

# Various Applications of Operator Theory to Physics and Engineering

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# Outline

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# Abstract

## Abstract

This talk presents applications of Operator Theory to the "Quantization of Scalar Fields on Curved Space-Times," "The Stability of Stars," from Newtonian Astrophysics, "The Stability of Rotating Black Holes," from General Relativity, the "Formulation of Outgoing Boundary Conditions" to partial differential equations, the "Diffusion Equation with Rough Coefficients," the "Description of Visco-Elastic Damping" and the "Description of Crack Propagation in Materials" in the framework of "Peridynamics."

# Quantum Field Theory in Curved Space-Times

- ▶ A main lesson taught by the field of Quantum Field Theory in Curved Space-Times is that
- ▶ the definition of the admissible states of the theories uses information that is non-local in the sense of differential geometry.
- ▶ My work, [Beyer, 1991], proved this in the particular case of the quantization (due to Fulling) of the free massive neutral scalar field,
- ▶ restricted to an open submanifold  $R$  (the ‘right wedge’) of Minkowski spacetime,
- ▶ in the context of the Algebraic Quantum Field Theory.

# Quantum Field Theory in Curved Space-Times

- ▶ In particular, a new approach to a rigorous definition of that quantization was given,
- ▶ using the functional calculus associated with the governing operator of the classical wave equation,
- ▶ and for the first time the associated Wightman two-point distribution was calculated.
- ▶ From the latter certain scaling limits were computed, for the vector states of Fulling's theory at each point in the closure of  $R$ .

# Quantum Field Theory in Curved Space-Times

- ▶ As a consequence, it was proved that for certain bounded open subsets of  $R$ ,
- ▶ whose boundary includes points of the "edge of  $R$ ",
- ▶ the Fulling representation and the usual representation of the corresponding subalgebra of the Weyl algebra,
- ▶ associated with the algebra of the canonical commutation relations, CCR,
- ▶ are not quasi-equivalent.
- ▶ Finally, further scaling limits are stated for the vector states of the standard theory as well as Fulling's theory.
- ▶ These scaling limits differ at points of the remaining part of the boundary of  $R$ .

# Stability of Stars

$$\begin{aligned}
 -\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= L\boldsymbol{\xi} \\
 &= -\nabla \left( \frac{p\Gamma_1}{\rho^2} \nabla \cdot (\rho\boldsymbol{\xi}) \right) - \frac{p\Gamma_1}{\rho^2} A [\nabla \cdot (\rho\boldsymbol{\xi})] \mathbf{e}_r \\
 &\quad + \nabla \left( \frac{p\Gamma_1}{\rho^2} A(\rho\boldsymbol{\xi})_r \right) + \frac{p\Gamma_1}{\rho} (\ln \rho)' A \xi_r \mathbf{e}_r \\
 &\quad + G \nabla \left[ \int_{star} \frac{[\nabla \cdot (\rho\boldsymbol{\xi})](\mathbf{y}) d^3 y}{|\mathbf{x} - \mathbf{y}|} \right].
 \end{aligned}$$

**Figure 1:** The governing equation for linearized adiabatic stellar oscillations around a spherically symmetric stellar equilibrium model, with mass density  $\rho$ , pressure  $p$ , adiabatic index  $\Gamma_1$  and buoyancy term  $A := (\ln \rho)' - \Gamma_1^{-1}(\ln p)'$ . Here the unknown function is  $\vec{\xi} : \mathbb{R} \times \text{star} \rightarrow \mathbb{R}^3$ , a Lagrangian displacement variable.

