Topological recursion :

a recursive way of counting surfaces

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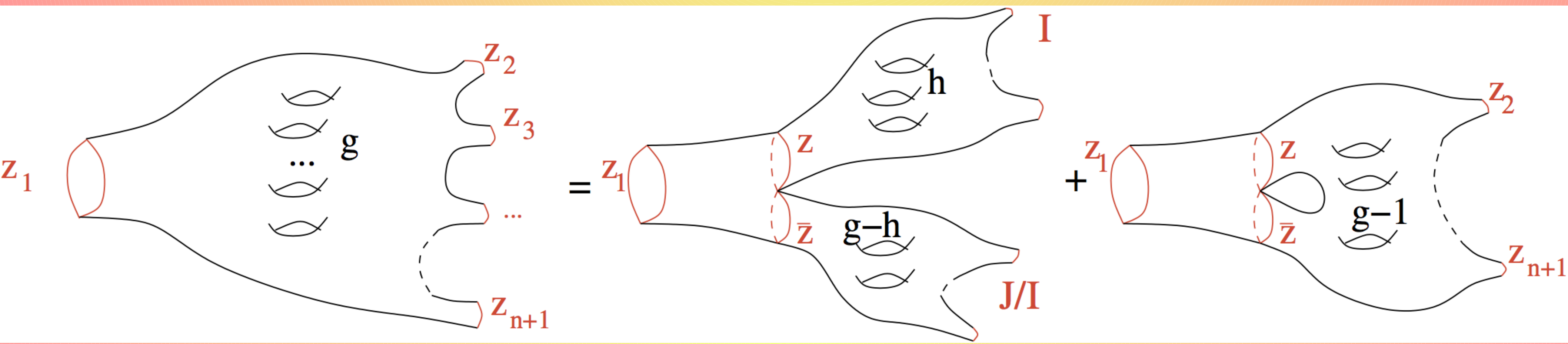
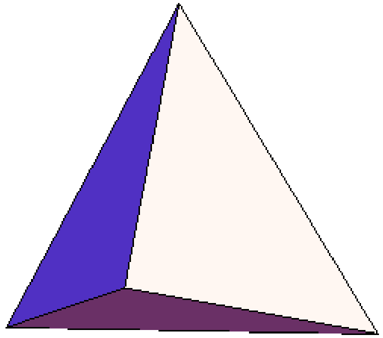
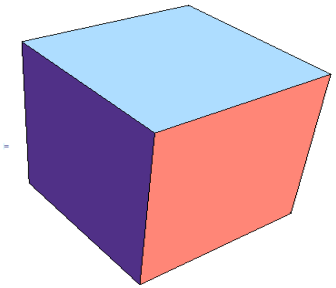
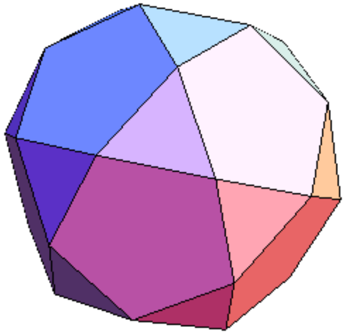
Isle sur Sorgues octobre 2018



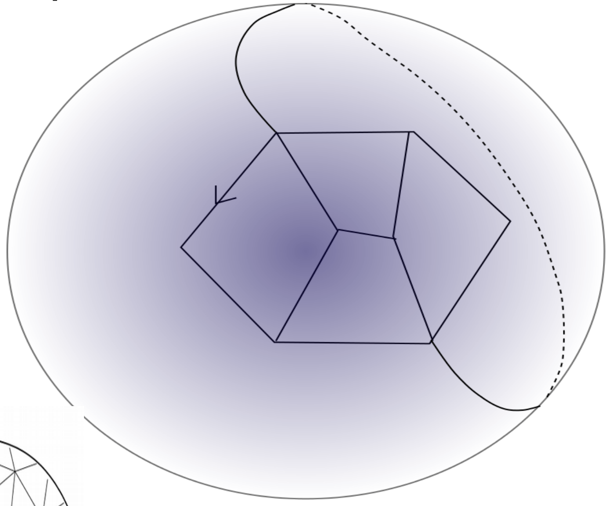


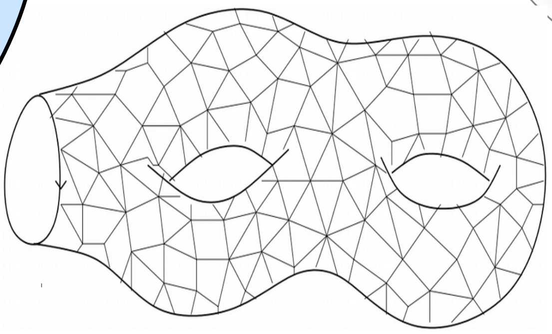




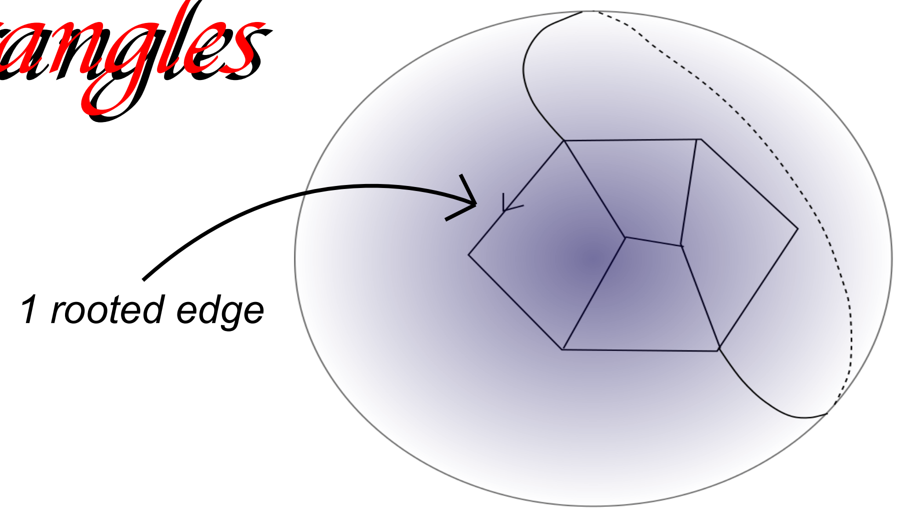
I. Counting discrete surfaces

Also called « maps »





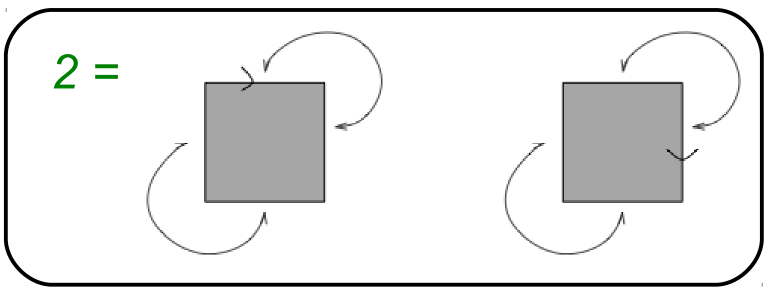
Surfaces made of quadrangles

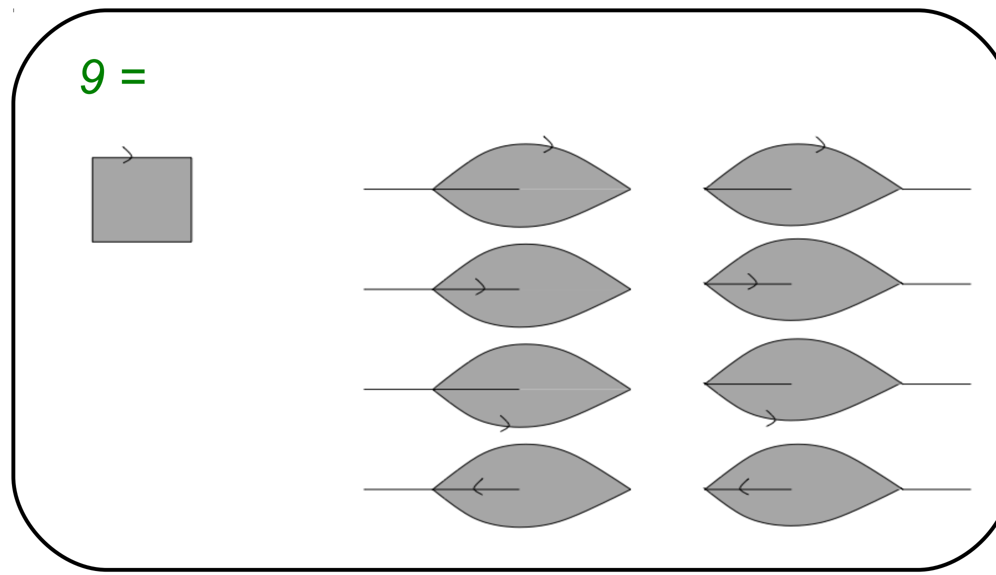


W. Tutte's *formula [1960's]*

Number of rooted quadrangulations of the sphere with k faces = $\frac{2.3^k(2k)!}{k!(k+2)!}$

2, 9, 54, 378, 2916, 24057, ...





