Turbulence of Gravitational Waves

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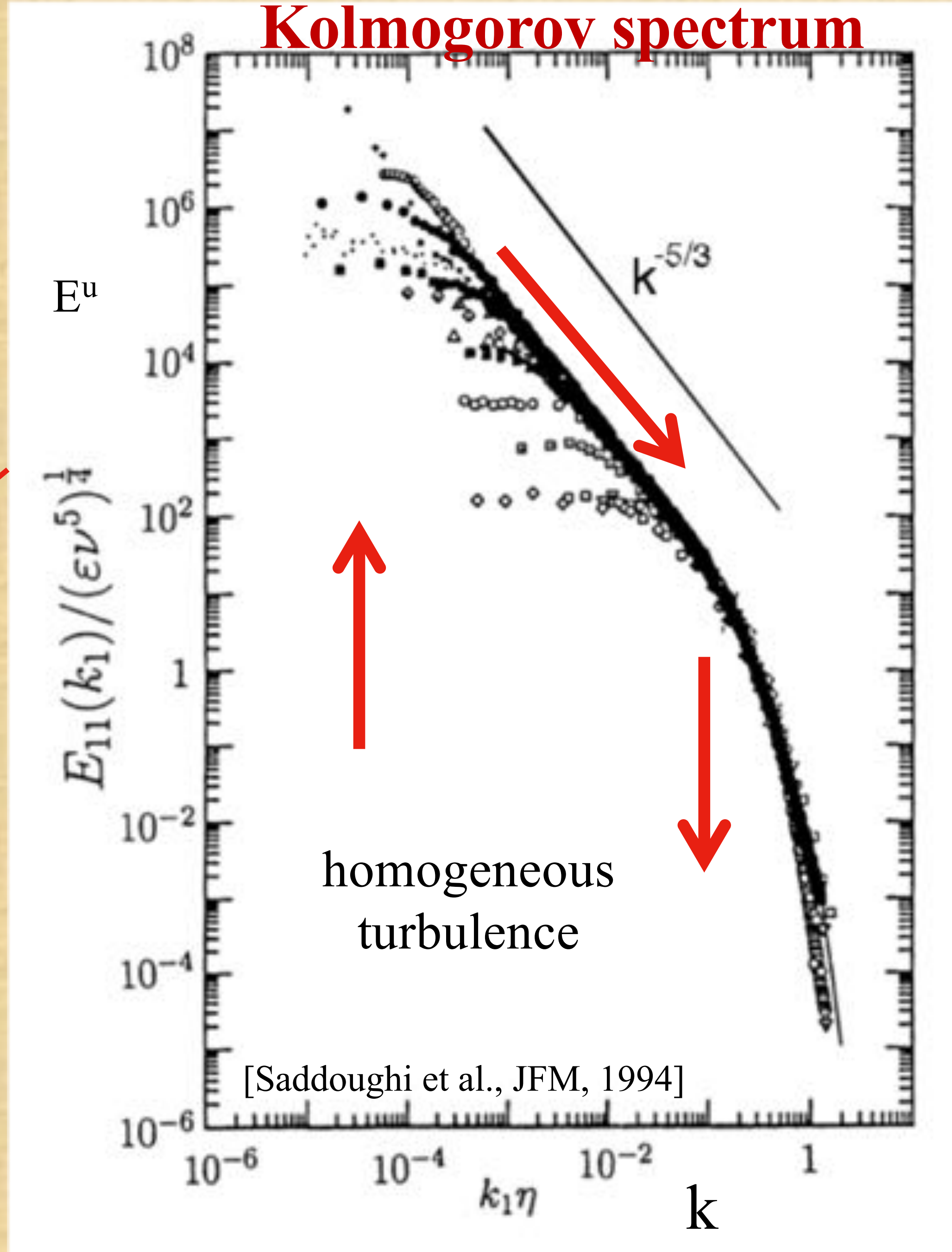
Turbulence in 3D hydrodynamics



Turbulence in 3D hydrodynamics

Incompressible
Navier-Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \Delta \mathbf{u} + \mathbf{f}$$

