \partial/\partial\bar{v} \tag{7}$$

$$
\begin{aligned}
F^{+}(z) &\equiv z^2(F_1^{+} + iF_2^{+}) - 2zF_3^{+} \; - \\
&\quad - \; (F_1^{+} - iF_2^{+}) \; , \qquad\qquad (8) \\
F^{-}(\bar{z}) &\equiv \bar{z}^2(F_1^{-} - iF_2^{-}) - 2\bar{z}F_3^{-} \; - \\
&\quad - \; (F_1^{-} + iF_2^{-}) \; , \\
J(z,\bar{z}) &\equiv \bar{z}z(J_0 + J_3) + \bar{z}(J_1 - iJ_2) \; + \\
&\quad + \; z(J_1 + iJ_2) + (J_0 - J_3)
\end{aligned}
$$

where we continue to suppress the x_μ, resp., $x_\pm, v, \bar{v}$, dependence in F and J. (The conjugation mentioned above is standard and in our terms it is : $I_0^+ \longleftrightarrow I_0^-$, $F^+(z) \longleftrightarrow F^-(\bar{z})$.)

It is easy to recover (3) from (5) - just note that both sides of each equation are first order polynomials in each of the two variables z and $\bar{z}$, then comparing the independent terms in (5) one gets at once (3).

Writing the Maxwell equations in the simple form (5) has also important conceptual meaning. The point is that each of the two scalar operators I_0^+, I_0^- is indeed a single object, namely it is an intertwiner of the conformal group, while the individual components in (1) - (3) do not have this interpretation. This is also the simplest way to see that the Maxwell equations are conformally invariant, since this is equivalent to the intertwining property.

To check the intertwining property we rewrite the operators (6) in the following form:

$$\begin{aligned} I_0^+ &= \frac{1}{2}\Big(2I_1 I_2 - 3I_2 I_1\Big), \\ I_0^- &= \frac{1}{2}\Big(2I_3 I_2 - 3I_2 I_3\Big) \end{aligned} \tag{9}$$

where

$$I_1 \equiv \partial_z\,,\ I_2 \equiv \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\,,\ I_3 \equiv \partial_{\bar{z}} \tag{10}$$

Let us briefly recall that a *Verma module* V^Λ over $\mathcal{G}^\mathbb{C}$ $(=\mathcal{G}_+\oplus\mathcal{H}\oplus\mathcal{G}_-)$ is defined as the highest weight module over $\mathcal{G}^\mathbb{C}$ with highest weight $\Lambda\in\mathcal{H}^*$ and highest weight vector $v_0\in V^\Lambda$, induced from the one-dimensional representation $V_0\cong\mathbb{C}v_0$ of $U(\mathcal{B})$, ($\mathcal{B}=\mathcal{H}\oplus\mathcal{G}_+$ is a Borel subalgebra of $\mathcal{G}^\mathbb{C}$), such that:

$$\begin{aligned} X\ v_0 &= 0, \quad \forall\, X\in\mathcal{G}_+ \\ H\ v_0 &= \Lambda(H)\ v_0\,, \quad \forall\, H\in\mathcal{H} \end{aligned} \tag{11}$$

Verma modules are generically irreducible. A Verma module V^Λ is reducible [BGG] iff there exists a root $\beta\in\Delta^+$ and $m\in\mathbb{N}$ such that

$$(\Lambda+\rho,\ \beta^\vee)\ =\ m \tag{12}$$

holds, where $\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\ \alpha$. If (12) holds then the reducible Verma module V^Λ contains an invariant submodule which is also a Verma module $V^{\Lambda'}$ with shifted weight $\Lambda'=\Lambda-m\beta$. This statement is equivalent to the fact that

V^{Λ} contains a *singular vector* $v_s \in V^{\Lambda}$, such that $v_s \neq \xi v_0$, $(0 \neq \xi \in \mathbb{C})$, and :

$$\begin{aligned} X\, v_s &= 0, \quad \forall\, X \in \mathcal{G}_+ && (13) \\ H\, v_s &= \Lambda'(H)\, v_s\,, \quad \Lambda' = \Lambda - m\beta, \quad \forall\, H \in \mathcal{H} \end{aligned}$$

Restricting to the $sl(4)$ case, let's denote the simple roots by α_k, (k=1,2,3). Explicitly, the singular vectors of weight $\alpha_{12} \equiv \alpha_1 + \alpha_2$, $\alpha_{23} \equiv \alpha_2 + \alpha_3$. are:

$$\begin{aligned} v_s^{12} &= const.\Big(2X_1^- X_2^- - 3X_2^- X_1^-\Big)v_0, \\ v_s^{23} &= const.\Big(2X_3^- X_2^- - 3X_2^- X_3^-\Big)v_0 \quad (14) \end{aligned}$$

where X_k^- are the simple negative root generators.

Now it remains to pass from generators X_k^- to their right action $\pi_R(X_k^-)$ on the coset $\mathcal{Y} = SL(4)/B$, where B is a Borel subgroup of $SL(4)$ consisting of all upper diagonal matrices. The local coordinates of $\mathcal{Y}$ may be given in the following matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ z & 1 & 0 & 0 \\ v & x_- & 1 & 0 \\ x_+ & \bar{v} & \bar{z} & 1 \end{pmatrix} \tag{15}$$

Then we can derive (possibly after change of variables) the relation between (10) and $\pi_R(X_k^-)$:

$$I_k = \pi_R(X_k^-) \tag{16}$$

Now we see that the variables $z, \bar{z}$ which we introduced as book-keeping device, together with the Minkowski variables $x_\pm, v, \bar{v}$, have definite group-theoretical meaning as six local coordinates on the flag manifold $\mathcal{Y}$. Under a natural conjugation this is also a flag manifold of the conformal group $SU(2,2)$.

Thus, we have seen an example of the main group-theoretical ingredient: that every Verma module singular vector provides an invariant differential operator. [Da,DI]

To put the example in the general picture we recall the physically relevant representations T^χ of the 4-dimensional conformal algebra $su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where n_1, n_2 are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being $(n_1+1)(n_2+1)$), and d is the conformal weight (or dimension, or energy).

[In the literature these Lorentz representations are labelled also by $(j_1, j_2) \equiv (n_1/2, n_2/2)$.] [For the conformal group we also have to take into account a central element taking sign values, however, these are fixed for the reducible cases we consider.]

Then the intertwining properties of the operators in (6) and some more are presented in the following diagram:

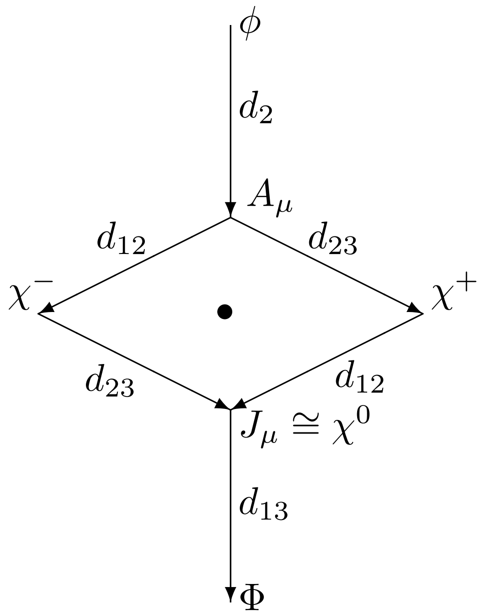

Fig. 1. Intertwining diagram for $su(2,2)$ involving Maxwell equations

where the representations have the signatures:

$$\begin{aligned}
&\chi(\phi) \ \sim\ [0,0;0], \quad \chi(\Phi) \ \sim\ [0,0;4], \\
&\chi(A_\mu) = [1,1;1], \quad \chi(J_\mu) = [1,1;3], \\
&\chi^- = [0,2;2], \quad \chi^+ = [2,0;2]
\end{aligned} \tag{17}$$

The invariant differential operators in the diagram are denoted by d_2, d_{jk} corresponding to the singular vectors of weight α_2, α_{jk}, resp.

Formulae (17) are part of an infinite hierarchy of couples of first order intertwiners. Explicitly, instead of (17) we have [Db,DI]:

$$\chi_k^+ = [k+2, k; 2] \ , \quad \chi_k^- = [k, k+2; 2] \ ,$$
$$\chi_k^0 = [k+1, k+1; 3] \ , \quad k \in \mathbb{Z}_+ \tag{18}$$

while instead of (5) we have:

$$I_k^+ \ F_k^+(z,\bar z) \ = \ J_k(z,\bar z) \ , \tag{19a}$$
$$I_k^- \ F_k^-(z,\bar z) \ = \ J_k(z,\bar z) \tag{19b}$$

where ($k \in \mathbb{Z}_+$)

$$\begin{aligned} I_k^+ \quad &= \quad \frac{k+2}{2}\Big(\bar z \partial_+ + \partial_v\Big) \ - \qquad\qquad (20a) \\ &\quad - \ \frac{1}{2}\Big(\bar z z \partial_+ + z\partial_v + \bar z \partial_{\bar v} + \partial_-\Big)\partial_z \ , \\ I_k^- \quad &= \quad \frac{k+2}{2}\Big(z \partial_+ + \partial_{\bar v}\Big) \ - \qquad\qquad (20b) \\ &\quad - \ \frac{1}{2}\Big(\bar z z \partial_+ + z\partial_v + \bar z \partial_{\bar v} + \partial_-\Big)\partial_{\bar z} \end{aligned}$$

while $F_k^+(z,\bar z)$, $F_k^-(z,\bar z)$, $J_k(z,\bar z)$, are polynomials in $z,\bar z$ of degrees $(k+2,k)$, $(k,k+2)$, $(k+1,k+1)$, resp., as explained above.

Note that we can rewrite (20) as (9):

$$I_k^+ = \frac{1}{2}\Big((k+2)I_1 I_2 - (k+3)I_2 I_1\Big),$$
$$I_k^- = \frac{1}{2}\Big((k+2)I_3 I_2 - (k+3)I_2 I_3\Big) \quad (21)$$

If we want to use the notation with indices as in (1), then $F_k^+(z,\bar{z})$ and $F_k^-(z,\bar{z})$ correspond to $F_{\mu\nu,\alpha_1,\ldots,\alpha_k}$ which is antisymmetric in the indices μ,ν, symmetric in $\alpha_1,\ldots,\alpha_k$, and traceless in every pair of indices, while $J_k(z,\bar{z})$ corresponds to $J_{\mu,\alpha_1,\ldots,\alpha_k}$ which is symmetric and traceless in every pair of indices. Note, however, that the analogs of (1) would be much more complicated if one wants to write explicitly all components. The crucial advantage of (19) is that the operators $I_k^\pm$ $(k>0)$ are given just by a slight generalization of $I_0^\pm$.