- Make **generating series** :

$2t+9t^2+54t^3+378t^4+2916t^5+\ldots$

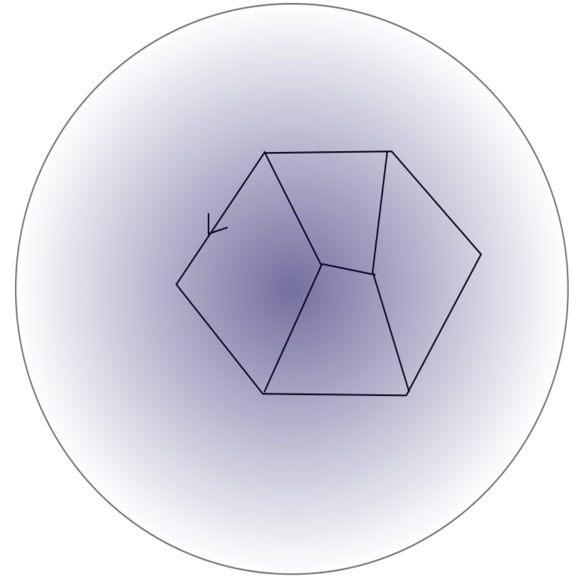
⇨ gives :

$$\sum_k t^k \frac{2.3^k(2k)!}{k!(k+2)!} = \frac{1-r^2}{4t}\,\frac{1+3r}{3\,(1+r)^3}$$

$$= \frac{1+3\sqrt{1-12t}}{(1+\sqrt{1-12t})^3}$$

where : $r=\sqrt{1-12t}$

Surfaces made of quadrangles



$2l=6$
$k=4$

$\to 225$

Number of quadrangulations
of the sphere with
1 rooted face of length $2l$
and k unmarked faces :

$$\sum_k \sum_l \frac{t^k}{z^{2l+1}}\, 3^k\, \frac{2l\,!}{l\,!\,(l-1)\,!}\, \frac{(2k+l-1)\,!}{k\,!\,(l+k+1)\,!} = \frac{1}{2}\left(z - t z^3 - \left(t z^2 - \frac{2+r}{3}\right)\sqrt{z^2 - \frac{8}{1+r}}\right)$$

$$= y = \boxed{W_{0,1}(z;t)}$$

map $t \to r$

$r = \sqrt{1-12t}$

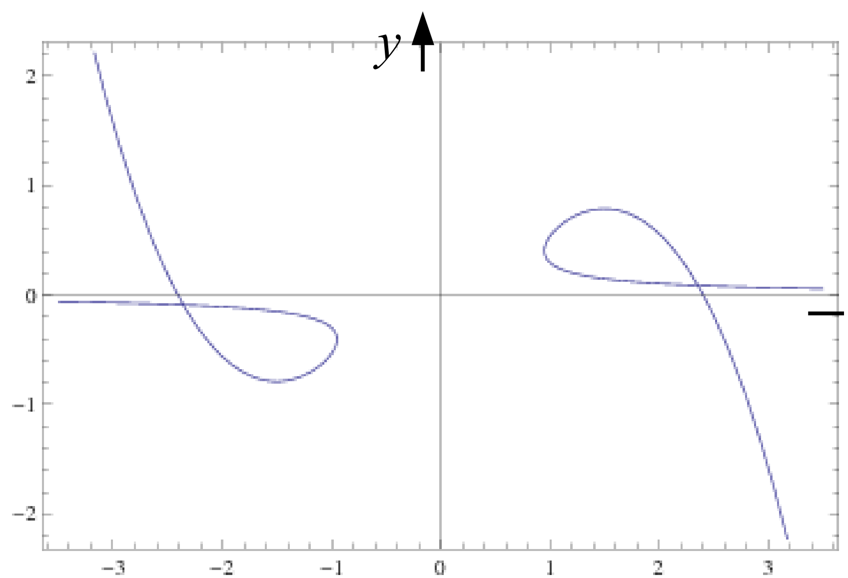
« Mirror map »

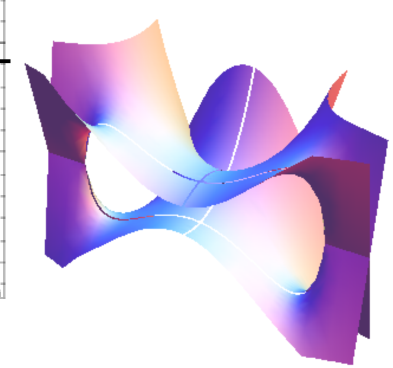
Spectral Curve

Algebraic relation

$$P(z,y,r) = 0$$

$P \in \mathbf{Z}[z,y,r]$

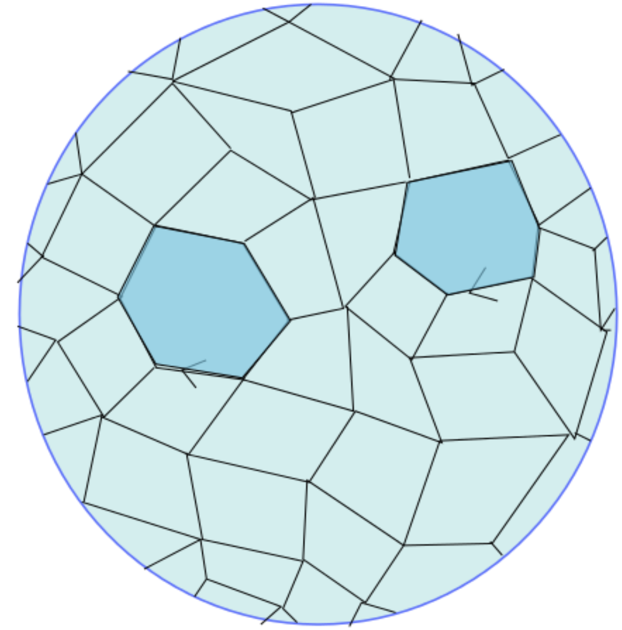




Surfaces made of quadrangles

Sphere

2 rooted marked faces

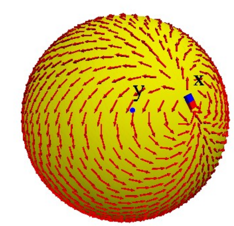


2 rooted faces of lengths $l_1=\frac{a+b}{2}, l_2=\frac{a-b}{2}$
and k unmarked faces :

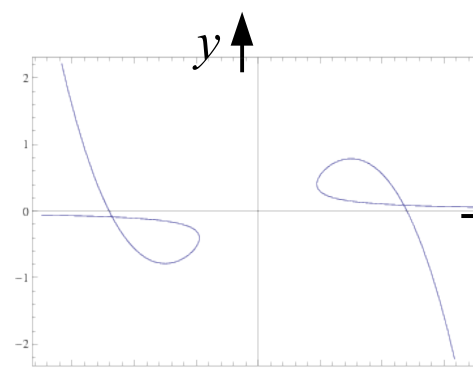
$$\sum_k \sum_{l_1,l_2} \frac{t^k}{z_1^{l_1+1} z_2^{l_2+1}} \; 3^k a \frac{(2k+a-1)!}{k!(k+a)!} \sum_{j=0}^{\frac{a-b}{2}} \frac{(a-b-2j)\;(a+b)!(a-b)!}{j!(b+j)!(a-j)!(a-b-j)!}$$

$$= \boxed{W_{0,2}(z_1,z_2;t)} = \frac{1}{2(z_1-z_2)^2}\left(1-\frac{z_1 z_2-\frac{8}{1+r}}{\sqrt{(z_1^2-\frac{8}{1+r})(z_2^2-\frac{8}{1+r})}}\right)$$

$$r=\sqrt{1-12t}$$



$W_{0,2}$ = Green function" on the Spectral Curve $W_{0,1}$



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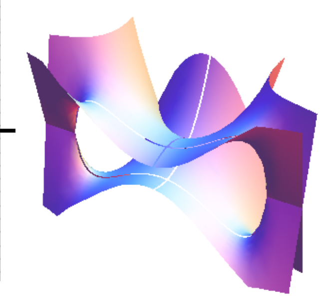