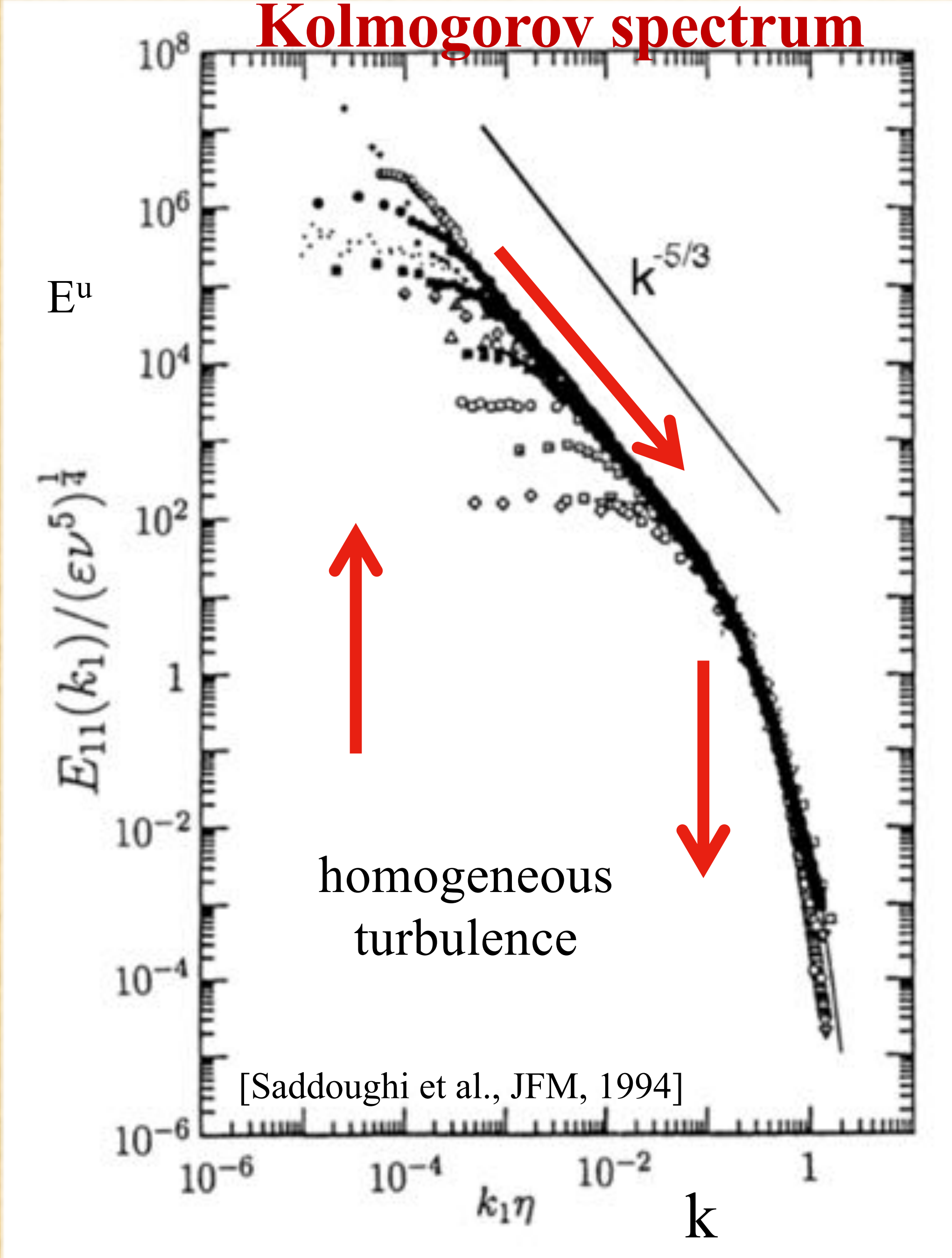
Fourier
transform

$$\partial_t \hat{u}_j(\mathbf{k}) = -i \int \left(\delta_{jn} - \frac{k_j k_n}{k^2} \right) q_m \hat{u}_m(\mathbf{p}) \hat{u}_n(\mathbf{q}) \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) d\mathbf{p} d\mathbf{q}$$

