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RECENT RESULTS ON POST-NEWTONIAN WAVEFORMS

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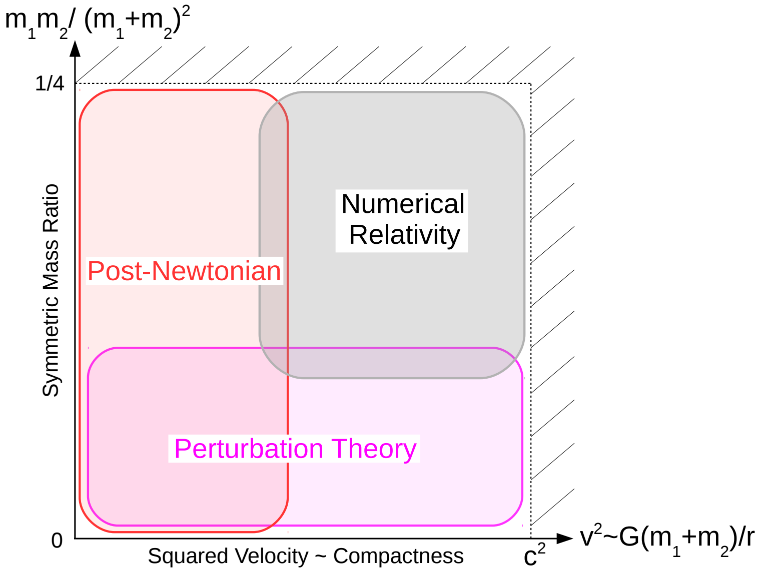
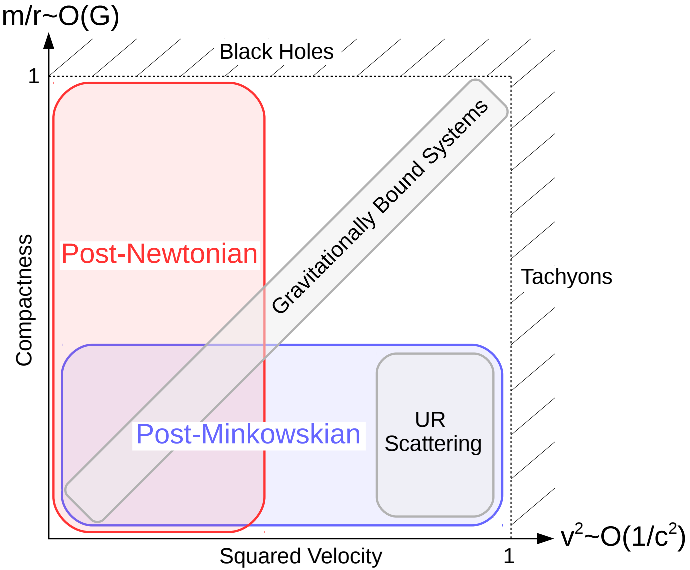
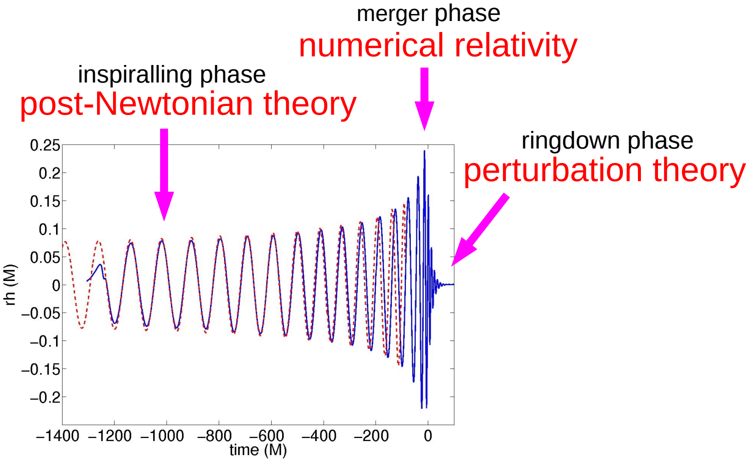
Methods to compute GW templates