We call the hierarchy of equations (19) the *Maxwell hierarchy*. The Maxwell equations are the zero member of this hierarchy.

Everything above is part of the general classification scheme:

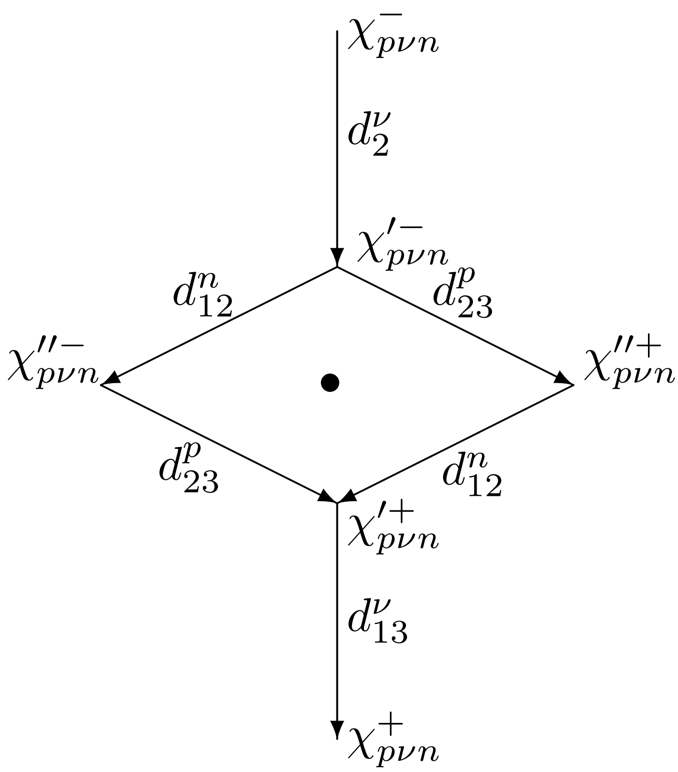

Fig. 2. General intertwining diagram $su(2,2) \cong so(4,2)$ (valid also for $so(5,1)$ and $so(3,3)$)

The signatures in the figure are:

$$\begin{aligned}
\chi^-_{p\nu n} &= (p,n;\ 2-\nu-(p+n)/2) \qquad (22)\\
\chi^+_{p\nu n} &= (n,p;\ 2+\nu+(p+n)/2)\\
\chi'^-_{p\nu n} &= (p+\nu,n+\nu;\ 2-(p+n)/2)\\
\chi'^+_{p\nu n} &= (n+\nu,p+\nu;\ 2+(p+n)/2)\\
\chi''^-_{p\nu n} &= (\nu,p+\nu+n;\ 2+(p-n)/2)\\
\chi''^+_{p\nu n} &= (p+\nu+n,\nu;\ 2+(n-p)/2)\\
& \quad p,\nu,n\in\mathbb{N}
\end{aligned}$$

The invariant differential operators in the diagram are denoted by d^ν_2, d^n_{12}, d^p_{23}, d^ν_{13} corresponding to the singular vectors of weight $\nu\alpha_2$, $n\alpha_{12}$, $p\alpha_{23}$, $\nu\alpha_{13}$, resp.

We exhibit also an alternative of the same situation showing also the integral intertwining operators in Fig. 2':

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\chi^{-}_{p\nu n} \arrow[rr, leftrightarrow] \arrow[d, "d^{\nu}_{2}"'] & & \chi^{+}_{p\nu n} \\
\chi'^{-}_{p\nu n} \arrow[rr, leftrightarrow] \arrow[d, "d^{n}_{12}"'] \arrow[drr, "d^{p}_{23}"' near start] & & \chi'^{+}_{p\nu n} \arrow[u, "d^{\nu}_{13}"'] \\
\chi''^{-}_{p\nu n} \arrow[rr, leftrightarrow] \arrow[urr, "d^{p}_{23}" near end] & & \chi''^{+}_{p\nu n} \arrow[u, "d^{n}_{12}"']
\end{tikzcd}
$$

Fig. 2′. Alternative of Fig. 2 with integral operators shown as horizontal dashed lines

The above diagram appeared first for the Euclidean conformal case $so(5,1) \cong su^*(4)$ in [DP]. In the case of $so(4)$ (and $so(n)$) symmetric traceless tensors only the top four representations with $n = p$ are relevant [DMPPT]. In particular, the cancelation of kinematical poles in the derivation of the OPE mentioned in the talk of G. Mack uses the partial equivalence between the representations $\chi^{+}_{n\nu n}$ and $\chi'^{+}_{n\nu n}$ provided by the operator d^{ν}_{13}.

Besides in the sextets the reducible VMs occur in some doublets which we omit for the lack of space here. The sextets and the doublets exhaust all reducible VMs induced from finite-dimensional Lorentz irreps.

Next we show the cases $so(p,q)$:

$$
\begin{tikzcd}
 & \Lambda_1^- \arrow[d, "d_1"] & \\
 & \Lambda_2^- & \\
 & \vdots & \\
 & \Lambda_{h-1}^- \arrow[d, "d_{h-1}"] & \\
 & \Lambda_h^- \arrow[dl, "d_h"'] \arrow[dr, "d_{h+1}"] & \\
\Lambda_{h+1}^- \arrow[dr, "d_{h+1}"'] & \bullet & \Lambda_{h+1}^+ \arrow[dl, "d_h"] \\
 & \Lambda_h^+ \arrow[d, "d'_{h-1}"] & \\
 & \Lambda_{h-1}^+ & \\
 & \vdots & \\
 & \Lambda_2^+ \arrow[d, "d'_1"] & \\
 & \Lambda_1^+ &
\end{tikzcd}
$$

Fig. 3. Diagram for the cases $so(p, q)$, $p + q = 2h + 2$, even

$$
\begin{tikzcd}
\Lambda_1^- \arrow[d, "d_1"] \\
\Lambda_2^- \\
\vdots \\
\Lambda_h^- \arrow[d, "d_h"] \\
\Lambda_{h+1}^- \arrow[d, "d_{h+1}"] \\
\Lambda_{h+1}^+ \arrow[d, "d'_h"] \\
\Lambda_h^+ \\
\vdots \\
\Lambda_2^+ \arrow[d, "d'_1"] \\
\Lambda_1^+
\end{tikzcd}
$$

Fig. 4. Diagram for the cases $so(p,q)$, $p+q=2h+3$, odd

Conformal Lie algebras

We start with the class of *Hermitian symmetric spaces*. The algebraic criterion is that the *maximal compact subalgebra* $\mathcal{K}$ of $\mathcal{G}$ is of the form:

$$\mathcal{K} \ = \ so(2) \oplus \mathcal{K}'$$

The Lie algebras from this class are:

$$so(p,2), \quad sp(p,R), \quad su(p,q),$$

$$so^*(2p), \quad E_{6(-14)}\,, \quad E_{7(-25)}$$

These groups/algebras have *highest/lowest weight representations*, and relatedly *(anti-)holomorphic discrete series representations*.

We already considered the *conformal algebras* so(p,2) in p-dimensional Minkowski space-time. In that case, there is a maximal *Bruhat decomposition* that has direct physical meaning:

$$so(p,2) \ = \ \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}} \ ,$$

$$\mathcal{M} \ = \ so(p-1,1) \ , \quad \text{Lorentz subalgebra}$$

$$\mathcal{A}, \quad \dim \mathcal{A} = 1, \quad \text{dilatations,}$$

$$\mathcal{N}, \quad \dim \mathcal{N} = p, \quad \text{translations,}$$

$$\tilde{\mathcal{N}}, \quad \dim \tilde{\mathcal{N}} = p, \quad \text{special conformal transformations}$$

Another special feature of the conformal algebra: the complexification of the maximal compact subalgebra $\mathcal{K}$ is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$\mathcal{K}^{\mathbb{C}} \ = \ so(p,\mathbb{C}) \oplus so(2,\mathbb{C}) \ \cong$$

$$\cong \ so(p-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}}$$

In particular, the coincidence of the complexification of the semi-simple subalgebras:

$$\mathcal{K}'^{\mathbb{C}} \ = \ \mathcal{M}^{\mathbb{C}} \qquad (*)$$

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $\mathcal{K}'$.

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of $so(p,2)$.

This subclass consists of:

$$so(p,2), \quad sp(p,\mathbb{R}), \quad su(p,p),$$

$$so^*(4p), \quad E_{7(-25)}$$

In view of applications to physics, we proposed to call these algebras '*conformal Lie algebras*', (or groups).
This class was identified by G. Mack (2007) from different configurations.

The Lie algebra $su(p,p)$

Let $\mathcal{G} = su(p,p)$, $p \geq 2$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(p) \oplus su(p)$, while $\mathcal{M} = sl(p,\mathbb{C})_{\mathbb{R}}$. The number of ERs in the corresponding diagram (multiplet) is equal to

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| \ = \ \binom{2p}{p}$$

The signature of the ERs of $\mathcal{G}$ is:

$$\chi \ = \ \{\, n_1\,, \dots, n_{p-1}\,, n_{p+1}\, \dots, n_{2p-1}\,; c\,\}\ , \qquad n_j \in \mathbb{N}\ , \quad c = d - p$$

Below we give the diagrams for $su(p,p)$ for $p = 3, 4$ ($p = 2$ was considered as $so(4,2)$). These are diagrams also for $sl(2p,\mathbb{R})$ and for $p = 2k$ these are diagrams also for $su^*(4k)$ [Dj,DI].

We use the following conventions. Each invariant differential operator is represented by an arrow accompanied by a symbol $i_{j\ldots k}$ encoding the root $\beta_{j\ldots k}$ and the numbers $m_i \in \mathbb{N}$ which encode the reducibility (12) of the corresponding Verma module.