# Stability of Stars

The physical boundary condition for  $\xi$ , the vanishing of the Lagrangian variation of the pressure at the boundary of the background star, is:

$$\delta p(t, \boldsymbol{x}) = -(p\Gamma_1 \nabla \cdot \boldsymbol{\xi})(t, \boldsymbol{x}) \longrightarrow 0 \quad \text{for } |\boldsymbol{x}| \longrightarrow R, \quad (2)$$

Figure 2



# Stability of Stars

Results [Beyer, 1995, 1995, 2000], [Beyer & Schmidt, 1995]:

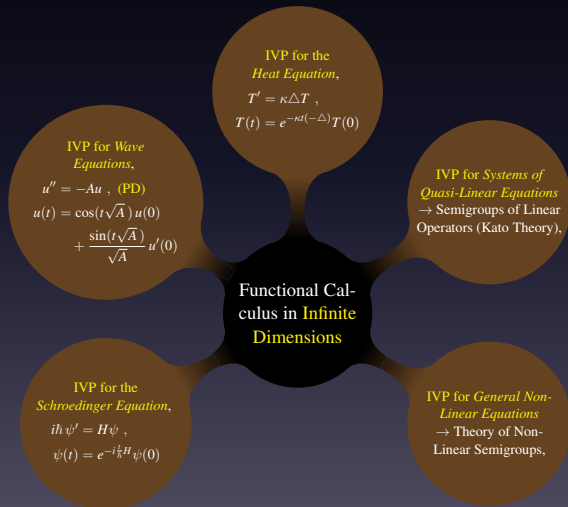
- ▶ *For a polytropic equation of state,*
- ▶  *$L$  induces a densely-defined, linear and symmetric operator  $\hat{L}_0$  in a weighted  $L^2$ -space  $(X, \langle | \rangle)$ .*
- ▶ *The physical boundary condition selects a uniquely determined self-adjoint extension,  $\hat{L}$ , of  $\hat{L}_0$ .*
- ▶ *The spectrum of  $\hat{L}$  is a pure point spectrum,*
- ▶ *i.e., in particular the eigenfunctions of  $\hat{L}$  are complete.*

# Stability of Stars

## Results:

- ▶ *Pure point spectra are known to be quite unstable against small perturbations of an operator.*
- ▶ *Therefore, a numerical treatment of stellar oscillations likely leads to occurrence of false continuous spectra.*
- ▶ *In a relativistic treatment, indeed continuous parts occur inside the physical spectrum!*
- ▶ Work of [Kojima, 1998] and [Beyer & Kokkotas, 1999].

# Applications of the Functional Calculus that is Associated with Self-Adjoint Linear Operators in Infinite Dimensional Hilbert Spaces



# Stability of Rotating Black Holes

- ▶ *The stability of the Schwarzschild black hole was demonstrated by [Kay & Wald, 1987],*
- ▶ who showed the boundedness of all solutions of the wave equation corresponding to  $C^\infty$  data of compact support.
- ▶ Their proof rests on the positivity of the conserved energy.

# Stability of Rotating Black Holes

- ▶ The problem is more subtle for Kerr space time and *still open*.
- ▶ A conserved energy exists, but the energy density is negative inside the ergosphere.
- ▶ Hence the total energy could be finite while the field still might grow exponentially in parts of the spacetime.

# Stability of Rotating Black Holes

- ▶ The reduced Klein-Gordon equation in a Kerr background,
- ▶ of mass  $M \geq 0$  and angular parameter  $a \in [0, M]$ ,
- ▶ corresponding to a mass  $m_0 \geq 0$
- ▶ and governing solutions of the form

$$\psi(t, r, \theta, \phi) = e^{im\varphi} u(t, r, \theta) ,$$

- ▶ where  $m$  runs through all integers and
- ▶  $t \in (-\infty, \infty)$ ,  $r \in (r_+, \infty)$ ,  $\theta \in (0, \pi)$ ,  $\varphi \in (0, 2\pi)$
- ▶ are the Boyer-Lindquist coordinates