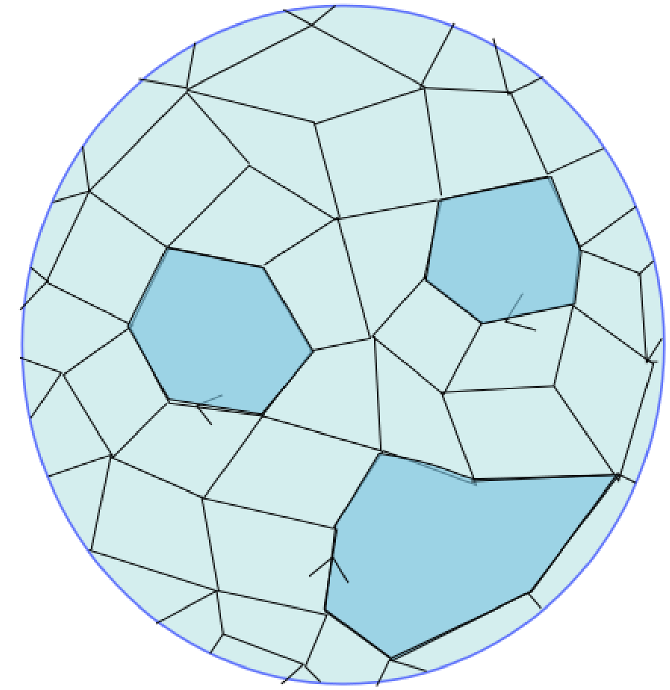
Fundamental 2nd kind form

Bergmann kernel

Surfaces made of quadrangles

Surface with g holes

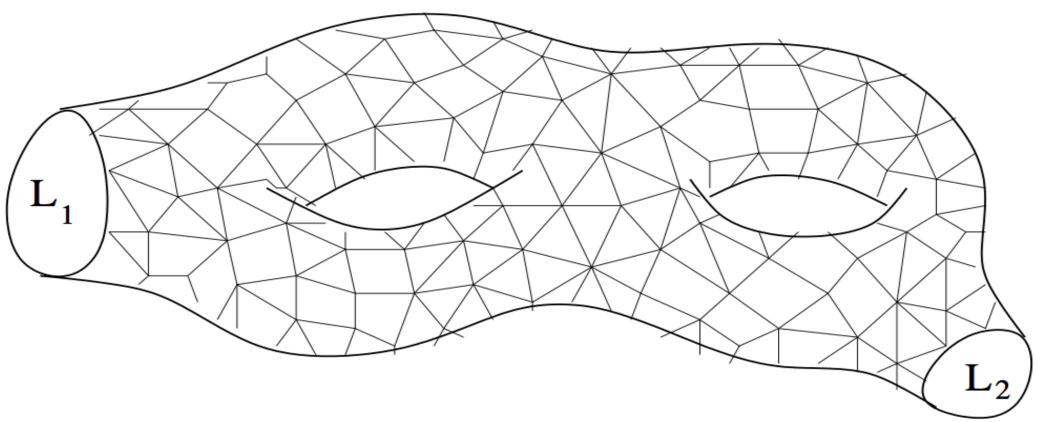
n rooted marked faces



n rooted faces of lengths $l_1, l_2, l_3, \ldots, l_n$

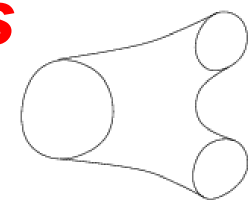
and k unmarked faces :

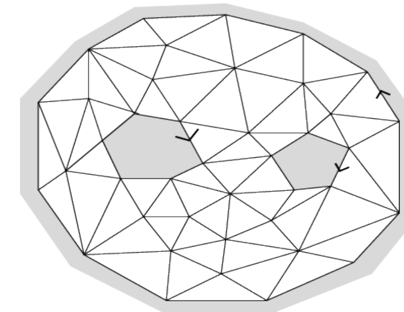
$$W_{g,n}(z_1, z_2, \ldots, z_n; t)$$



Question : what is the formula for $W_{g,n}$?

Formulas for Higher topologies

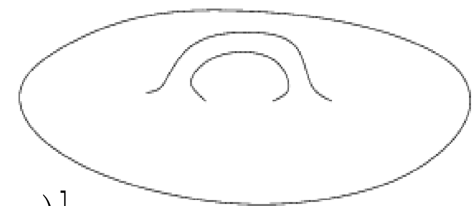




0 holes, 3 marked faces

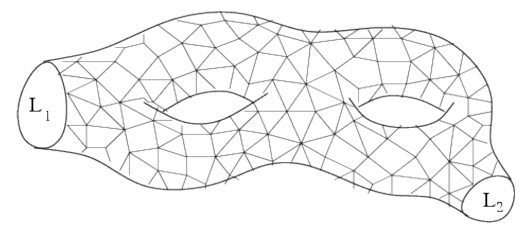
$$W_{0,3}(z_1,z_2,z_3)=\underset{z\to\pm\sqrt{\frac{8}{1+r}}}{Res}\frac{\sqrt{z^2-\frac{8}{1+r}}}{2(z_1-z)\sqrt{z_1^2-\frac{8}{1+r}}\,W_{0,1}(z)}\left[2W_{0,2}(z,z_2)W_{0,2}(z,z_3)\right]$$

1 hole, 1 marked face



$$W_{1,1}(z_1)=\underset{z\to\pm\sqrt{\frac{8}{1+r}}}{Res}\frac{\sqrt{z^2-\frac{8}{1+r}}}{2(z_1-z)\sqrt{z_1^2-\frac{8}{1+r}}\,W_{0,1}(z)}\left[W_{0,2}(z,z)\right]$$

And in general   *g* holes, *n* marked faces



$$W_{g,n+1}(z_1,\dots,z_{n+1})=\underset{z\to\pm\sqrt{\frac{8}{1+r}}}{Res}\frac{\sqrt{z^2-\frac{8}{1+r}}}{2(z_1-z)\sqrt{z_1^2-\frac{8}{1+r}}\,W_{0,1}(z)}*$$

$$\left[W_{g-1,n+2}(z,z,z_2,\dots,z_{n+1})+\sum W_{g_1,|I_1|+1}(z,I_1)W_{g_2,|I_2|+1}(z,I_2)\right]$$

$\exists$ Operator kernel $K(z_1,z)$ Such that

$$W_{g,n+1}(z_1,\dots,z_{n+1})=K(z_1,z)*\left[W_{g-1,n+2}(z,z,z_2,\dots,z_{n+1})+\sum W_{g_1,|I_1|+1}(z,I_1)W_{g_2,|I_2|+1}(z,I_2)\right]$$

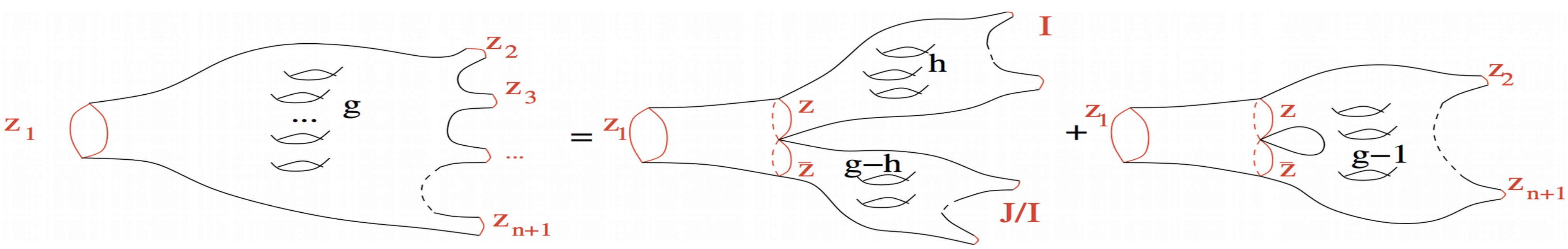
Recursion relation

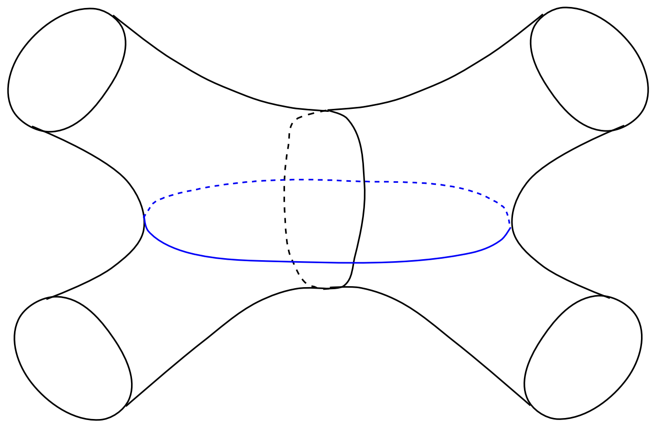
Theorem [E04]

∃ Operator kernel $K(z_1, z)$ Such that

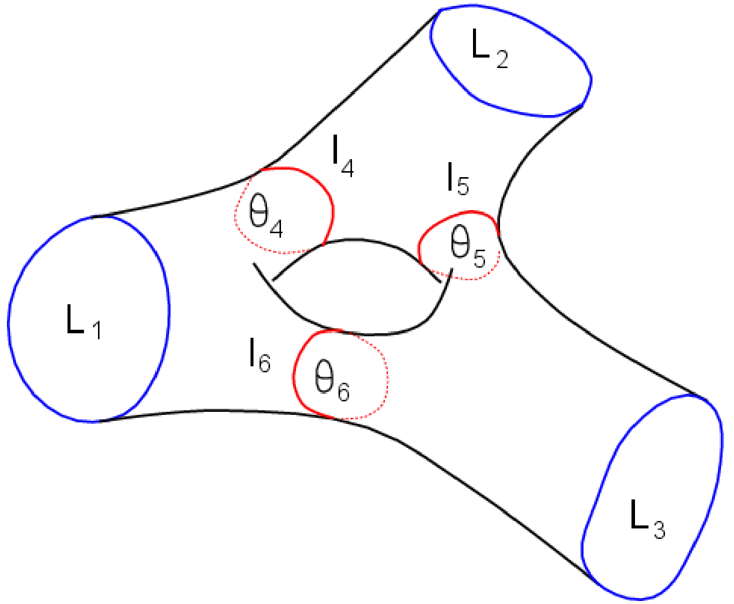
$$W_{g,n+1}(z_1,\ldots,z_{n+1}) = K(z_1,z) * \left[W_{g-1,n+2}(z,z,z_2,\ldots,z_{n+1}) + \sum W_{g_1,|I_1|+1}(z,I_1) W_{g_2,|I_2|+1}(z,I_2) \right]$$

- It allows to compute recursively all the $W_{g,n}(z_1,\ldots,z_n)$
- Recursion on $2g+n-2$ → allows to find all (g,n) by gluing pieces of (0,2) and (0,1).