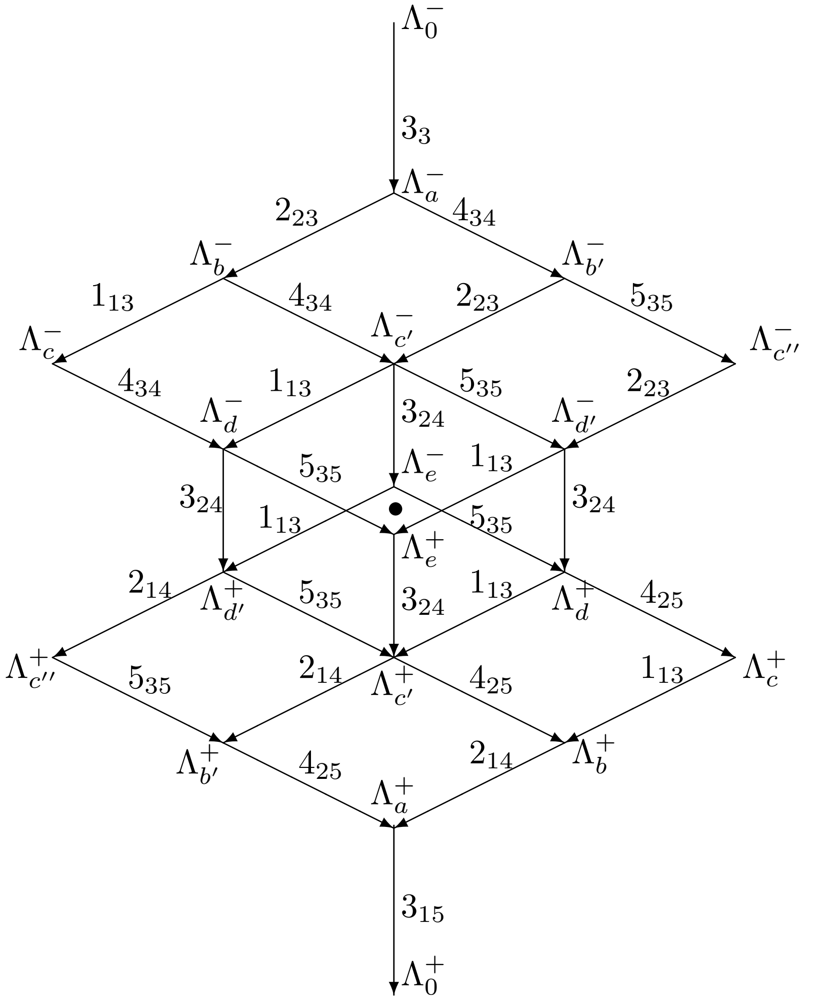

Fig. 5. Pseudo-unitary symmetry $su(3,3)$

The pseudo-unitary symmetry $su(p,p)$ is similar to conformal symmetry in p^2 dimensional space, for $p=2$ coincides with conformal 4-dimensional case. By parabolic relation the $su(3,3)$ diagram above is valid also for $sl(6,R)$.

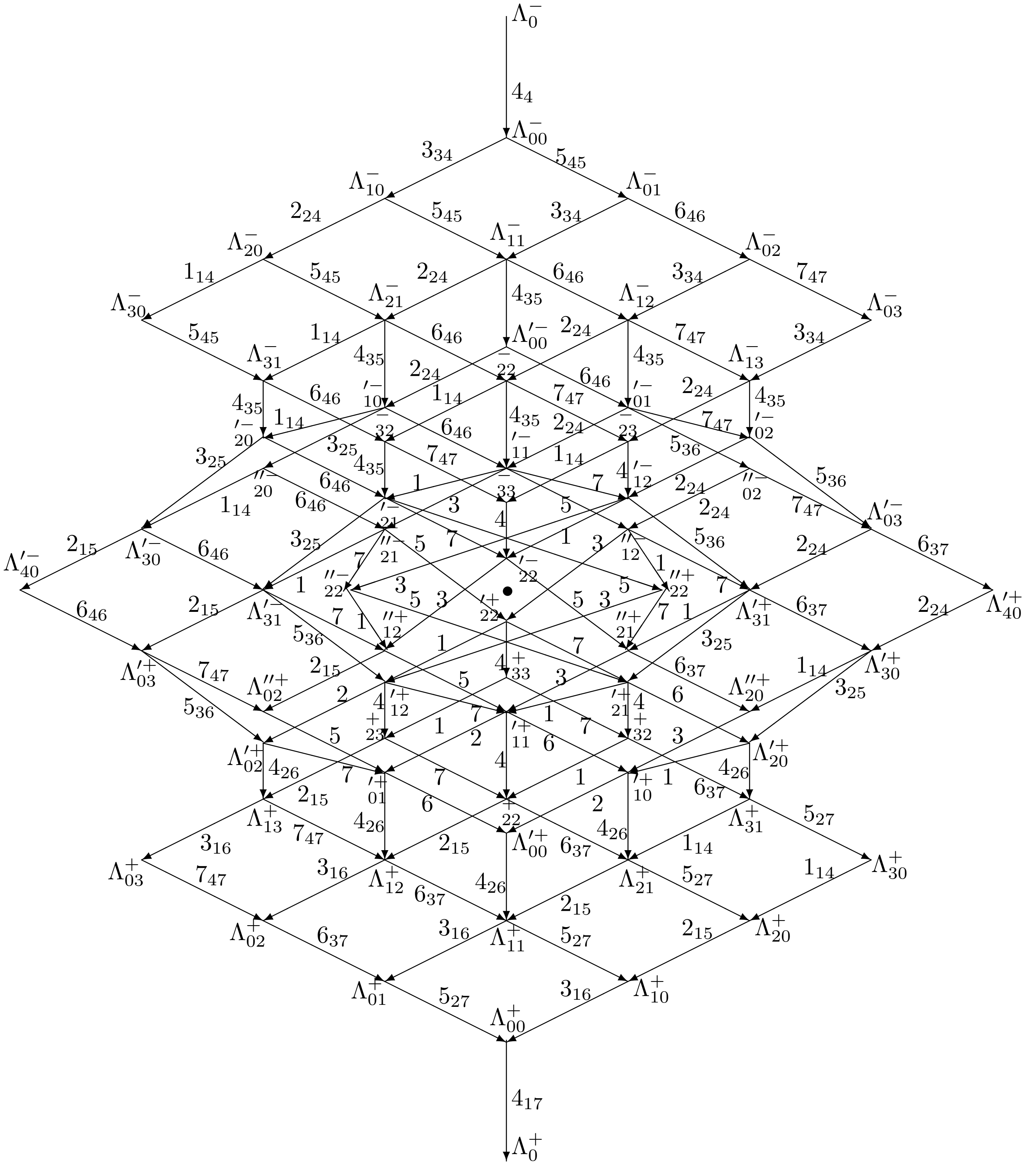

Fig. 6. Pseudo-unitary symmetry in 16-dimensional space.

By parabolic relation the $su(4,4)$ diagram above is valid also for $sl(8,R)$ and $su^*(8)$.

The Lie algebras $sp(n,\mathbb{R})$ and $sp(\frac{n}{2},\frac{n}{2})$ (n–even)

Let $n \geq 2$. Let $\mathcal{G} = sp(n,\mathbb{R})$, the split real form of $sp(n,\mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subalgebra is $\mathcal{K} \cong u(1) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{R})$. The number of ERs in the corresponding diagram (multiplet) is:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}}_m)| \;=\; 2^n$$

The signature of the ERs of $\mathcal{G}$ is:

$$\chi \;=\; \{\, n_1\,, \ldots, n_{n-1}\,;\, c\,\}\;, \qquad n_j \in \mathbb{N}\;,$$

Below we give pictorially the multiplets for $sp(n,\mathbb{R})$ for $n = 3,4,5,6$ ($n = 2$ was already considered as $so(3,2)$). For $n = 2r$ these are also multiplets for $sp(r,r)$ [Dj,DI].