# Stability of Rotating Black Holes

is given by

$$\begin{aligned} & \frac{\partial^2 u}{\partial t^2} + \frac{\Delta}{(r^2 + a^2)\Sigma + 2Ma^2r \sin^2 \theta} \cdot \\ & \left[ \frac{4imMar}{\Delta} \frac{\partial u}{\partial t} - \frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right. \\ & \left. + \frac{m^2}{\Delta \sin^2 \theta} (\Delta - a^2 \sin^2 \theta) + m_0^2 \Sigma \right] u = 0 , \end{aligned}$$

and

$$\Delta := r^2 + a^2 - 2Mr , \quad \Sigma := r^2 + a^2 \cos^2 \theta , \quad r_+ := M + \sqrt{M^2 - a^2} .$$

The quantity  $r_+$  is the largest zero of  $\Delta$ .

# Stability of Rotating Black Holes

## Results:

- ▶ The wave equation is of the form

$$(u')'(t) + iBu'(t) + (A + C)u(t) = 0 \quad (5.1)$$

for every  $t \in I$ ,

- ▶ for suitable operators  $A, B$  and  $C$ .
- ▶ The stability of the solutions is governed by the spectrum of the corresponding operator polynomial

$$A - \lambda B - \lambda^2 \quad (5.2)$$

where  $\lambda \in \mathbb{C}$ .



# Stability of Rotating Black Holes

- ▶ The papers [Beyer, 2001, 2011], among others, proves that the solutions of (5.1) are *stable* if the mass  $m_0$  satisfies the inequality

$$m_0 \geq \frac{|m|a}{2Mr_+} \sqrt{1 + \frac{2M}{r_+}} . \quad (5.3)$$

- ▶ There was a mounting evidence that the solutions of (5.1) are *unstable* if the (5.3) is violated.
- ▶ In particular, [Beyer, Alcubierre & Megevand, 2013] create a closely related model that *indicates the existence of unstable eigenvalues* of (5.2).
- ▶ Afterwards, this has been proved, [Shlapentokh-Rothman, 2014].

# Stability of Rotating Black Holes

- ▶ In connection with the problem of the stability of black holes,
- ▶ often *quasi-normal frequencies and modes appear*.
- ▶ My paper, [Beyer, 1999], gave an *operator theoretic interpretation* of these frequencies as
- ▶ *resonances* of the governing operator of the wave equation in question.

# The Formulation of Outgoing Boundary Conditions to PDE

In the following, we give a well-posed formulation of the initial-boundary value problem for

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + Vu = 0$$

on the interval  $I = (0, a)$ , where  $a > 0$ , for standard **Sommerfeld outgoing boundary conditions**

$$\left( \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} \right) \Big|_{x=0} = 0, \quad \left( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right) \Big|_{x=a} = 0$$

# The Formulation of Outgoing Boundary Conditions to PDE

and Engquist-Majda outgoing boundary conditions

$$\begin{aligned}\left(\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{V}{2}u\right)(0) &= 0, \\ \left(\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial x^2} - \frac{V}{2}u\right)(a) &= 0.\end{aligned}$$

# The Formulation of Outgoing Boundary Conditions to PDE

## Definition 6.1

We define the linear operator

$$A_V : D(A_V) \rightarrow W_{\mathbb{C}}^1(I) \times L_{\mathbb{C}}^2(I)$$

in  $X := W_{\mathbb{C}}^1(I) \times L_{\mathbb{C}}^2(I)$  by

$$D(A_V) := \{(f, g) \in C^2(\bar{I}, \mathbb{C}) \times C^1(\bar{I}, \mathbb{C}) : f_0' - g_0 = f_a' + g_a = 0\}$$

and

$$A_V(f, g) := (-g, -f'' + Vf)$$

for all  $(f, g) \in D(A_V)$ .

# The Formulation of Outgoing Boundary Conditions to PDE

Results: [Beyer, 2007], among others

## Theorem 6.2

- (i)  $A_V$  is a densely-defined, linear and accretive operator in  $X$ .
- (ii)  $\bar{A}_V$  generates a contractive strongly continuous semigroup  $T_V : [0, \infty) \rightarrow L(X, X)$ .