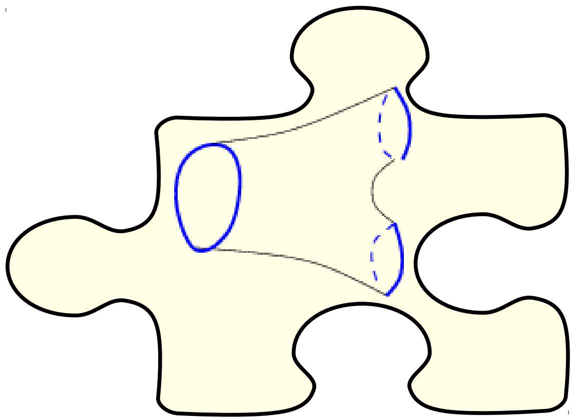
Hyperbolic surfaces



Count hyperbolic surfaces

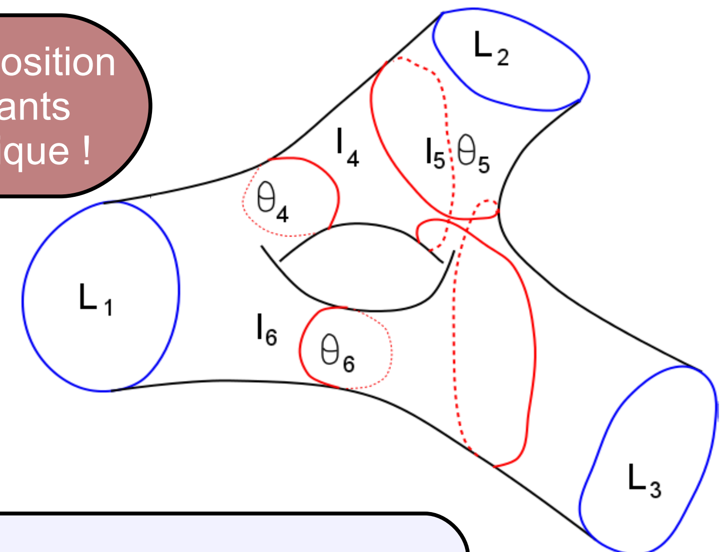
$M_{g,n}(L_1,\ldots,L_n) = \left\{ \begin{array}{l} \text{surfaces with constant curvature R=-1} \\ \text{with n geodesic boundaries of lengths } L_1,\ldots,L_n \end{array} \right\}$

- Decompose into « *pairs of pants* »



Decomposition into pants Not unique !

(g=1, n=3)



- Measure Volume (*Weil - Petersson*) :

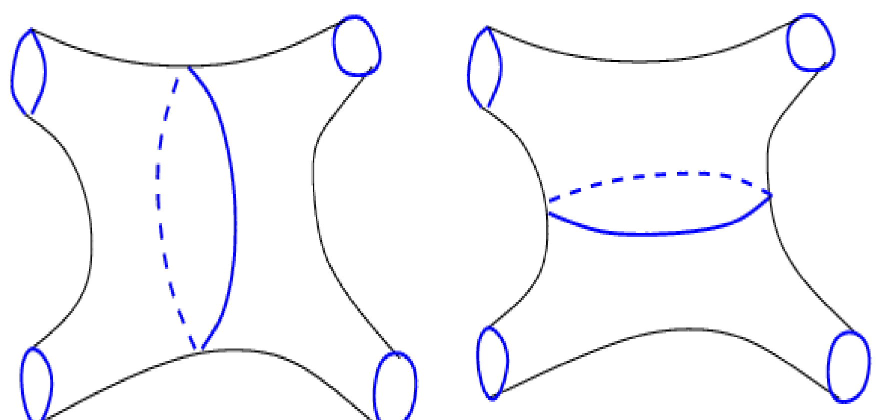
$$V_{g,n}(L_1,\ldots,L_n) = \int_{M_{g,n}(L_1,\ldots,l_n)} \prod_{edges} dl \wedge d\theta$$

$$V_{0,3}(L_1,L_2,L_3) = 1$$

$$V_{1,1}(L_1) = \frac{1}{24}(2\pi^2+L_1^2)$$

$$V_{0,4}(L_1,\ldots,L_4) = 2\pi^2+\frac{1}{2}\sum_i L_i^2$$

(g=0, n=4)



Recursive counting

- Laplace transforms of volumes

$$W_{g,n}(z_1,\dots,z_n) = \int_0^\infty \dots \int_0^\infty L_1 dL_1 e^{-z_1 L_1} \dots L_n dL_n e^{-z_n L_n} \; V_{g,n}(L_1,\dots,L_n)$$

Theorem : *[Mirzakhani 2004]* *Laplace transformed [EO 2006]*

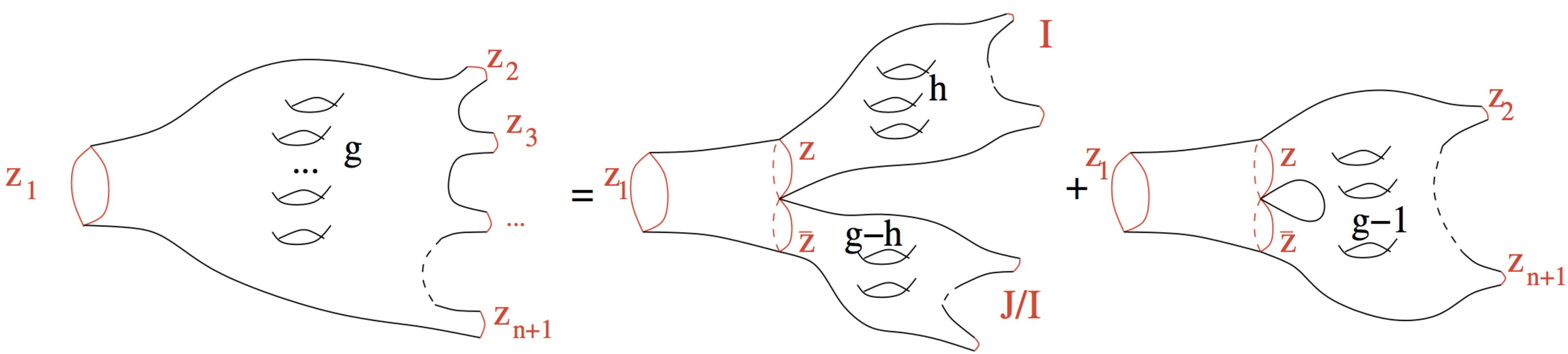
$$W_{g,n}(z_1,\dots,z_n) = Res_{z\to 0} K(z_1,z) \left[W_{g-1,n+1}(z,-z,z_2,\dots,z_n) + \sum W_{g_1,|I_1|+1}(z,I_1) W_{g_2,|I_2|+1}(-z,I_2) \right]$$

$$K(z_1,z) = \frac{\pi}{(z_1^2 - z^2)\sin(2\pi z)} \qquad W_{0,2}(z_1,z_2) = \frac{1}{(z_1-z_2)^2}$$

Interpretation :

- Can count surfaces of genus *g*, with *n* boundaries by recursion on their Euler characteristics $\chi = 2-2g-n$

Topological Recursion



Recursive counting

Theorem : *[Mirzakhani 2004]* *Laplace transformed [EO 2006]*

$$W_{g,n}(z_1,\dots,z_n)=Res_{z\to 0}K(z_1,z)\left[W_{g-1,n+1}(z,-z,z_2,\dots,z_n)+\sum W_{g_1,|I_1|+1}(z,I_1)W_{g_2,|I_2|+1}(-z,I_2)\right]$$

$$K(z_1,z)=\frac{\pi}{(z_1^2-z^2)\sin(2\pi z)} \qquad W_{0,2}(z_1,z_2)=\frac{1}{(z_1-z_2)^2}$$

- Example $W_{1,1}(z_1)=\int_0^\infty L_1\,dL_1\,e^{-z_1L_1}\left(\frac{L_1^2}{24}+\frac{2\pi^2}{24}\right)=\frac{3}{24z_1^4}+\frac{2\pi^2}{24z_1^2}$

$$W_{1,1}(z_1)=Res_{z\to 0}K(z_1,z)\left[W_{0,2}(z,-z)\right]$$

$$=Res_{z\to 0}\frac{\pi}{(z_1^2-z^2)\sin(2\pi z)}\frac{1}{4z^2}$$

$$=\frac{1}{8z_1^2}Res_{z\to 0}\frac{1}{z^3}\left(1-\frac{z^2}{z_1^2}\right)^{-1}\left(1-\frac{1}{6}(2\pi z)^2+\dots\right)^{-1}$$