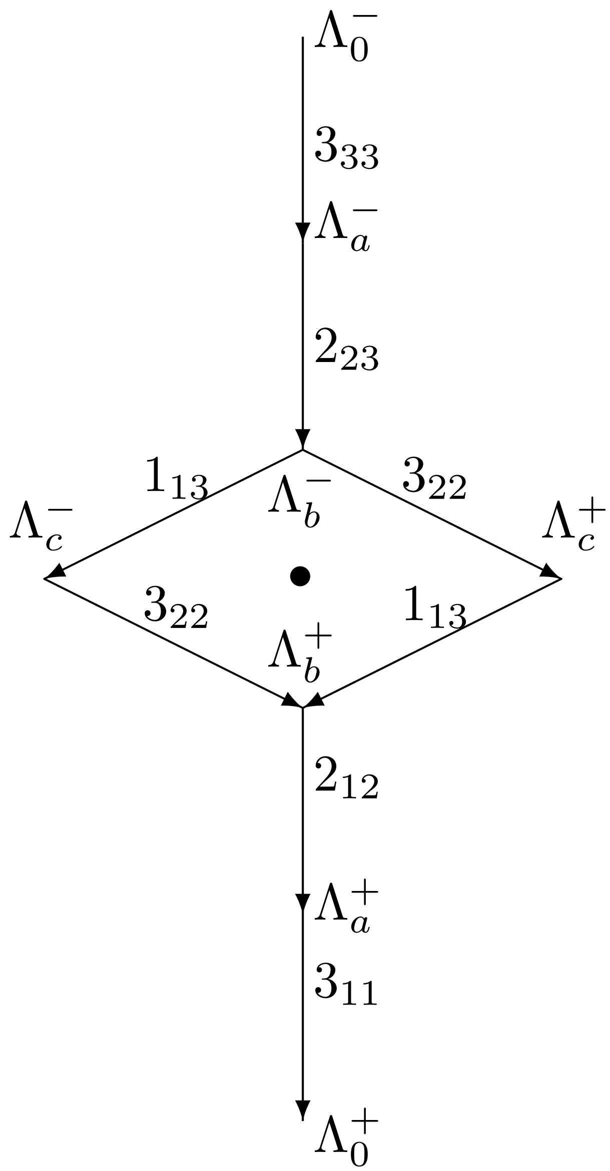

Fig. 7. Main multiplets for $Sp(3,\mathbb{R})$

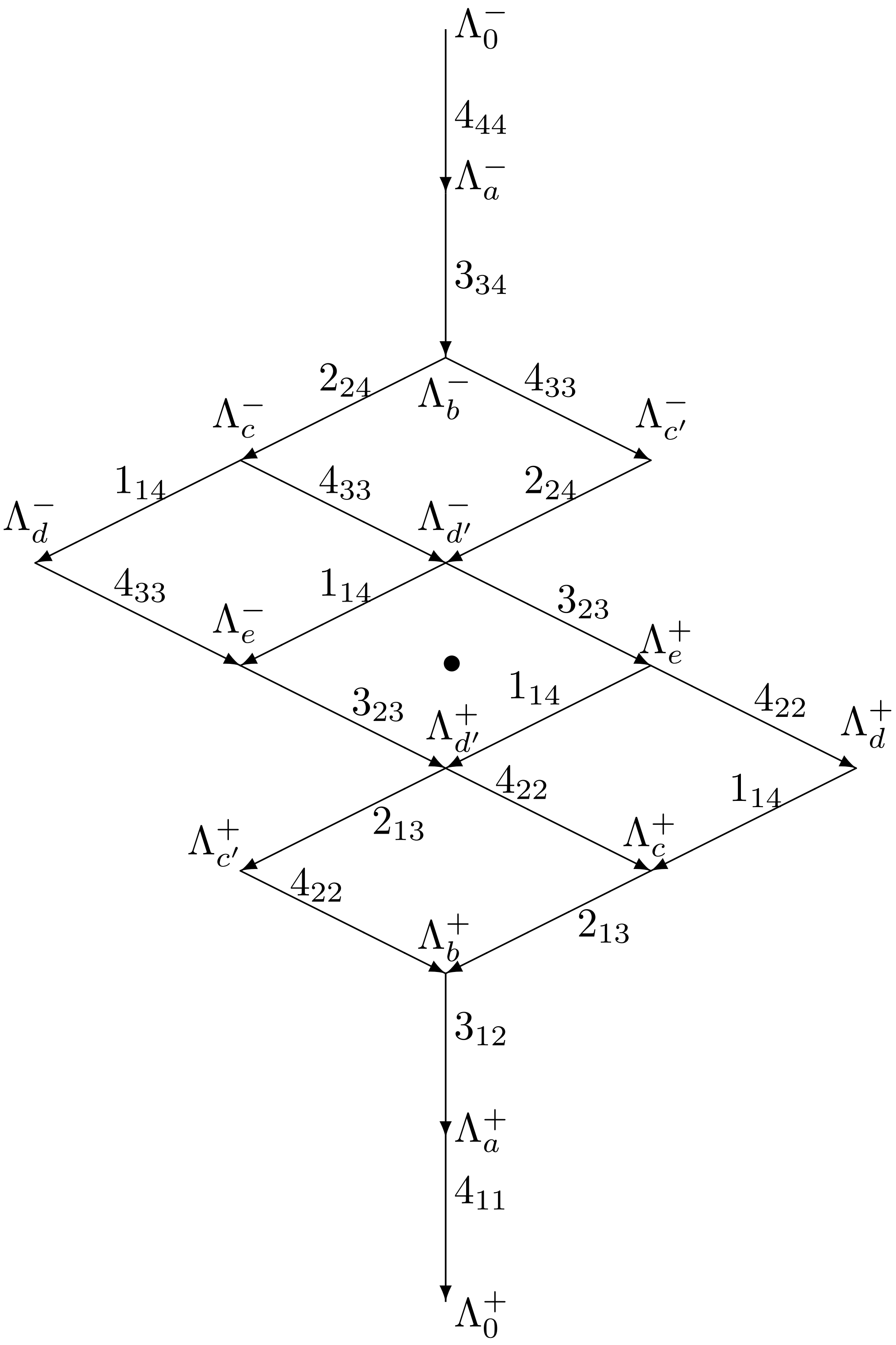

Fig. 8. Main multiplets for $sp(4,I\!\!R)$ and $sp(2,2)$

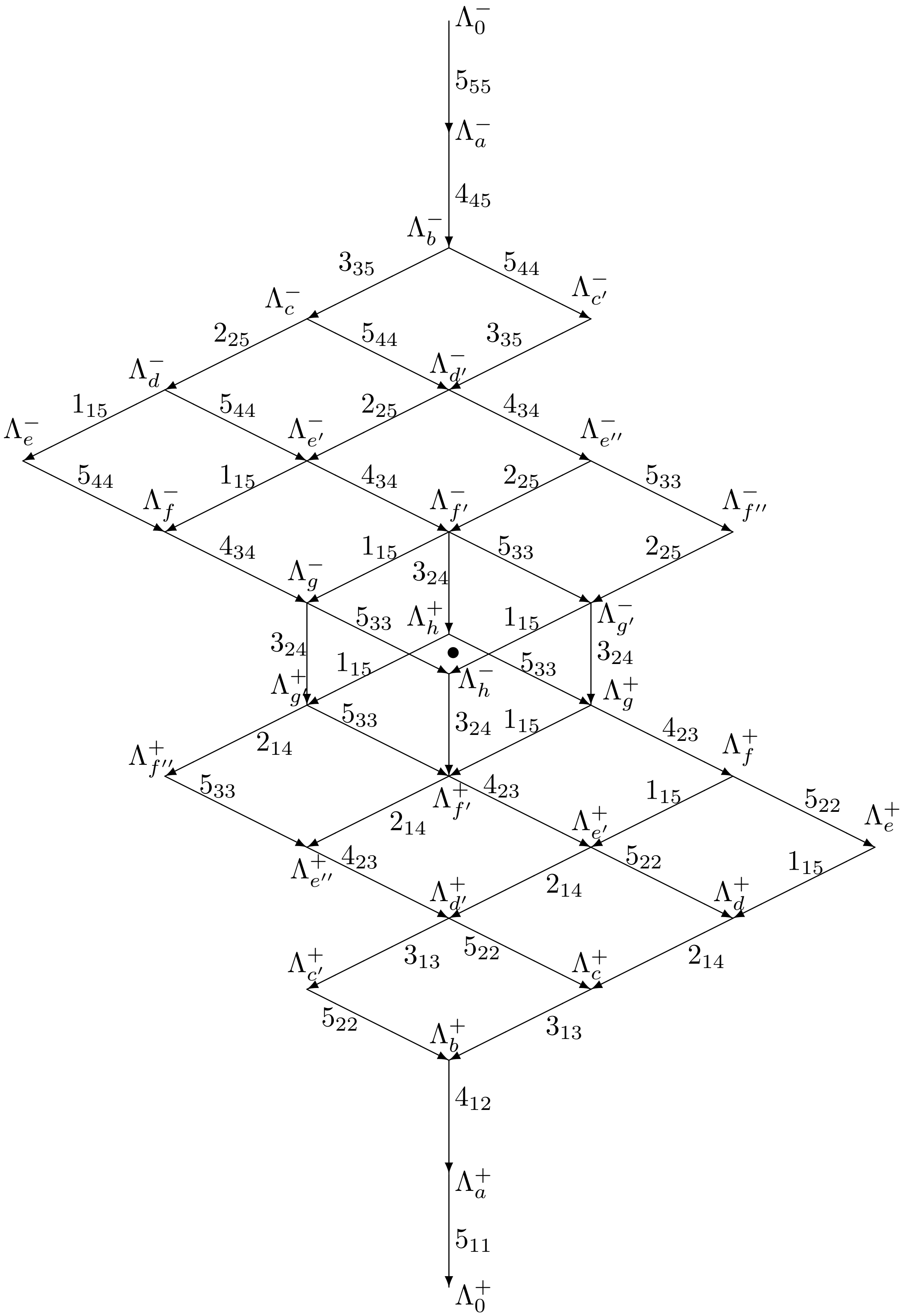

Fig. 9. Main multiplets for $Sp(5, I\!\!R)$

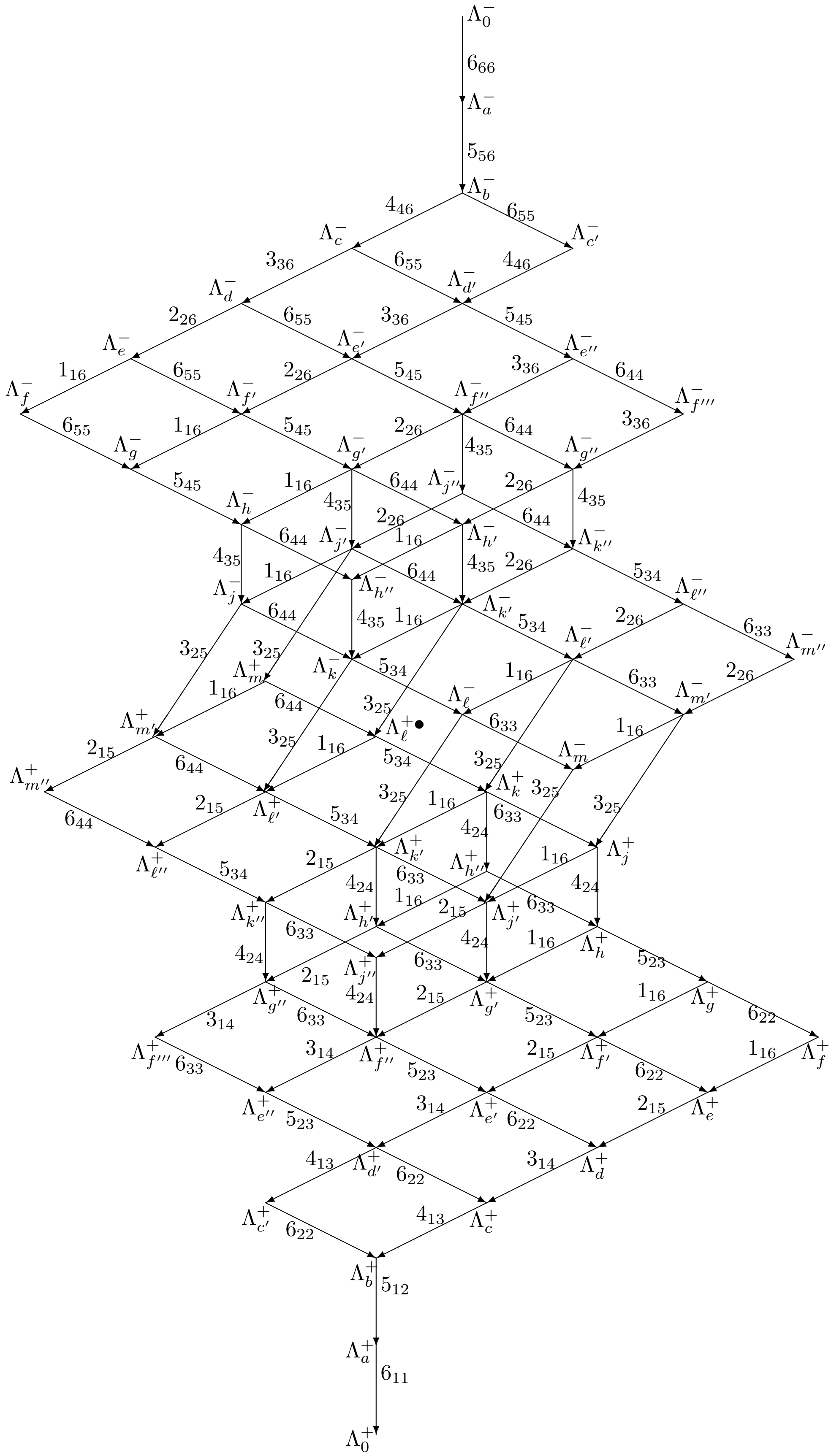

Fig. 10. Main multiplets for $Sp(6, I\!R)$ and $sp(3,3)$

The Lie algebras $so^*(2n)$

The Lie algebra $\mathcal{G} = so^*(2n)$ is given by:

$$\begin{aligned} so^*(2n) &\doteq \{X \in so(2n,\mathbb{C}) : J_n C X = X J_n C\} = \\ &= \{ \ X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \ \mid \ a, b \in gl(n,\mathbb{C}), \\ &\qquad {}^t a = -a, \quad b^\dagger = b \ \} \ . \end{aligned} \tag{23}$$