triadic interactions

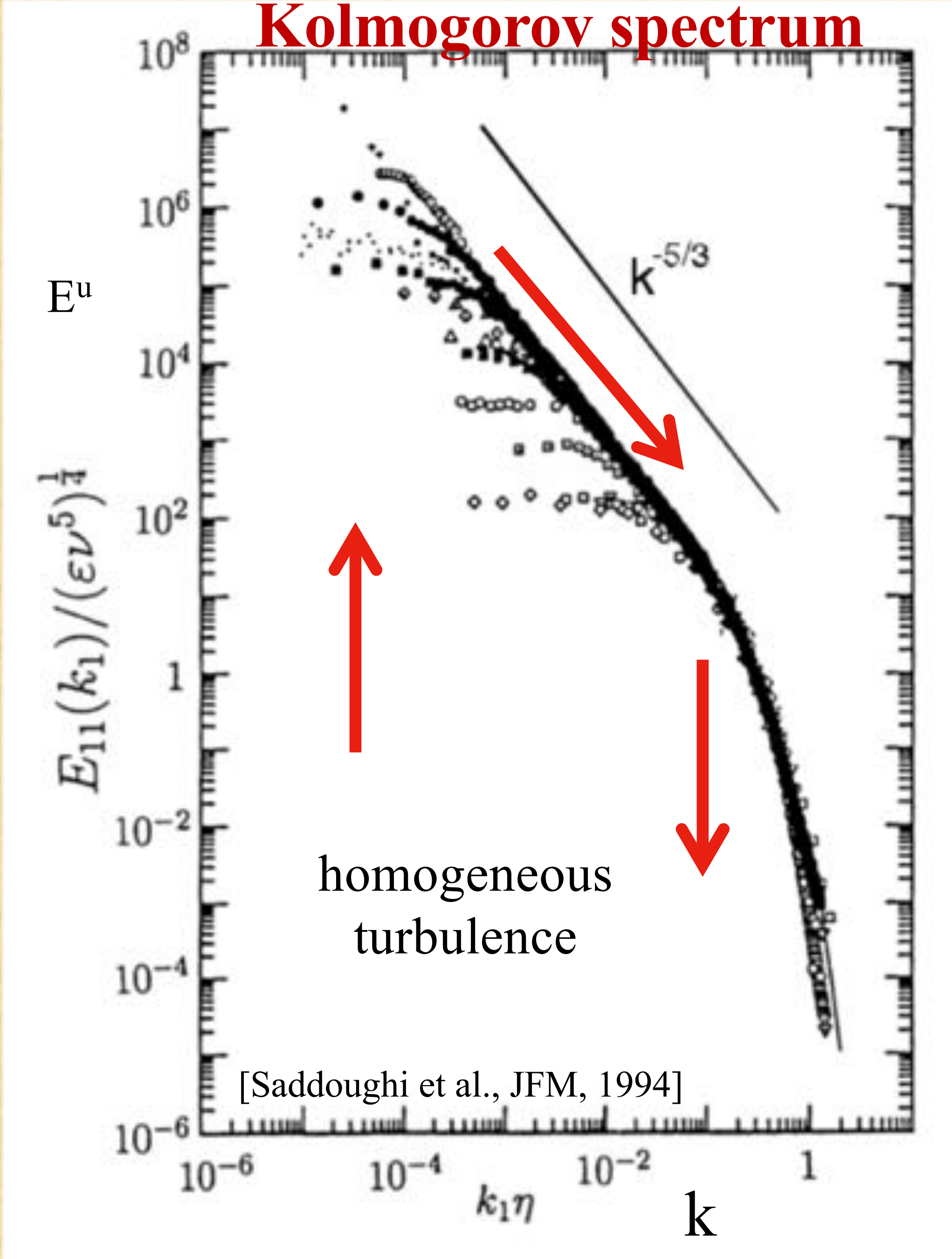
nonlinear
analysis

$$\partial_t E(k) = NL - 2\nu k^2 E(k) + F(k)$$



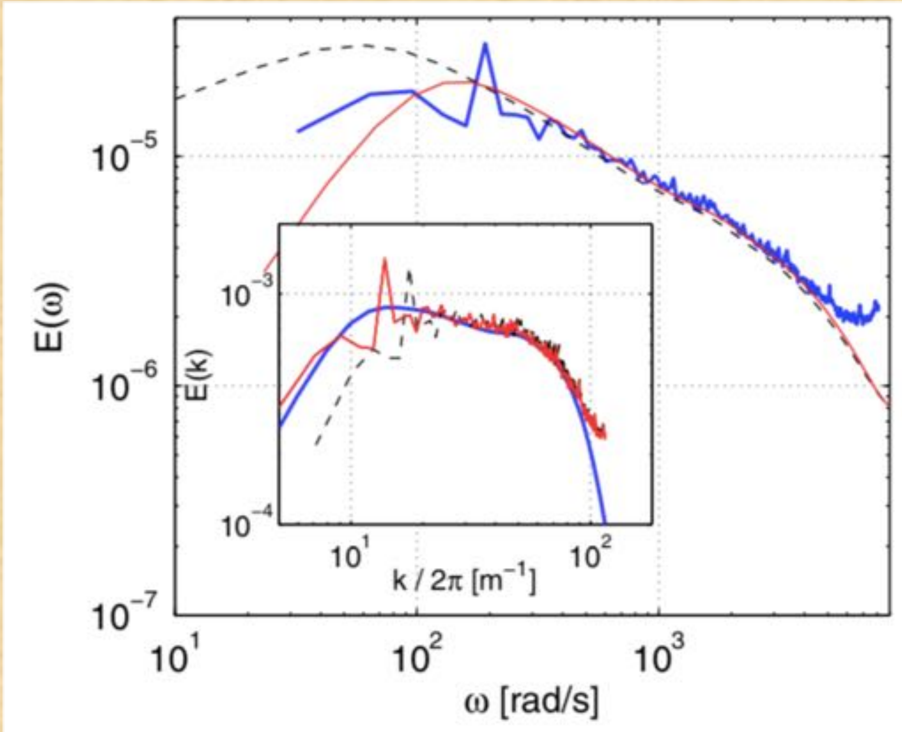
Turbulence in 3D hydrodynamics

Wave turbulence



The sound of wave turbulence: the vibrating elastic plates

[During+, PRL, 2006; Cobelli+, PRL, 2009; Mordant+, PRE, 2017; Haissaini+, 2018]