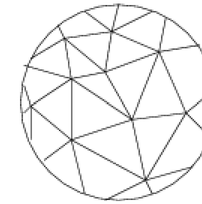
$$=\frac{1}{8z_1^2}Res_{z\to 0}\frac{1}{z^3}\left(1+\frac{z^2}{z_1^2}+\frac{1}{6}(2\pi z)^2+\dots O(z^4)\right)$$

$$=\frac{1}{8z_1^2}\left(\frac{1}{z_1^2}+\frac{1}{6}(2\pi)^2\right)=\frac{3}{24z_1^4}+\frac{2\pi^2}{24z_1^2}$$

Topological Recursion

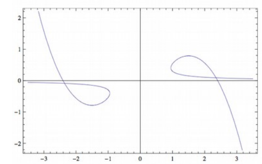
New Invariants

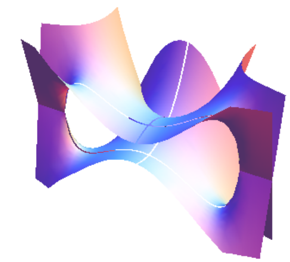
Summary and generalization



- Disc amplitude $W_{0,1}(z)$ = « number » of ***discs*** of parameter z

  $\rightarrow$ spectral curve (= surface in $\mathbb{C}\times\mathbb{C}$)

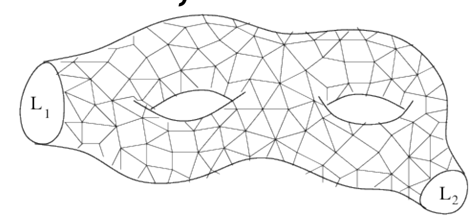




- $W_{0,2}(z_1, z_2)$ = « number » of ***cylinders*** of parameters z_1, z_2

  = fundamental form on the surface $y = W_{0,1}(z)$

- $W_{g,n}(z_1, \ldots, z_n)$ = « number » of ***surfaces of genus g and n boundaries,*** of parameters $z_1, \ldots, z_n$



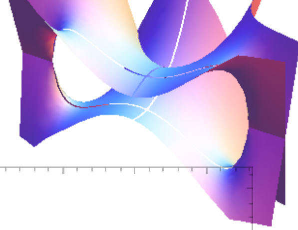
By the « topological recursion » on 2g-2+n :

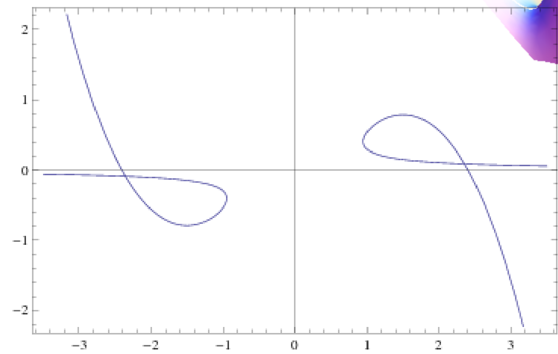
$$K(z_1, z) = \frac{1}{2} \frac{\int_{z'=\bar{z}}^{z} W_{0,2}(z_1, z')}{W_{0,1}(z) - W_{0,1}(\bar{z})}$$

$$W_{g,n}(z_1, \ldots, z_n) = Res_{z \to a} K(z_1, z) \left[W_{g-1,n+1}(z, -z, z_2, \ldots, z_n) + \sum W_{g_1, |I_1|+1}(z, I_1) W_{g_2, |I_2|+1}(-z, I_2) \right]$$

- Can we start from an arbitrary function $W_{0,1}(z) = f(z)$ **?**

Invariants of a curve





Pick an arbitrary

Spectral curve $S = \{ (x,y),\ P(x,y)=0 \}$

Define its ***invariants***

$$W_{0,1}(z) = y\,dx$$

$W_{0,2}(z_1,z_2) =$ fundamental 2nd kind differential

and then recursively (*Topological recursion*)

$$W_{g,n}(z_1,\dots,z_n)=K(z_1,z)*\left[W_{g-1,n+1}(z,\bar{z},z_2,\dots,z_n)+\sum W_{g_1,|I_1|+1}(z,I_1)\,W_{g_2,|I_2|+1}(\bar{z},I_2)\right]$$

$$K(z_1,z) = \frac{1}{2}\,\frac{\int_{z'=\bar{z}}^{z} W_{0,2}(z_1,z')}{W_{0,1}(z)-W_{0,1}(\bar{z})}$$

Universal : no parameter

Invariants of a curve

Topological recursion defines invariants of a spectral curve :

$$S \xrightarrow{\text{T.R.}} W_{g,n}$$

$W_{g,n}$ = symmetric n form on S^n

$W_{g,0} = F_g \in \mathbb{C}$

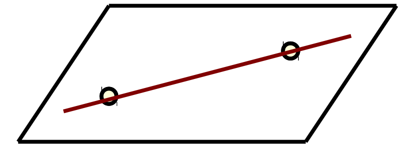
Plenty of beautiful mathematical properties

- Modular invariants
- Seiberg-Witten equations
- Integrability $\rightarrow$ τ - function
- Virasoro, W, link to CFT
- Commute with limits

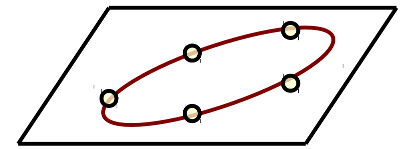
Gromov-Witten Invariants

Count surfaces in a target space

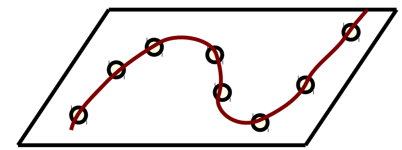
- How many lines through 2 points in CP^2 ? $\Longrightarrow$ 1



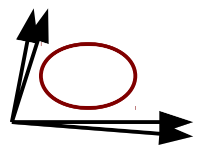
- How many conics through 5 points in CP^2 ? $\Longrightarrow$ 1



- How many degree 3 rational curves through 8 points in CP^2 ? $\Longrightarrow$ 12



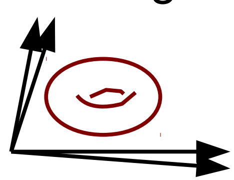
- How many degree *d* rational curves through *3d-1* points in CP^2 ? $\Longrightarrow$ $N_0(CP^2, d)$

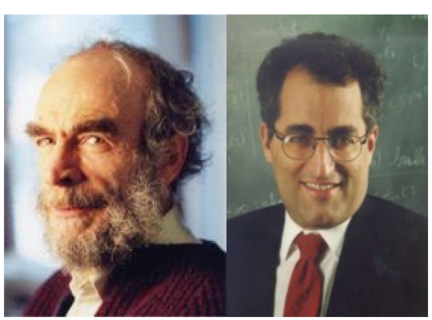


= *1, 1, 12, 620, 87304, ...*

- How many degree *d* elliptic curves through *3d* points in CP^2 ? $\Longrightarrow$ $N_1(CP^2, d)$

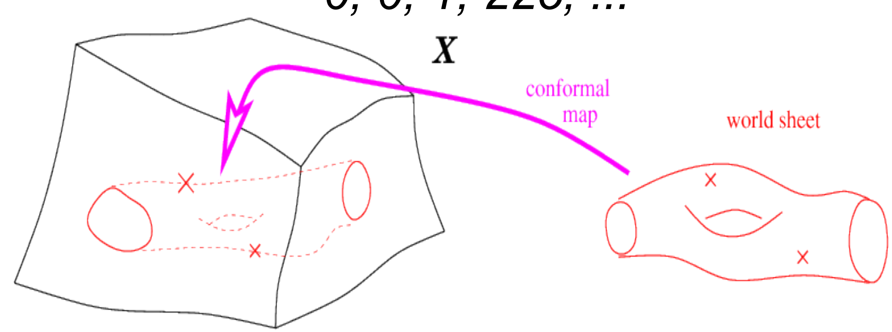
= *0, 0, 1, 225, ...*





$\Longrightarrow$ **Gromov-Witten** numbers

$N_g(X, \beta)$



Count surfaces in a target space

$X =$ Calabi Yau space, $\beta \in H_2(X;\mathbb{Z})$ = degree

Gromov-Witten invariants

$$M_{g,0}(X,\beta) = \{(\Sigma_g, f) \mid f: \Sigma_g \to X,\ f(\Sigma_g) \in \beta\}$$

$$N_{g,0}(X,\beta) = \int_{M_{g,0}(X,\beta)} 1$$

= Number of holomorphic (*stable*) maps

$L =$ *(special)* Lagrangian submanifold (brane), $\beta \in H_2(X,L;\mathbb{Z})$ $w_i \in H_1(L;\mathbb{Z})$ = degrees

Open Gromov-Witten invariants

$N_{g,n}(X,L,\beta,\vec{w})$ = Number of holomorphic (*stable*) maps $f: \Sigma_{g,n} \to X$, $f(\Sigma_{g,n}) \in \beta$, $f(\partial_i \Sigma_{g,n}) \in w_i$