$\dim_R \mathcal{G} = n(2n-1)$, $\mathrm{rank}\,\mathcal{G} = n$. The maximal compact subalgebra is $\mathcal{K} \cong u(n)$. For even $n = 2r$ the algebra $\mathcal{G} = so^*(4r)$ belongs to the class of 'conformal Lie algebras' (since $\mathcal{M} = su^*(n)$).
Further we restrict to the case $\mathcal{G} = so^*(12)$, since $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(8) \cong so(6,2)$.

The number of ERs/GVMs in the corresponding multiplets is [Dj]:

$$\begin{aligned} &|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| / |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| = \\ &= |W(so(12,\mathbb{C}))| / |W(sl(6,\mathbb{C}))| = 32 \end{aligned}$$

where $\mathcal{H}$ is a Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$.

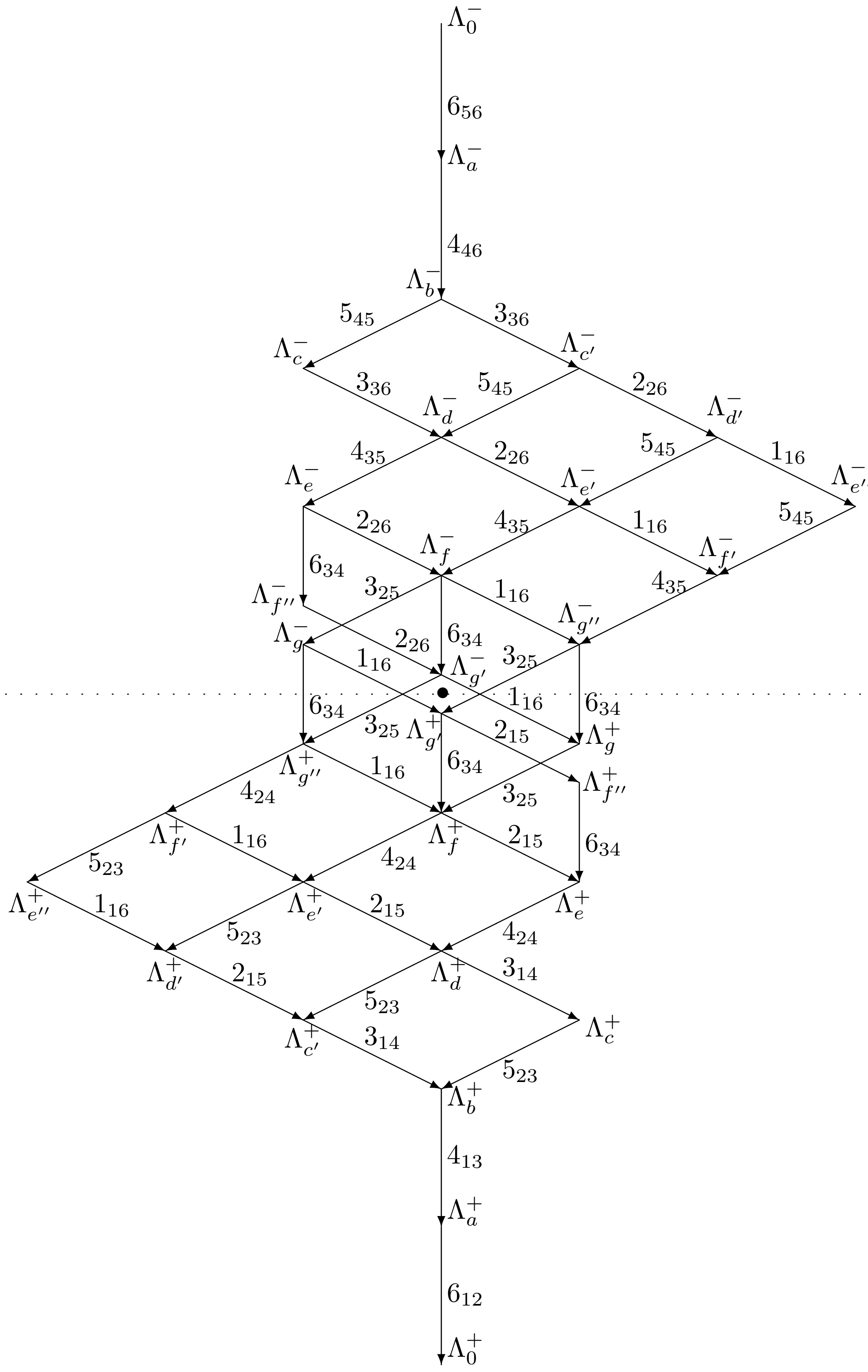

Fig. 11. $SO^*(12)$ main multiplets

The Lie algebras $E_{7(-25)}$ and $E_{7(7)}$

Let $\mathcal{G} = E_{7(-25)}$. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$, while $\mathcal{M} \cong E_{6(-6)}$.

The signatures of the ERs of $\mathcal{G}$ are:

$$\chi \;=\; \{\, n_1 \,, \ldots, n_6 \,; c \,\} \;, \qquad n_j \in \mathbb{N} \;.$$

The same can be used for the parabolically related exceptional Lie algebra $E_{7(7)}$ [Dj,DI].

The number of ERs in these main multiplets is:

$$\frac{|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})|}{|W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})|} \;=\; \frac{|W(E_7)|}{|W(E_6)|} \;=\; \frac{2^{10}\,3^4\,5.7}{2^7\,3^4\,5} \;=\; 56 \tag{24}$$

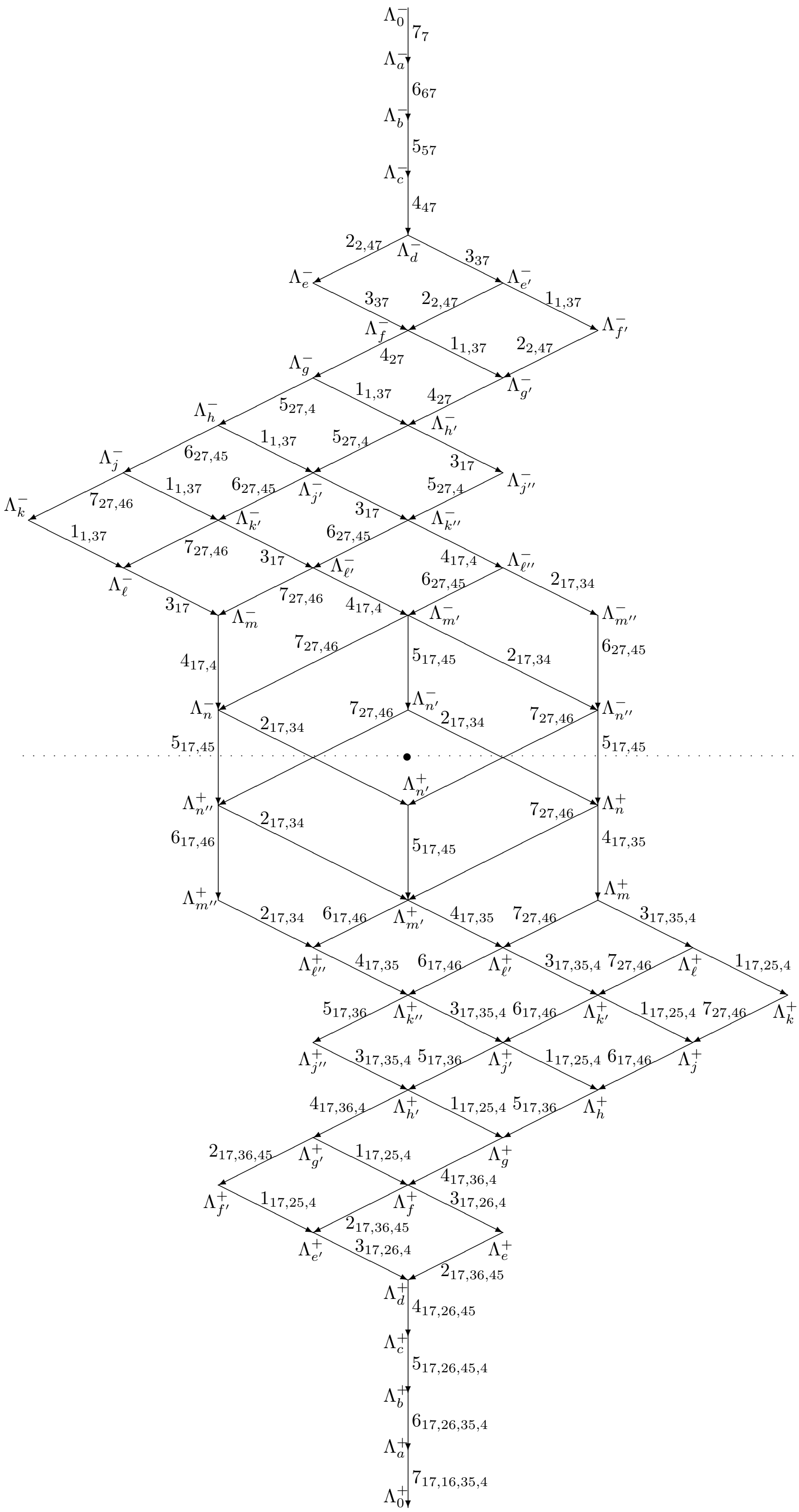

Fig. 12. Main Type for $E_{7(-25)}$

The Lie algebras $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$

Let $\mathcal{G} = E_{6(-14)}$. The maximal compact subalgebra is $\mathcal{K} \cong so(10) \oplus so(2)$, while $\mathcal{M} \cong su(5,1)$.