# Properties of the Diffusion Equation with Rough Coefficients

- ▶ The diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(p \operatorname{grad} u) + f \quad (7.1)$$

describes general diffusion processes,

- ▶ including the propagation of heat, and flows through porous media.
- ▶ Here  $u$  is the density of the diffusing material,
- ▶  $p$  is the diffusivity of the material,
- ▶ and the function  $f$  describes the distribution of sources and sinks.

# Properties of the Diffusion Equation with Rough Coefficients

- ▶ [Aksoylu & Beyer, 2009] and [Aksoylu & Beyer, 2010] focus on stationary solutions of (7.1) satisfying

$$-\operatorname{div}(p \operatorname{grad} u) = f . \quad (7.2)$$

- ▶ For instance, the fictitious domain method
- ▶ and composite materials are sources of rough coefficients.
- ▶ Important current applications deal with composite materials whose components have nearly constant diffusivity,
- ▶ but vary by several orders of magnitude.
- ▶ In composite material applications, it is quite common to idealize the diffusivity by a piecewise constant function
- ▶ and also to consider limits where the values of that function approach zero or infinity in parts of the material.



# Properties of the Diffusion Equation with Rough Coefficients

DEFINITION 4.1. Let  $p : \Omega \rightarrow \mathbb{R}$  be measurable. We define the linear operator  $A : D(A) \rightarrow L^2_{\mathbb{C}}(\Omega)$  in  $L^2_{\mathbb{C}}(\Omega)$  by

$$D(A) := \{u \in H^1_0(\Omega) : p \nabla_w u \in D(\nabla_0^*)\}$$

and

$$Au := \nabla_0^* p \nabla_w u$$

for every  $u \in D(A)$ .

**Figure 3:** Definition of the diffusion operator. Here,  $n \in \mathbb{N}^*$  and  $\Omega$  is some non-empty open subset of  $\mathbb{R}^n$ .

DEFINITION 4.2. We define the subset  $\mathcal{L}$  of  $L^\infty(\Omega)$  to consist of those elements  $p$  for which there are real  $C_1, C_2$  satisfying  $C_2 \geq C_1 > 0$  and such that  $C_1 \leq p \leq C_2$  a.e. on  $\Omega$ . Note that the last inequality also implies that  $1/p \in \mathcal{L}$  and, in particular, that  $1/C_2 \leq 1/p \leq 1/C_1$  a.e. on  $\Omega$ .

**Figure 4:** Definition of a suitable class of diffusivities.

# Properties of the Diffusion Equation with Rough Coefficients

**THEOREM 4.3.** *Let  $p \in \mathcal{L}$ . Then  $A$  is a densely defined, linear, and self-adjoint operator in  $L^2_{\mathbb{C}}(\Omega)$ .*

**Figure 5:** Basic properties of the diffusion operator I

**COROLLARY 4.4.** *Let  $p \in \mathcal{L}$  and, in addition,  $\Omega$  be bounded. Then  $A$  has a purely discrete spectrum.*

**Figure 6:** Basic properties of the diffusion operator II

**DEFINITION 5.1.** *Let  $\varpi \in L^\infty(\Omega)$ . We define the densely defined, linear operator  $\hat{A} : H^1_0(\Omega) \times D(\nabla_0^*) \rightarrow L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$  in  $L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$  by*

$$\hat{A}(u, q) := (\nabla_0^* q, \nabla_w u - \varpi q)$$

*for every  $(u, q) \in H^1_0(\Omega) \times D(\nabla_0^*)$ .*

**THEOREM 5.2.** *The operator  $\hat{A}$  is self-adjoint.*

**Figure 7:** Associated first-order operator.

# Properties of the Diffusion Equation with Rough Coefficients

- ▶ In  $n$ -dimensions,  $n \geq 1$ , by assuming a weak notion of convergence on the set of diffusivities,
- ▶ we prove the strong sequential continuity of the solution maps.
- ▶ In 1 dimension, we prove a stronger result, i.e., the unique extendability of the map of solution operators, associating to each diffusivity the corresponding solution operator,
- ▶ to a sequentially continuous map in the operator norm on a set containing "diffusivities" assuming infinite values in parts of the medium.
- ▶ In this case, we also give explicit estimates on the convergence behavior of the map.
- ▶ In the end, we provide connections to preconditioning.