Generating series

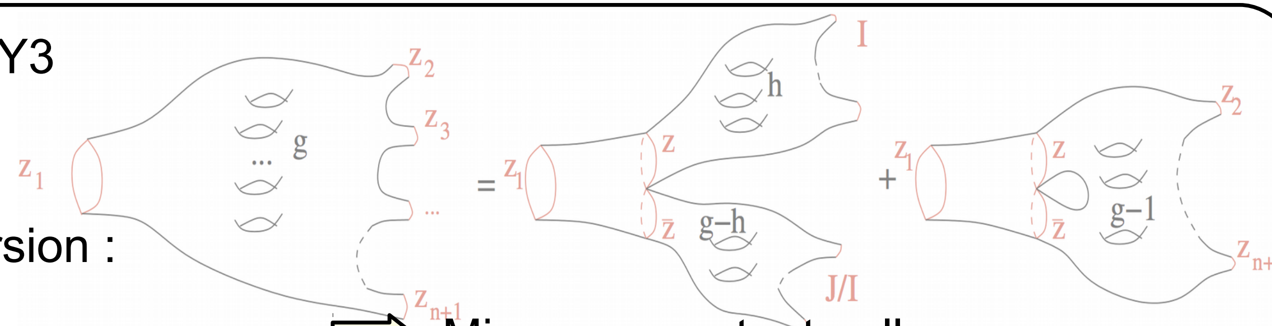
$$W_{g,n}(X,L,t,z_1,\dots,z_n) = \sum_{\beta,\vec{w}} e^{t.\beta}\, N_{g,n}(X,L,\beta,\vec{w}) \prod_{i=1}^{n} e^{w_i z_i} w_i\, dz_i$$

<u>Theorem</u> for X = toric CY3

[EO 2012, BLZ 2014]

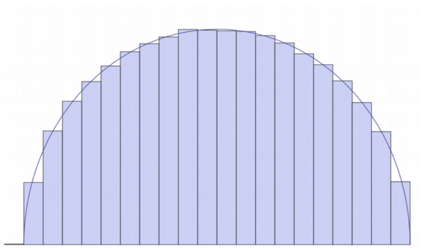
$W_{g,n}$ satisfy the topological recursion :

for spectral curve = mirror of X

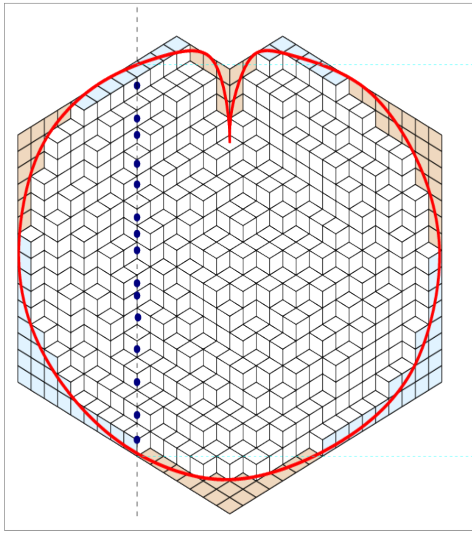


⇨ Mirror symmetry to all genus

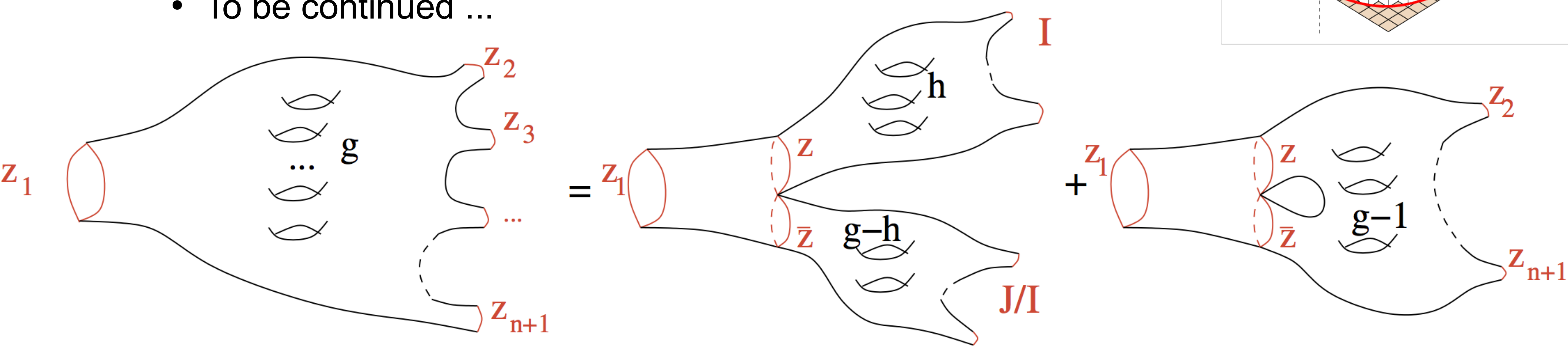
Conclusion



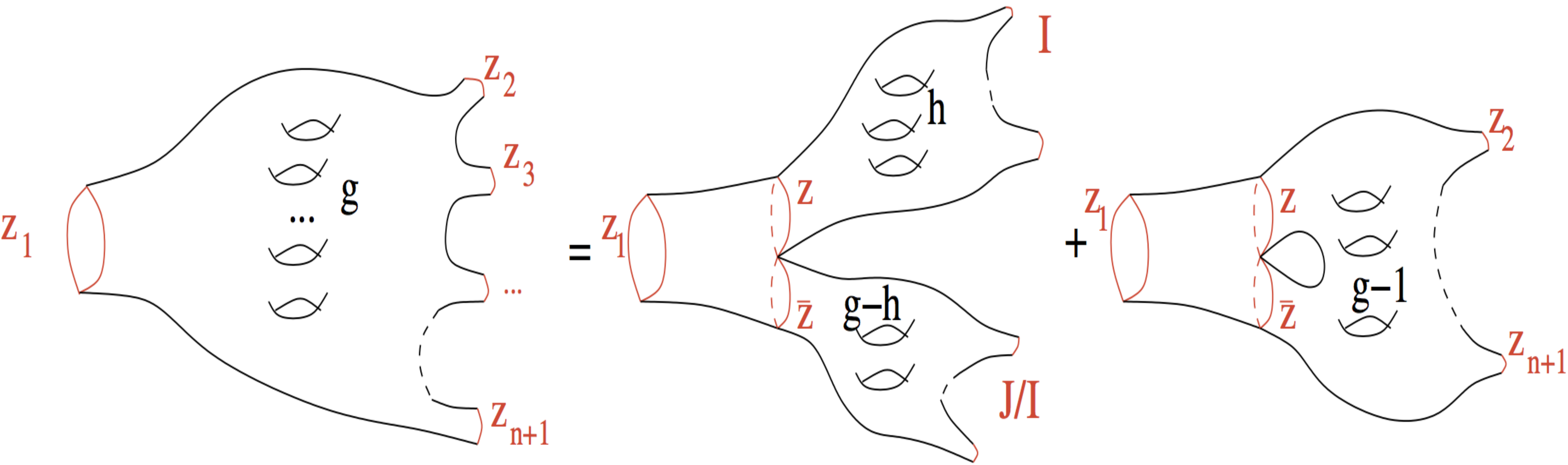
The topological recursion is



- A universal equation, satisfied by many geometric invariants
- Computational : easy to implement on a computer
- A definition of new invariants : the invariants of spectral curves
- Many beautiful mathematical properties
- Has found many applications, more to come ...
  Hitchin systems, moduli spaces of connections
- To be continued ...



Thank you for your attention



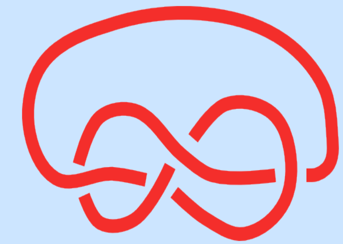
Application : knot theory

- Consider a knot 

S = ***A*** – Polynomial of the knot $A(e^x, e^y) = 0$

Example : fig 8 knot

$$A(e^x, e^y) = e^y + e^{-y} - e^{2x} - 2^{-2x} + e^x + e^{-x} + 2$$



A – Polynomial $\xrightarrow{\text{T.R.}}$ $W_{g,n}$

$W_{g,n}$ $\longrightarrow$ $\psi_\hbar(x) = e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int^x \dots \int^x W_{g,n}} \times \theta_{[\chi]}(\frac{1}{\hbar}\ \nu, \tau)$

Volume conjecture (extended) :

$\psi_\hbar(x)$ is the asymptotic expansion of the ***Jones polynomial*** $J_N(q)$

$\hbar = \ln q \qquad x = N\hbar$

(more precisely : it is the asymptotic expansion of a solution of the same holnomic equation as the Jones polynomial)

Application : CFT

Liouville theory — central charge $c = 1 - 6Q^2$ — $Q = b + 1/b$

Vertex operators $V_{\alpha_i}(z_i)$

Currents $J(x_i)$ — Stress energy tensor $T(x_i) = -J(x_i)^2 + Q\,\partial J(x_i)$

Amplitudes $\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \ldots V_{\alpha_N}(z_N) \rangle$

$$W_n(x_1, \ldots, x_n) = \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \ldots V_{\alpha_N}(z_N) \;\; J(x_1) \ldots J(x_n) \rangle$$