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{ n_1 , n_3 , n_4 , n_5 , n_6 \, ; \, c\} \ , \quad c = d - \tfrac{11}{2} \ .$$

The above can be used for the parabolically related exceptional Lie algebras $E_{6(6)}$ and $E_{6(2)}$ [Dj,DI].

(The algebra $E_{6(-14)}$ does not belong to the class of conformal Lie algebras and the formula as (24) for the main multiplet does not hold here.)

There are 70 VMs in the main multiplet:

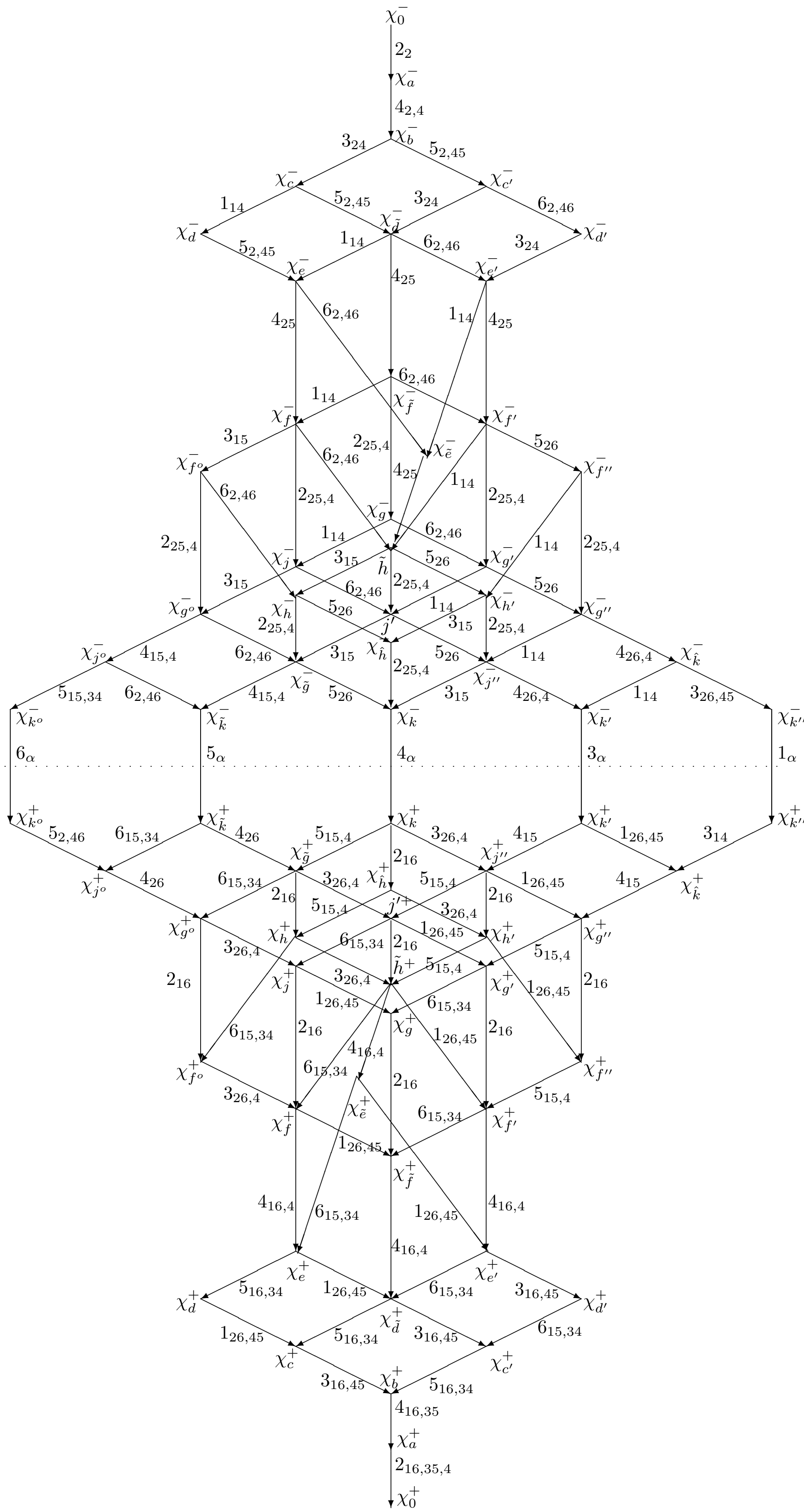

Fig. 13. Main Type for $E_{6(-24)}$

Quantum groups

We go back to the 4D conformal case. The form (9) is that we generalize for the deformed case. In fact, we can write at once the q-deformed form [Dh,DII]:

$$\begin{aligned} \hat{I}^+_n &= \frac{1}{2}\Big([n+2]_q\, \hat{I}_1 \hat{I}_2 - [n+3]_q\, \hat{I}_2 \hat{I}_1 \Big), \\ \hat{I}^-_n &= \frac{1}{2}\Big([n+2]_q\, \hat{I}_3 \hat{I}_2 - [n+3]_q\, \hat{I}_2 \hat{I}_3 \Big) \end{aligned} \tag{25}$$

where $[m]_q \equiv \frac{q^m - q^{-m}}{q - q^{-1}}$ are the ubiquitous q-numbers.
Here $\hat{I}^\pm_n$ are obtained from the lowest possible singular vectors of $U_q(sl(4))$, namely, those of weight α_{12}, α_{23} [Dh,DII].

To proceed further, we should make this form explicit by first generalizing the variables, then the functions and the operators.

Quantum Minkowski space-time

We know from [Dh,DII] what are the properties of the non-commutative coordinates on the $SL_q(4)$ coset. Thus, we obtain for the commutation rules of the q-Minkowski space-time coordinates

$$\begin{aligned} x_{\pm} v &= q^{\pm 1} v x_{\pm}\,, \quad x_{\pm}\bar{v} = q^{\pm 1}\bar{v} x_{\pm}\,, \\ x_+ x_- - x_- x_+ &= \lambda v \bar{v}\,, \quad \bar{v} v = v \bar{v} \end{aligned} \tag{26}$$

The q-Minkowski length ℓ_q is defined as the q-determinant of $M = \begin{pmatrix} x_+ & \bar{v} \\ v & x_- \end{pmatrix}$:

$$\ell_q \doteq \det{}_q\, M = x_+ x_- - q \bar{v} v \tag{27}$$

and hence it commutes with the q-Minkowski coordinates. It has the correct classical limit $\ell_{q=1} = x_0^2 - \vec{x}^2$.

For q phase ($|q| = 1$) the commutation relations (26) are preserved by an anti-linear anti-involution ω acting as:

$$\omega(x_\pm) = x_\pm \,, \quad \omega(v) = \bar{v} \tag{28}$$

from which follows also that $\omega(\ell_q) = \ell_q$.

The commutation rules involving the spin variables $z, \bar{z}$ are:

$$\begin{aligned}
&\bar{z}z = z\bar{z}\,, \\
&x_+ z = q^{-1} z x_+ \,, \quad x_- z = q z x_- - \lambda v\,, \\
&vz = q^{-1} z v\,, \quad \bar{v} z = q z \bar{v} - \lambda x_+ \,, \\
&\bar{z} x_+ = q x_+ \bar{z}\,, \quad \bar{z} x_- = q^{-1} x_- \bar{z} + \lambda \bar{v}\,, \\
&\bar{z} v = q^{-1} v \bar{z} + \lambda x_+ \,, \quad \bar{z}\bar{v} = q \bar{v} \bar{z}\,, \\
&z \ell_q = \ell_q z\,, \quad \bar{z} \ell_q = \ell_q \bar{z}
\end{aligned} \tag{29}$$

Certainly, the commutation relations (29) are also preserved (for q phase) by the conjugation ω - supplementing (28) by $\omega(z) = \bar{z}$.

With this conjugation $\mathcal{Y}_q$ becomes a coset of $SU_q(2,2)$.

Quantum Maxwell equations hierarchy

The normally ordered basis of the q - coset $\mathcal{Y}_q$ considered as an associative algebra is :

$$\begin{aligned} \widehat{\varphi}_{ijk\ell mn} \quad &= \quad z^i\ v^j\ x_-^k\ x_+^\ell\ \bar{v}^m\ \bar{z}^n\,, \\ & \qquad i,j,k,\ell,m,n \in \mathbb{Z}_+ \end{aligned} \tag{30}$$

We introduce now the representation spaces C^χ , $\chi = [n_1, n_2; d]$. The elements of C^χ , which we shall call (abusing the notion) functions, are polynomials in $z, \bar{z}$ of degrees n_1, n_2, resp., and formal power series in the quantum Minkowski variables. Namely, these functions are given by:

$$\widehat{\varphi}_{n_1,n_2}(\bar{Y}) \quad = \quad \sum_{\substack{i,j,k,\ell,m,n\in\mathbb{Z}_+ \\ i\leq n_1,\ n\leq n_2}} \mu^{n_1,n_2}_{ijk\ell mn}\ \widehat{\varphi}_{ijk\ell mn} \tag{31}$$

where $\bar{Y}$ denotes the set of the six coordinates on $\mathcal{Y}_q$. Thus the quantum analogs of $F_n^{\pm}$, J_n, cf. (19), are :

$$\hat{F}_n^{+} = \hat{\varphi}_{n+2,n}(\bar{Y}) , \quad \hat{F}_n^{-} = \hat{\varphi}_{n,n+2}(\bar{Y}) ,$$
$$\hat{J}_n = \hat{\varphi}_{n+1,n+1}(\bar{Y}) \tag{32}$$