# Application of Fractional Derivatives to Visco-Elastic Damping

- ▶ Equation of motion for a single degree of freedom oscillator with viscous damping is given by

$$(D^2 + aD + b)x = f ,$$

- ▶  $x$  denotes the displacement at time  $t$  with respect to the equilibrium position for  $f = 0$ ,
- ▶  $f$  is the external force per mass,
- ▶  $a, b \in \mathbb{R}$  are the damping and the stiffness constants per mass
- ▶ and  $D$  denotes the derivative.

# Application of Fractional Derivatives to Visco-Elastic Damping

- ▶ It has been suggested by [Caputo, 1976],
- ▶ to substitute  $D$  in the damping term by a fractional derivative,  $D^q$ ,  $0 < q < 1$ ,
- ▶ in the sense of Riemann-Liouville
- ▶ and to solve the generalized equation

$$(D^2 + aD^q + b)x = f , \quad (8.1)$$

by a Fractional Calculus as established by [Ross, 1975, 1977], [Oldham & Spanier, 1974] see also [Bride, 1975] and [Nishimoto, 1984, 1987].

# Application of Fractional Derivatives to Visco-Elastic Damping

The **major reasons** for replacing the viscous damping law by one with a fractional derivative are the following.

- ▶ The **curve fitting properties of measured response functions**, in time and frequency domain, for viscoelastic damping laws **improve, in particular, qualitatively**.
- ▶ **Less parameters are sufficient for the curve fit**, compared to an improvement of the damping law that takes into account higher order integer derivatives.
- ▶ The **violation of causality in time domain response**, which sometimes occurs with generalized damping laws, such as structural (constant hysteretic) damping, **can be avoided**.

# Application of Fractional Derivatives to Visco-Elastic Damping

## Results:

- ▶ [Beyer & Kempfle, 1995], use a functional-analytic approach for the treatment of integro-partial differential equations.
- ▶ Represent (8.1) as an operator equation

$$Ax = f, \quad (8.2)$$

for an operator  $A : D(A) \rightarrow X$ ,  $D(A) \subset X$ , in an appropriate function space (“data space”)  $X$ ,

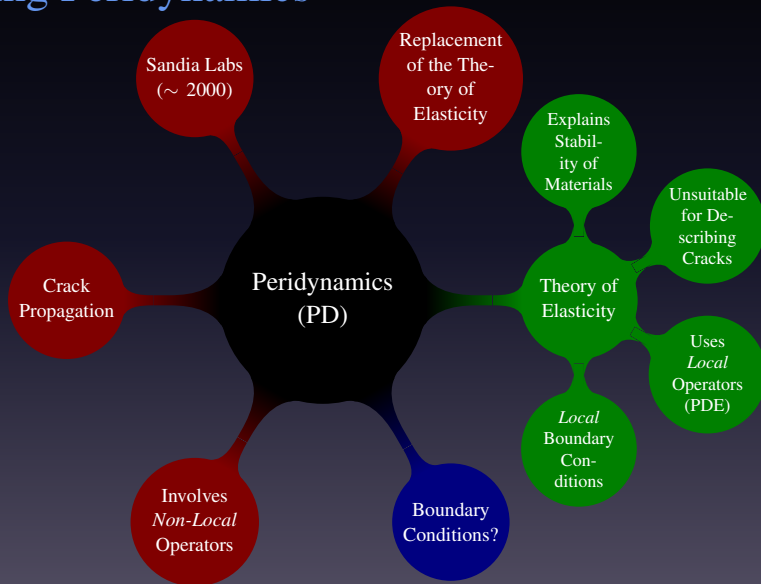
- ▶ Prove well-posedness of the problem, or what is the same, “the stability of the solutions”, i.e., that there is a unique solution  $x \in X$  to (8.2) depending continuously on the forcing term  $f \in X$ ,