Heavy limit $\log \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \ldots V_{\alpha_N}(z_N) \rangle \sim \sum_g Q^{2-2g} F_g$

$$W_n(x_1, \ldots, x_n) \sim \sum_g Q^{2-2g-n} W_{g,n}(x_1, \ldots, x_n)$$

Amplitudes $W_{g,n}(x_1, \ldots, x_n)$ computed from *Topological Recursion* :

from Spectral curve : $(y^2 - t(x)) \cdot \psi(x) = 0$

$$t(x) = \frac{1}{Q^2} \sum_{i=1}^{N} \frac{\alpha_i (Q - \alpha_i)}{(x - z_i)^2} + \frac{\beta_i}{x - z_i}$$

$y = \dfrac{d}{dx}$ $\quad [x, y] = 1$

String Theory

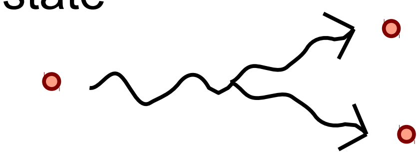
in a (very small) nut shell

Classical Physics : *particle =*

- Particles are « points » that move along trajectories (lines)

Quantum Physics :

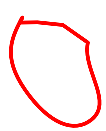
- Probabilistic : probability of final state, knowing initial state

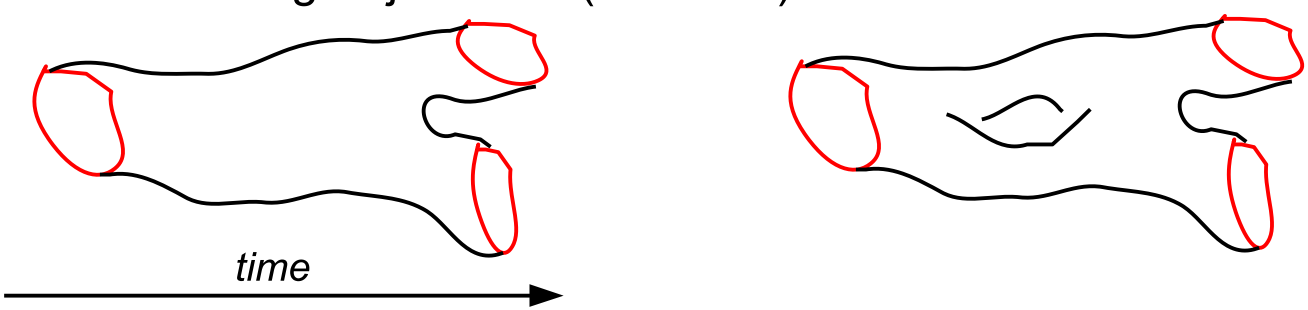


Classical strings :

- Particles are « strings » that move along trajectories (surfaces)

particle =

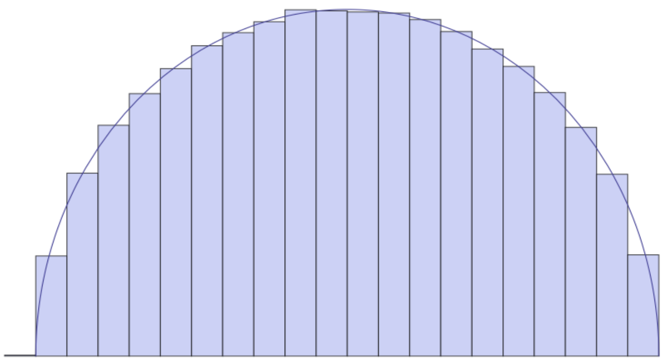


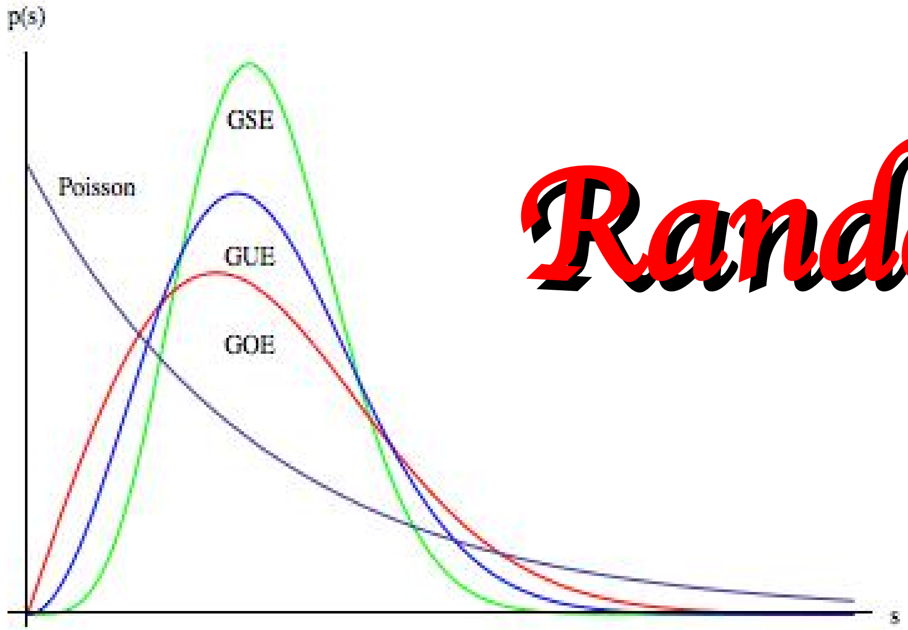


Quantum strings :

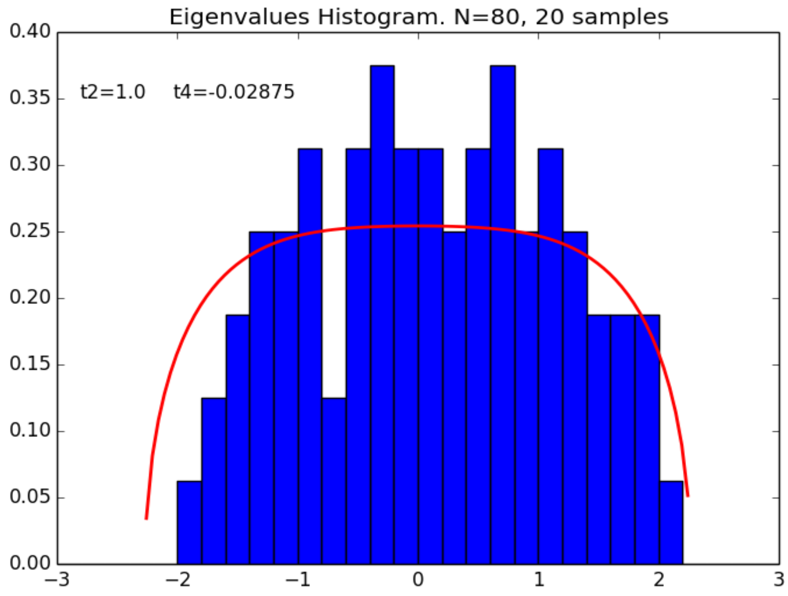
- Probabilistic : probability law on surfaces knowing the boundaries

⇒ Necessity to « count » (*measure*) sets of surfaces





Random matrices

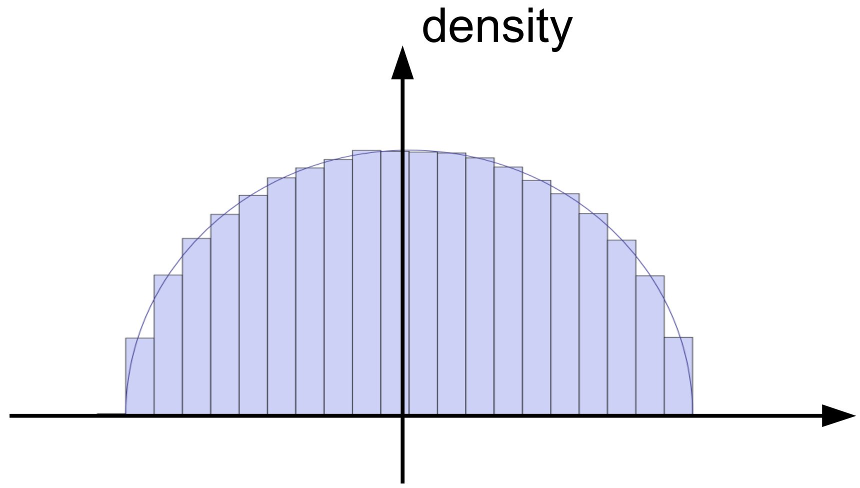


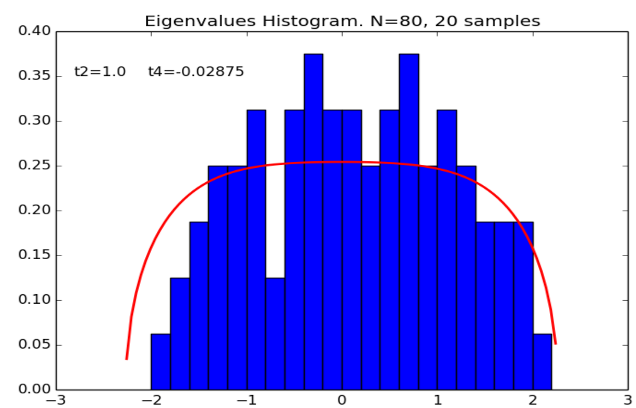
Random matrix

Choose the entries at random

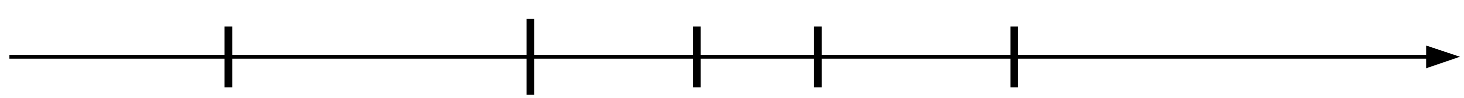
(i.i.d. normal law)