Methods to compute GW templates

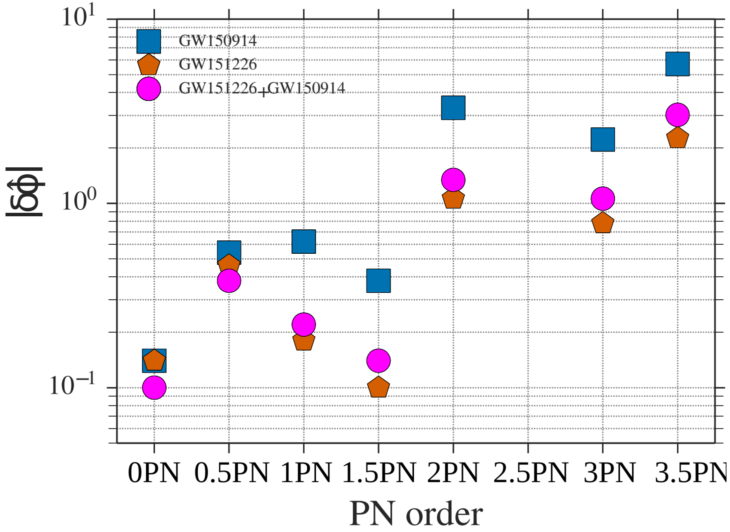


The gravitational chirp of compact binaries

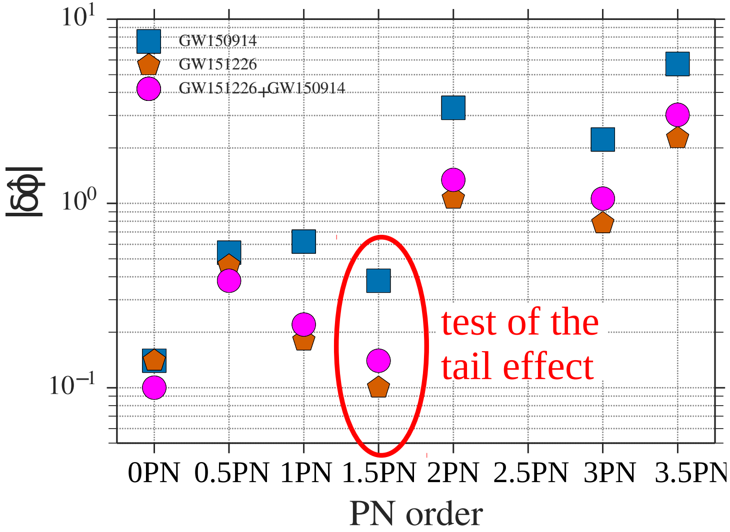


Effective methods such as EOB and IMR that interpolate between the PN and NR are also very important notably for the data analysis of black hole binaries

Measurement of PN parameters [LIGO/Virgo collaboration 2016]



Measurement of PN parameters [LIGO/Virgo collaboration 2016]



Methods to compute PN equations of motion

1. ADM Hamiltonian canonical formalism [Ohta *et al.* 1973; Schäfer 1985]
2. EOM in harmonic coordinates [Damour & Deruelle 1985; Blanchet & Faye 1998, 2000]
3. Extended fluid balls [Grishchuk & Kopeikin 1986]
4. Surface-integral approach [Itoh, Futamase & Asada 2000]
5. Effective-field theory (EFT) [Goldberger & Rothstein 2006; Foffa & Sturani 2011]

- EOM derived in a general frame for arbitrary orbits
- Dimensional regularization is applied for UV divergences[1]
- Radiation-reaction dissipative effects added separately by matching
- Spin effects can be computed within a pole-dipole approximation
- Tidal effects incorporated at leading 5PN and sub-leading 6PN orders

[1] Except in the surface-integral approach

Methods to compute PN radiation field

1. Multipolar-post-Minkowskian (MPM) & PN [Blanchet-Damour-Iyer 1986, . . . , 1998]
2. Direct iteration of the relaxed field equations (DIRE) [Will-Wiseman-Pati 1996, . . .]
3. Effective-field theory (EFT) [Hari Dass & Soni 1982; Goldberger & Ross 2010]

- Involves a machinery of tails and related non-linear effects
- Uses dimensional regularization to treat point-particle singularities
- Phase evolution relies on balance equations valid in adiabatic approximation
- Spin effects are incorporated within a pole-dipole approximation
- Provides polarization waveforms for DA & spin-weighted spherical harmonics decomposition for NR