# Application of Fractional Derivatives to Visco-Elastic Damping

- ▶ Prove that solutions are physically reasonable, i.e., in this case, mainly that the solutions are causal.
- ▶ In addition, we calculate the solutions explicitly in terms of confluent hypergeometric functions and the exponential integral.



# Description of Crack Propagation in Materials, using Peridynamics



# Description of Crack Propagation in Materials, using Peridynamics

The **formal** *peridynamic wave equation in 1-space dimension* is given by

$$\rho \frac{\partial^2 u}{\partial t^2}(x, t) = \int_{-\infty}^{\infty} C(x' - x) (u(x', t) - u(x, t)) dx' + b(x, t) ,$$

(9.1)

where  $x, t \in \mathbb{R}$ ,

$C$  is the *micromodulus*, assumed to be a real-valued *even* function,

$\rho > 0$  is the *mass density*,

$b$  is the prescribed *body force density*,

and  $u$  is the *displacement field*.

# Description of Crack Propagation in Materials, using Peridynamics

For comparison, *the corresponding equation in classical elasticity is given by*

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} + b ,$$

where  $E > 0$  is the so called "Young's modulus," and is *describing compression waves in a rod.*

# Description of Crack Propagation in Materials, using Peridynamics:

If  $C \in L^1(\mathbb{R})$ , we can rewrite (9.1) as

$$\rho \frac{\partial^2 u}{\partial t^2}(x, t) = - \left[ \int_{-\infty}^{\infty} C(x') dx' \right] u(x, t) - (C * u(\cdot, t))(x) + b(x, t) ,$$

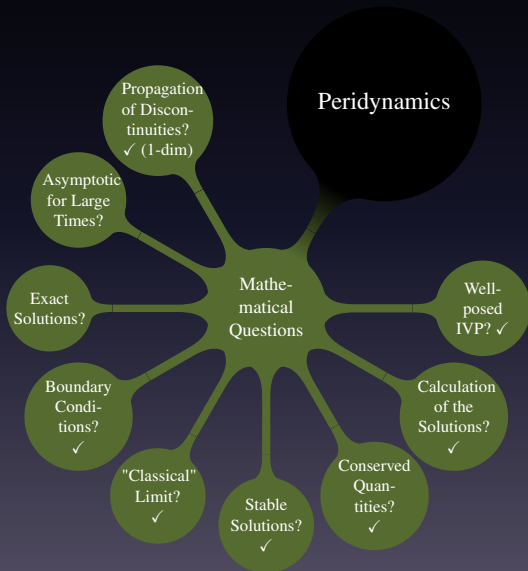
for all  $x, t \in \mathbb{R}$ .

# Description of Crack Propagation in Materials, using Peridynamics

## Results:

- ▶ We, [Beyer & Aksoylu, 2016], [Aksoylu, Beyer & Celiker 2017, 2017], consider (9.1) in  $n \in \mathbb{N}^*$  dimensions.
- ▶ For this purpose, we use methods from operator theory
- ▶ and solve the following problems.

# Description of Crack Propagation in Materials, using Peridynamics



# Description of Crack Propagation in Materials, using Peridynamics

The development of the extension to the *vectorial* equations from PD in 2D/3D dimensions is work in progress! Mathematically, such extension leads to "*operator matrices*" as indicated in Remark 4 in [Beyer & Aksoylu, 2016].

Thank You!



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