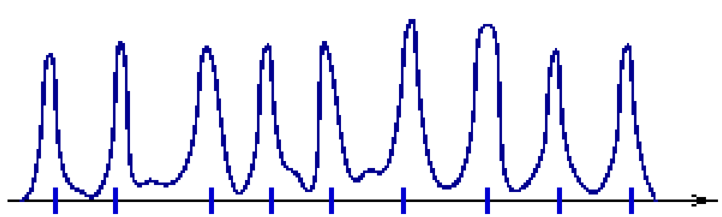
3	6	17	4.5	2
6	7	-3	4	1
17	-3	0.3	6.1	18
4.5	4	6.1	5	-0.3
2	1	18	-0.3	3





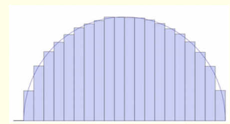
Plot the eigenvalues





Density and correlations

- Density of eigenvalues $\rho(\lambda)$



- Correlation of 2 eigenvalues $\rho_2(\lambda_1, \lambda_2)$
- Correlation of n eigenvalues $\rho_n(\lambda_1, \lambda_2, \ldots, \lambda_n)$

- Cumulants correlation

$$\rho_2^c(\lambda_1, \lambda_2) = \rho_2(\lambda_1, \lambda_2) - \rho(\lambda_1)\rho(\lambda_2)$$

$$\rho_n(\lambda_1, \lambda_2, \ldots, \lambda_n) \rightarrow \rho_n^c(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Definition :

- Stieljes transform of density of eigenvalues :

$$W_1(z) = \int_{supp(\rho)} \frac{\rho(\lambda)\, d\lambda}{z - \lambda}$$

- Stieljes transform of correlations

$$W_n(z_1, \ldots, z_n) = \int_{supp(\rho)} \ldots \int_{supp(\rho)} \frac{\rho_n^c(\lambda_1, \ldots, \lambda_n)\, d\lambda_1 \ldots d\lambda_n}{(z_1 - \lambda_1) \ldots (z_n - \lambda_n)}$$

$\rightarrow$ turns functions of real λ_i into analytic functions of complex variables z_i

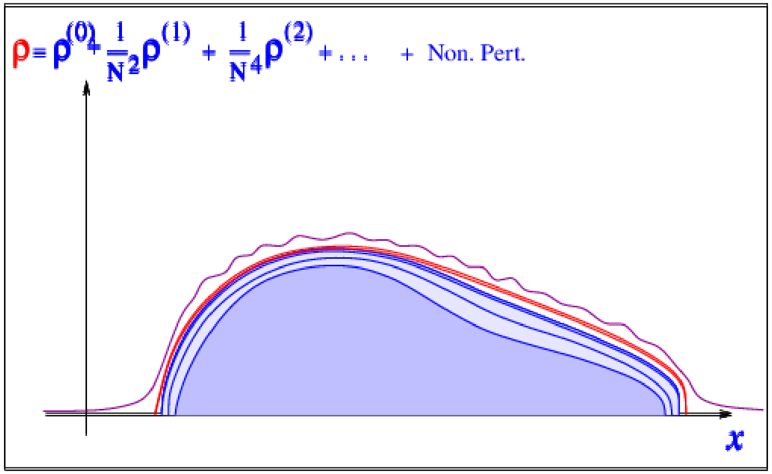
Density and correlations

- Large size expansion $N \to \infty$

$$\rho(\lambda) \sim N\rho^{(0)}(\lambda) + \frac{1}{N}\rho^{(1)}(\lambda) + \frac{1}{N^3}\rho^{(2)}(\lambda)$$

$$\dots + \frac{1}{N^{2g-1}}\rho^{(g)}(\lambda) + \dots$$

$$+ e^{-NA} \dots$$



- For Stieljes transforms

$$W_n(z_1, \dots, z_n) \sim \sum_{g=0}^{\infty} N^{2-2g-n} W_{g,n}(z_1, \dots, z_n)$$

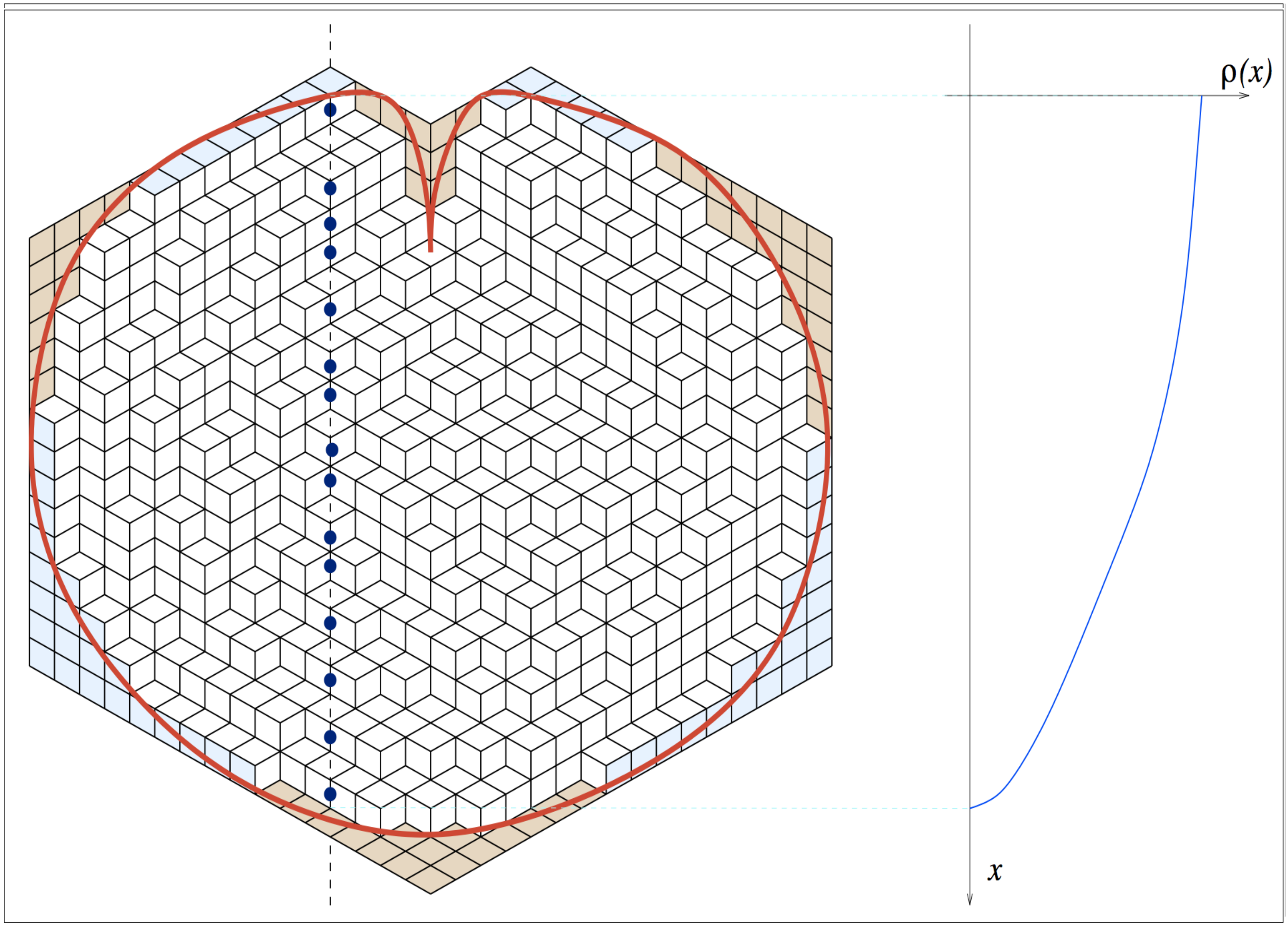
The coefficients $W_{g,n}$ of the asymptotic expansions obey a recursion relation :

= *Topological Recursion*

$$W_{g,n}(z_1, \dots, z_n) = K(z_1, z) * \left[W_{g-1,n+1}(z, \bar{z}, z_2, \dots, z_n) + \sum W_{g_1,|I_1|+1}(z, I_1) W_{g_2,|I_2|+1}(\bar{z}, I_2) \right]$$

Application : crystal growth

- Crystal $Z(\text{shape}) = \sum_{\pi} q^{\text{boxes}}$



Spectral curve : $y = W_{0,1}(x) = \int \frac{\rho(x')\,dx'}{x - x'}$

Smallest degree algebraic curve inscribed in shape

Question : $\ln Z \sim \sum_g \ln(q)^{2g-2} F_g$ $F_g = W_{g,0}(S)$