Summary of known PN orders

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN)	4PN non-spin 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN + 4.5PN 4PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian	4PN non-spin 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (NL) SSS		
Effective Field Theory (EFT)	3PN 2.5PN (NL) SO 4PN (NNL) SS	2PN 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE)	2.5PN 1.5PN (L) SO 2PN (L) SS	2PN 1.5PN (L) SO 2PN (L) SS	2PN 1.5PN (L) SO 2PN (L) SS
Surface Integral	3PN		

Many works devoted to spins:

- Spin effects (SO, SS, SSS) are known in EOM up to 4PN order
- SO effects are known in radiation field up to 4PN
- SS in radiation field known to 3PN

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2} n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \cdots\right] n_{12}^i + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN

[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
[Foffa & Sturani 2011]	Effective field theory

4PN

[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
[Foffa & Sturani 2012, 2013] (partial results)	Effective field theory

First ambiguity-free derivation of the 4PN EOM

[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2017ab]

- Fokker Lagrangian in harmonic coordinates
- Non-local treatment for the 4PN tail term
- Dimensional regularization for UV divergences (appear at 3PN order)
- Dimensional regularization for IR divergences (appear at 4PN order)

1. Problem of ambiguity parameters solved by means of a matching equation relating the near zone field to the asymptotic far zone field
2. Reproduce the 4PN energy and periastron advance in the test mass limit for circular orbits known from gravitational self-force (GSF) calculations
3. Equivalence with the end result of the 4PN ADM Hamiltonian [Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]
4. Extension at the 5PN order seems unattainable at present

Comparing 4PN with the 1PM approximation

[Blanchet & Fokas 2018]

1. The 1PM field equations of N particles in harmonic coordinates read

$$\Box h^{\mu\nu} = \frac{16\pi}{c^2}\sum_{a=1}^{N} G m_a \int_{-\infty}^{+\infty} \mathrm{d}\tau_a \, u_a^\mu u_a^\nu \delta^{(4)}(x - y_a)$$

2. The Lienard-Wiechert solution is

$$h^{\mu\nu}(x) = -\frac{4}{c^2}\sum_a \frac{G m_a \, u_a^\mu u_a^\nu}{r_a^{\text{ret}}\,(ku)_a^{\text{ret}}}$$

where $r_a^{\text{ret}} = |\boldsymbol{x} - \boldsymbol{x}_a^{\text{ret}}|$ and $(ku)_a^{\text{ret}}$ is the redshift factor

3. In small 1PM terms trajectories are straight lines hence the retardations can be explicitly performed

$$h^{\mu\nu}(\boldsymbol{x}, t) = -\frac{4}{c^2}\sum_a \frac{G m_a \, u_a^\mu u_a^\nu}{r_a\sqrt{1 + (n_a u_a)^2}}$$

Comparing 4PN with the 1PM approximation

[Blanchet & Fokas 2018]

1. This yields the 1PM equations of motion but in PN like form[2]

$$\boxed{\begin{aligned}\frac{\mathrm{d}\boldsymbol{v}_a}{\mathrm{d}t} = -\gamma_a^{-2}\sum_{b\neq a}\frac{Gm_b}{r_{ab}^2\,y_{ab}^{3/2}}\Bigg[&(2\epsilon_{ab}^2-1)\boldsymbol{n}_{ab}\\ &+\gamma_b\Big(-4\epsilon_{ab}\gamma_a(n_{ab}v_a)+(2\epsilon_{ab}^2+1)\gamma_b(n_{ab}v_b)\Big)\frac{\boldsymbol{v}_{ab}}{c^2}\Bigg]\end{aligned}}$$

2. These equations of motion are conservative and admit a conserved energy

$$\boxed{\begin{aligned}E = \sum_a m_a c^2\gamma_a + \sum_a\sum_{b\neq a}\frac{Gm_am_b}{r_{ab}\,y_{ab}^{1/2}}\Bigg\{&\gamma_a\left(2\epsilon_{ab}^2+1-4\frac{\gamma_b}{\gamma_a}\epsilon_{ab}\right)\\ &+\frac{\gamma_b^2}{\gamma_a}(2\epsilon_{ab}^2-1)\frac{\dot{r}_{ab}(n_{ab}v_b)-(v_{ab}v_b)}{(v_{ab}^2-\dot{r}_{ab}^2)y_{ab}+\frac{\gamma_b^2}{c^2}\Big(\dot{r}_{ab}(n_{ab}v_b)-(v_{ab}v_b)\Big)^2}\Bigg\}\end{aligned}}$$