Using the above machinery we can present a deformed version of the Maxwell hierarchy of equations. For this we use that the operators $\hat{I}_a$ are given by the right action of $U_q(sl(4))$ on $\mathcal{Y}_q$:

$$\hat{I}_a = \pi_R(X_a^-) \tag{33}$$

Explicitly, we have:

$$\hat{I}_1 = \hat{\mathcal{D}}_z \ T_z \ T_v \ T_+ \ (T_- T_{\bar{v}})^{-1} \tag{34a}$$

$$\begin{aligned} \hat{I}_2 = & \Big(q \ \hat{M}_z \ \hat{\mathcal{D}}_v \ T_-^2 \ + \ \hat{\mathcal{D}}_- \ T_- \ + \\ & + \ \hat{M}_z \ \hat{M}_{\bar{z}} \ \hat{\mathcal{D}}_+ \ T_- \ T_{\bar{v}} \ T_v^{-1} \ + \\ & + \ q^{-1} \ \hat{M}_{\bar{z}} \ \hat{\mathcal{D}}_{\bar{v}} \ - \\ & - \ \lambda \ \hat{M}_v \ \hat{M}_{\bar{z}} \ \hat{\mathcal{D}}_- \ \hat{\mathcal{D}}_+ \ T_{\bar{v}} \Big) \ T_{\bar{v}} \ T_{\bar{z}}^{-1} \end{aligned} \tag{34b}$$

$$\hat{I}_3 = \hat{\mathcal{D}}_{\bar{z}} \ T_{\bar{z}} \tag{34c}$$

where we us the q-shift operators T_κ:

$$\begin{aligned} T_z\,\widehat{\varphi}_{ijk\ell mn} &= q^i\,\widehat{\varphi}_{ijk\ell mn} \\ T_v\,\widehat{\varphi}_{ijk\ell mn} &= q^j\,\widehat{\varphi}_{ijk\ell mn} \\ T_-\,\widehat{\varphi}_{ijk\ell mn} &= q^k\,\widehat{\varphi}_{ijk\ell mn} \\ T_+\,\widehat{\varphi}_{ijk\ell mn} &= q^\ell\,\widehat{\varphi}_{ijk\ell mn} \\ T_{\bar{v}}\,\widehat{\varphi}_{ijk\ell mn} &= q^m\,\widehat{\varphi}_{ijk\ell mn} \\ T_{\bar{z}}\,\widehat{\varphi}_{ijk\ell mn} &= q^n\,\widehat{\varphi}_{ijk\ell mn} \end{aligned}$$

further the q-difference operators:

$$\widehat{\mathcal{D}}_\kappa = \frac{1}{q-q^{-1}}\, M_\kappa^{-1}\, \left(T_\kappa - T_\kappa^{-1}\right)$$

where:

$$\begin{aligned} M_z\,\widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{i+1,jk\ell mn} \\ M_v\,\widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{i,j+1,k\ell mn} \\ M_-\,\widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{ij,k+1,\ell mn} \\ M_+\,\widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{ijk,\ell+1,mn} \\ M_{\bar{v}}\,\widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{ijk\ell,m+1,n} \\ M_{\bar{z}}\,\widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{ijk\ell m,n+1} \end{aligned}$$

Note that for $q \to 1$ we have: $T_\kappa \to \mathbf{1}$, $\widehat{\mathcal{D}}_\kappa \to \partial_\kappa$.

With this we have now the q - Maxwell hierarchy of equations - it remains just to substitute the operators of (34) in (25). In fact, we can also rewrite these in the q-analog of (19). We have :

$$
\begin{aligned}
{}_qI_n^+ &= \tfrac{1}{2}\Bigg(\Big(q\widehat{\mathcal{D}}_v + \widehat{M}_{\overline{z}}\widehat{\mathcal{D}}_+(T_-T_v)^{-1}T_{\overline{v}}\Big)[n+2-N_z]_q \\
&\quad -q^{-n-2}\Big(\widehat{\mathcal{D}}_-T_- + q^{-1}\widehat{M}_{\overline{z}}\widehat{\mathcal{D}}_{\overline{v}} - \qquad\qquad (35a) \\
&\quad -\lambda\widehat{M}_v\widehat{M}_{\overline{z}}\widehat{\mathcal{D}}_-\widehat{\mathcal{D}}_+T_{\overline{v}}\Big)\widehat{\mathcal{D}}_z\Bigg)T_+T_-T_vT_zT_{\overline{z}}^{-1} \\
{}_qI_n^- &= \tfrac{1}{2}\Big(\widehat{\mathcal{D}}_{\overline{v}} + q\widehat{M}_z\widehat{\mathcal{D}}_+T_{\overline{v}}T_-T_v^{-1} - \qquad\qquad (35b) \\
&\quad -q\lambda\widehat{M}_v\widehat{\mathcal{D}}_-\widehat{\mathcal{D}}_+T_{\overline{v}}\Big)T_{\overline{v}}[n+2-N_{\overline{z}}]_q - \\
&\quad -\tfrac{1}{2}q^{n+3}\Big(\widehat{\mathcal{D}}_- + q\widehat{M}_z\widehat{\mathcal{D}}_vT_-\Big)\widehat{\mathcal{D}}_{\overline{z}}T_-T_{\overline{v}}
\end{aligned}
$$

Clearly, for $q = 1$ the operators in (34), (35) coincide with (10),(9), resp.

With this the final result for the q - Maxwell hierarchy of equations is:

$$
\begin{aligned}
{}_qI_n^{+}\ {}_qF_n^{+} &= {}_qJ_n\,, && \text{(36a)}\\
{}_qI_n^{-}\ {}_qF_n^{-} &= {}_qJ_n && \text{(36b)}
\end{aligned}
$$

q - d'Alembert equations hierarchy

Next we consider another one parameter hierarchy:

$$\chi_r^{d+} = [r, 0; \frac{r}{2} + 1], \qquad (37a)$$

$$\chi_r^{d0+} = [r-1, 1; \frac{r}{2} + 2], \quad r \in \mathbb{N},$$

$$\chi_r^{d-} = [0, r; \frac{r}{2} + 1], \qquad (37b)$$

$$\chi_r^{d0-} = [1, r-1; \frac{r}{2} + 2], \quad r \in \mathbb{N},$$

where there are two conjugated equations:

$${}_qI_r^{+} \; F_r^{d+} = J_r^{d+} \;, \qquad (38a)$$

$${}_qI_r^{-} \; F_r^{d-} = J_r^{d-} \qquad (38b)$$

where ${}_qI_r^{\pm}$ are given by (25).

For the minimal possible value of the parameter $r = 1$ we obtain the two conjugate *q - Weyl equations*.

The case $r = 2$ gives the q-Maxwell equations (note that $J_2^{d+} = J_2^{d-}$). This is the only intersection of the present hierarchy with the q-Maxwell hierarchy.

We call this hierarchy *q - d'Alembert hierarchy* following the classical case, (cf. [De,DI,DII]), due to the following. We consider the representations $\chi_a^{d\pm}$ for the excluded above value $r = 0$, when they coincide. Thus, we set: $\chi^d \equiv \chi_0^{d\pm} = [0, 0; 1]$, $F^d \equiv F_0^{d\pm}$. Furthermore, the relevant equation is the q-d'Alembert equation [De,DII]:

$$\Box_q \ F^d \ = \ J^d \tag{39}$$

where $\chi^J \ = \ [0, 0; 3]$,

$$\Box_q \ = \ \left(\widehat{\mathcal{D}}_{\bar{v}}\widehat{\mathcal{D}}_v - q\widehat{\mathcal{D}}_-\widehat{\mathcal{D}}_+ T_v T_{\bar{v}}\right) T_v T_{\bar{v}} T_+ T_- \tag{40}$$

Weyl gravity equations hierarchy

Next we study another hierarchy which is given as follows:

$$\begin{tikzcd}[column sep=small] & C_m^+ \arrow[dr] & \\ C_m^h \arrow[ur] \arrow[dr] & & C_m^T \\ & C_m^- \arrow[ur] & \end{tikzcd} \tag{41}$$

where $m \in \mathbb{N}$, and the corresponding signatures are:

$$\begin{aligned} \chi_m^+ &= [2m, 0; 2], \quad \chi_m^- = [0, 2m; 2], \\ \chi_m^h &= [m, m; 2-m], \quad \chi_m^T = [m, m; 2+m] \end{aligned} \tag{42}$$

The arrows on (41) represent invariant differential operators of order m. It is a partial case of the general conformal scheme parametrized by three natural numbers p, ν, n, (cf. Fig. 3), setting there: $\nu = 1$, $p = n = m$. This hierarchy intersects with the Maxwell hierarchy for the lowest value $m = 1$. Below we consider the linear conformal gravity which is obtained for $m = 2$.

Linear conformal gravity

Linear conformal gravity is governed by the *Weyl tensor* $C_{\mu\nu\sigma\tau}$ which is given in terms of the Riemann curvature tensor $R_{\mu\nu\sigma\tau}$, Ricci curvature tensor $R_{\mu\nu}$, scalar curvature R :

$$\begin{aligned} C_{\mu\nu\sigma\tau} &= R_{\mu\nu\sigma\tau} - \tfrac{1}{2}(g_{\mu\sigma}R_{\nu\tau} + g_{\nu\tau}R_{\mu\sigma} - \\ &\quad - g_{\mu\tau}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\tau}) + \\ &\quad + \tfrac{1}{6}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})R \end{aligned} \tag{43}$$

where $g_{\mu\nu}$ is the metric tensor. Linear conformal gravity is obtained when the metric tensor is written as: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat Minkowski metric, $h_{\mu\nu}$ are small so that all quadratic and higher order terms are neglected. In particular:

$$R_{\mu\nu\sigma\tau} = \tfrac{1}{2}(\partial_\mu\partial_\tau h_{\nu\sigma} + \partial_\nu\partial_\sigma h_{\mu\tau} - \partial_\mu\partial_\sigma h_{\nu\tau} - \partial_\nu\partial_\tau h_{\mu\sigma})$$

The equations of linear conformal gravity are:

$$\partial^\nu \partial^\tau C_{\mu\nu\sigma\tau} = T_{\mu\sigma} \tag{44}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. From the symmetry properties of the Weyl tensor it follows that it has ten independent components. These may be chosen as follows (introducing notation for future use):

$$\begin{aligned}
&C_0 = C_{0123} \ , \quad C_1 = C_{2121} \ , \quad C_2 = C_{0202} \ , \\
&C_3 = C_{3012} \ , \quad C_4 = C_{2021} \ , \quad C_5 = C_{1012} \ , \\
&C_6 = C_{2023} \ , \quad C_7 = C_{3132} \ , \quad C_8 = C_{2123} \ , \\
&C_9 = C_{1213}
\end{aligned} \tag{45}$$

Furthermore, the Weyl tensor transforms as the direct sum of two conjugate Lorentz irreps, which we shall denote as $C^\pm$ (cf. (42) for $m = 2$). The tensors $T_{\mu\nu}$ and $h_{\mu\nu}$ are symmetric and traceless with nine independent components.