[2] $y_{ab} = 1+(n_{ab}u_a)^2$ and $\epsilon_{ab} = -(u_au_b)$

Gravitational radiation reaction calculations

1. For compact binaries the radiation reaction is under control up to 4PN order
2. The extension to 4.5PN order for compact binaries is in progress
3. For general matter systems the 4PN radiation reaction derives from radiation reaction potentials valid in a specific extension of the [Burke & Thorne 1971] gauge

$$\boxed{\begin{aligned} V^{\text{reac}} &= -\frac{G}{5c^5} x^{ij} I_{ij}^{(5)} + \frac{G}{c^7}\left[\frac{1}{189} x^{ijk} I_{ijk}^{(7)} - \frac{1}{70} r^2 x^{ij} I_{ij}^{(7)}\right] \\ &\quad \underbrace{- \frac{4G^2M}{5c^8} x^{ij} \int_0^{+\infty} \mathrm{d}\tau\, I_{ij}^{(7)}(t-\tau)\left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12}\right]}_{\text{4PN radiation reaction tail}} + \mathcal{O}\left(\frac{1}{c^9}\right) \\ V_i^{\text{reac}} &= \frac{G}{c^5}\left[\frac{1}{21} \hat{x}^{ijk} I_{jk}^{(6)} - \frac{4}{45} \epsilon_{ijk}\, x^{jl} J_{kl}^{(5)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right) \end{aligned}}$$

Radiation reaction derivation of balance equations

1. Well known results for the energy and angular momentum

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\frac{G}{c^5}\varepsilon_{ijk}\left(\frac{2}{5}I^{(2)}_{jl}I^{(3)}_{kl} + \frac{1}{c^2}\left[\frac{1}{63}I^{(3)}_{jlm}I^{(4)}_{klm} + \frac{32}{45}J^{(2)}_{jl}J^{(3)}_{kl}\right]\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

2. And for linear momentum (this effect responsible for the recoil of the source)

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{G}{c^7}\left[\frac{2}{63}I^{(4)}_{ijk}I^{(3)}_{jk} + \frac{16}{45}\varepsilon_{ijk}I^{(3)}_{jl}J^{(3)}_{kl}\right] + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3. However we find also for the center-of-mass position [Blanchet & Faye 2018]

$$\boxed{\frac{\mathrm{d}I_i}{\mathrm{d}t} = P_i - \frac{2G}{21c^7}\,I^{(3)}_{ijk}I^{(3)}_{jk} + \mathcal{O}\left(\frac{1}{c^9}\right)}$$

Strangely this formula does not seem to appear in the GW litterature

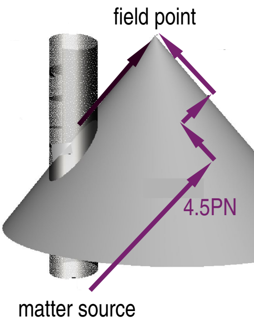
The 4.5PN radiative quadrupole moment

$$
\begin{aligned}
U_{ij}(t) = I^{(2)}_{ij}(t) &+ \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}} \\
&+ \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau\, I^{(3)}_{a<i} I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\} \\
&+ \underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}} \\
&+ \underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}} \\
&+ \mathcal{O}\left(\frac{1}{c^{10}}\right)
\end{aligned}
$$

4.5PN coefficient in the GW flux

[Marchand, Blanchet, Faye 2017]

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{4.5PN}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ \bigg(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\text{E} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3 \bigg) \pi x^{9/2} \Bigg\}$$



- The 4.5PN tail effect represents the complete 4.5PN coefficient in the GW energy flux in the case of circular orbits
- Perfect agreement with results from BH perturbation theory in the small mass ratio limit $\nu \to 0$ [Tanaka, Tagoshi & Sasaki 1996]
- Result confirmed by factorized and resummed waveforms [Messina & Nagar 2017]
- However the 4PN term in the flux is still in progress

Towards 4.5PN waveform and phase evolution

The 4PN EOM and high-order tails are now completed but it remains:

1. Difficult computation of the source multipole moments
   - preliminary calculation of the mass quadrupole moment at 4PN order with Hadamard regularization for the IR has bee done [in preparation]
   - needs to be corrected by dimensional regularization for the IR
   - computation of the current quadrupole moment at 3PN order
2. Control of various non-linear interactions such as between tail terms and memory effects at 4PN order
3. Control of the effect of past evolution in the tail integrals at 4PN order