Further, we shall use again the fact that a Lorentz irrep (spin-tensor) with signature (n_1, n_2)

may be represented by a polynomial $G(z,\bar{z})$ in $z,\bar{z}$ of order n_1, n_2, resp. More explicitly, for the Weyl gravity representations mentioned above we use:

$$\begin{aligned}
C^+(z) &= z^4C^+_4 + z^3C^+_3 + z^2C^+_2 + zC^+_1 + C^+_0 \ , \\
C^-(\bar{z}) &= \bar{z}^4C^-_4 + \bar{z}^3C^-_3 + \bar{z}^2C^-_2 + \bar{z}C^-_1 + C^-_0 \ , \\
T(z,\bar{z}) &= z^2\bar{z}^2T'_{22} + z^2\bar{z}T'_{21} + z^2T'_{20} + \\
&\quad + z\bar{z}^2T'_{12} + z\bar{z}T'_{11} + zT'_{10} + \\
&\quad + \bar{z}^2T'_{02} + \bar{z}T'_{01} + T'_{00} \ , \qquad (46) \\
h(z,\bar{z}) &= z^2\bar{z}^2h'_{22} + z^2\bar{z}h'_{21} + z^2h'_{20} + \\
&\quad + z\bar{z}^2h'_{12} + z\bar{z}h'_{11} + zh'_{10} + \\
&\quad + \bar{z}^2h'_{02} + \bar{z}h'_{01} + h'_{00}
\end{aligned}$$

The components $C^\pm_k$ are given in terms of the

Weyl tensor components as follows:

$$
\begin{aligned}
C_0^+ &= C_2 - \frac{1}{2}C_1 - C_6 + i(C_0 + \frac{1}{2}C_3 + C_7) \\
C_1^+ &= 2(C_4 - C_8 + i(C_9 - C_5)) \\
C_2^+ &= 3(C_1 - iC_3) \\
C_3^+ &= 8(C_4 + C_8 + i(C_9 + C_5)) \\
C_4^+ &= C_2 - \frac{1}{2}C_1 + C_6 + i(C_0 + \frac{1}{2}C_3 - C_7) \\
C_0^- &= C_2 - \frac{1}{2}C_1 - C_6 - i(C_0 + \frac{1}{2}C_3 + C_7) \\
C_1^- &= 2(C_4 - C_8 - i(C_9 - C_5)) \\
C_2^- &= 3(C_1 + iC_3) \\
C_3^- &= 2(C_4 + C_8 - i(C_9 + C_5)) \\
C_4^- &= C_2 - \frac{1}{2}C_1 + C_6 - i(C_0 + \frac{1}{2}C_3 - C_7)
\end{aligned}
\tag{47}
$$

while the components T'_{ij} are given in terms of

$T_{\mu\nu}$ as follows:

$$\begin{aligned}
T'_{22} &= T_{00} + 2T_{03} + T_{33} \\
T'_{11} &= T_{00} - T_{33} \\
T'_{00} &= T_{00} - 2T_{03} + T_{33} \\
T'_{21} &= T_{01} + iT_{02} + T_{13} + iT_{23} \\
T'_{12} &= T_{01} - iT_{02} + T_{13} - iT_{23} \\
T'_{10} &= T_{01} + iT_{02} - T_{13} - iT_{23} \\
T'_{01} &= T_{01} - iT_{02} - T_{13} + iT_{23} \\
T'_{20} &= T_{11} + 2iT_{12} - T_{22} \\
T'_{02} &= T_{11} - 2iT_{12} - T_{22}
\end{aligned} \tag{48}$$

and similarly for h'_{ij} in terms of $h_{\mu\nu}$.

In these terms all linear conformal Weyl gravity equations (44) (cf. also (41)) may be written in compact form as the following pair of equations:

$$I^+ \, C^+(z) \; = \; T(z,\bar{z}) \, , \qquad I^- \, C^-(\bar{z}) \; = \; T(z,\bar{z}) \tag{49}$$

where the operators $I^{\pm}$ are given as follows:

$$
\begin{aligned}
I^{+} \;=\;& \Big(z^2\bar{z}^2\partial_+^2 + z^2\partial_v^2 + \bar{z}^2\partial_{\bar{v}}^2 + \partial_-^2 + \\
& +2z^2\bar{z}\partial_v\partial_+ + 2z\bar{z}^2\partial_+\partial_{\bar{v}} + \\
& +2z\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \\
& +2\bar{z}\partial_-\partial_{\bar{v}} + 2z\partial_v\partial_- \Big)\partial_z^2 - \\
& -6\Big(z\bar{z}^2\partial_+^2 + z\partial_v^2 + 2z\bar{z}\partial_v\partial_+ + \bar{z}^2\partial_+\partial_{\bar{v}} + \\
& +\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \partial_v\partial_- \Big)\partial_z + \\
& +12\Big(\bar{z}^2\partial_+^2 + \partial_v^2 + 2\bar{z}\partial_v\partial_+ \Big) \;, \qquad (50) \\
I^{-} \;=\;& \Big(z^2\bar{z}^2\partial_+^2 + z^2\partial_v^2 + \bar{z}^2\partial_{\bar{v}}^2 + \partial_-^2 + \\
& +2z^2\bar{z}\partial_v\partial_+ + 2z\bar{z}^2\partial_+\partial_{\bar{v}} + \\
& +2z\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \\
& +2\bar{z}\partial_-\partial_{\bar{v}} + 2z\partial_v\partial_- \Big)\partial_{\bar{z}}^2 - \\
& -6\Big(z^2\bar{z}\partial_+^2 + \bar{z}\partial_{\bar{v}}^2 + 2z\bar{z}\partial_+\partial_{\bar{v}} + z^2\partial_v\partial_+ + \\
& +z(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \partial_-\partial_{\bar{v}} \Big)\partial_{\bar{z}} + \\
& +12\Big(z^2\partial_+^2 + \partial_{\bar{v}}^2 + 2z\partial_+\partial_{\bar{v}} \Big)
\end{aligned}
$$

To make more transparent the origin of (49) and in the same time to derive the quantum group deformation of (49), (50) we first introduce the following parameter-dependent operators:

$$\begin{aligned} I^+_n &= \frac{1}{2}\Big(n(n-1)I_1^2 I_2^2 - 2(n^2-1)I_1 I_2^2 I_1 \ + \\ &\quad + \ n(n+1)I_2^2 I_1^2\Big), \qquad (51) \\ I^-_n &= \frac{1}{2}\Big(n(n-1)I_3^2 I_2^2 - 2(n^2-1)I_3 I_2^2 I_3 \ + \\ &\quad + \ n(n+1)I_2^2 I_3^2\Big) \end{aligned}$$

where $I_1 = \partial_z$, $I_2 = \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-$, $I_3 = \partial_{\bar{z}}$, are from (10). We recall that group-theoretically the operators I_a correspond to the three simple roots of the root system of $sl(4)$, while the operators $I^\pm_n$ correspond to the singular vectors for the two non-simple non-highest roots. More precisely, the operator I^+_n is obtained from the $sl(4)$ formula for the singular vector of weight $m_{12}\alpha_{12} = 2\alpha_{12}$, while the operator I^-_n corresponds to weight $m_{23}\alpha_{23} = 2\alpha_{23}$. The parameter $n = \max(2j_1, 2j_2)$.

It is easy to check that we have the following relation:

$$I^{\pm} = I^{\pm}_4 \tag{52}$$

i.e., (49) are written as:

$$I^+_4 \, C^+(z) \ = \ T(z,\bar{z}) \ , \qquad I^-_4 \, C^-(\bar{z}) \ = \ T(z,\bar{z}) \tag{53}$$

Using the same operators we can write down the pair of equations which give the Weyl tensor components in terms of the metric tensor:

$$I^+_2 \, h(z,\bar{z}) \ = \ C^-(\bar{z}) \ , \qquad I^-_2 \, h(z,\bar{z}) \ = \ C^+(z) \tag{54}$$

We stress again the advantage of the indexless formalism due to which two different pairs of equations, (53), (54), may be written using the same parameter-dependent operator expressions by just specializing the values of a parameter.

The above equations are is immediately generalizable to the deformed case.

Using the $U_q(sl(4))$ formula for the singular vector given in [Dh,DII] we obtain for the q-analogue of (51):

$$\begin{aligned} {}_qI_n^+ &= \frac{1}{2}\Big([n]_q\,[n-1]_q\ {}_qI_1^2\,{}_qI_2^2\ - \\ &\quad -\ [2]_q\,[n-1]_q\,[n+1]_q\,{}_qI_1\,{}_qI_2^2\,{}_qI_1 + \\ &\quad +[n]_q\,[n+1]_q\,{}_qI_2^2\,{}_qI_1^2\Big)\ , \end{aligned} \tag{55}$$

$$\begin{aligned} {}_qI_n^- &= \frac{1}{2}\Big([n]_q\,[n-1]_q\,{}_qI_3^2\,{}_qI_2^2\ - \\ &\quad -\ [2]_q\,[n-1]_q\,[n+1]_q\,{}_qI_3\,{}_qI_2^2\,{}_qI_3 + \\ &\quad +[n]_q\,[n+1]_q\,{}_qI_2^2\,{}_qI_3^2\Big) \end{aligned}$$

where the q-deformed ${}_qI_a$ were given above.

Then the q-Weyl gravity equations are (cf. (53)):

$$ {}_qI_4^+\, C^+(z) \;=\; T(z,\bar{z})\,, \qquad {}_qI_4^-\, C^-(\bar{z}) \;=\; T(z,\bar{z}) \tag{56} $$

while q-analogues of (54) are:

$$ {}_qI_2^+\, h(z,\bar{z}) \;=\; C^-(\bar{z})\,, \qquad {}_qI_2^-\, h(z,\bar{z}) \;=\; C^+(z) \tag{57} $$

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