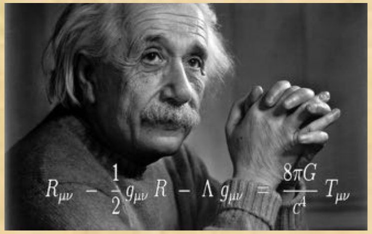
Analytical theory of weak wave turbulence

+ Natural **asymptotical closure** of the hierarchy of moment equations
  [Benney & Saffman, PRSLA, 1966; Benney & Newell, JMP, 1967]

+ The kinetic equations admits **exact stationary finite flux** solutions
  [Zakharov & Filonenko, SPD, 1967]

– Finite flux spectra not valid **over all k's** $\rightarrow$ strong turbulence
  [SG+, JPP, 2000; Meyrand+, PRL, 2016]

– Experiments and dns show **some limitations** in the predictions
  [Morize+, PoF, 2005; Nazarenko, NJP, 2007]



Gravitational waves

$\Lambda = 0$

Exact linear solutions in an empty – flat – Universe:

$$R_{\mu\nu} = 0$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1$$

Effect of a + gravitational wave on a ring of particles (h = 0.5)

$$\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$$

$$h^{+}_{\mu\nu} = a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Weakly nonlinear general relativity

$\Lambda=0$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1$$

$$R_{\mu\nu} = 0$$

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ~~\ldots~~$$

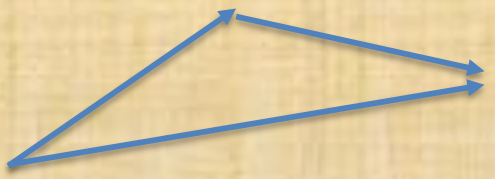
$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}$$

Triadic interactions:

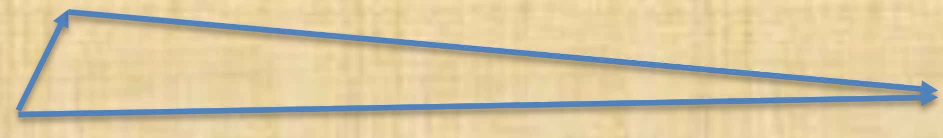
$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \text{ and } \omega_{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}$$

$$\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$$

$\tau_{lin} << \tau_{nonlin}$



local interactions



non-local interactions

We found that there is no contribution on the **resonant manifold**

Three-wave interactions in GW turbulence **are absent** !

[SG & Nazarenko, PRL, 2017]

Nonlinear general relativity equations

Einstein-Hilbert action: $S = \frac{1}{2}\int R\sqrt{-g}\,d^4x$

g is the determinant of $g_{\mu\nu}$

R is the scalar curvature

Diagonal space-time metric:

$$\partial/\partial z = 0$$

[Hadad & Zakharov, JGP, 2014]

$$g_{\mu\nu} = \begin{pmatrix} -(H_0)^2 & 0 & 0 & 0 \\ 0 & (H_1)^2 & 0 & 0 \\ 0 & 0 & (H_2)^2 & 0 \\ 0 & 0 & 0 & (H_3)^2 \end{pmatrix}$$

$$H_0 = e^{-\lambda}\gamma,\ \ H_1 = e^{-\lambda}\beta,\ \ H_2 = e^{-\lambda}\alpha,\ \ H_3 = e^{\lambda}$$

Spatial isotropy *is not* assumed

Lagrangian density

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2}\left[\underbrace{\frac{\alpha\beta}{\gamma}\dot{\lambda}^2 - \frac{\alpha\gamma}{\beta}(\partial_x\lambda)^2 - \frac{\beta\gamma}{\alpha}(\partial_y\lambda)^2}_{\text{Give the linear contribution}} - \frac{\dot{\alpha}\dot{\beta}}{\gamma} + \frac{(\partial_x\alpha)(\partial_x\gamma)}{\beta} + \frac{(\partial_y\beta)(\partial_y\gamma)}{\alpha}\right]$$

Give the linear contribution

$\alpha = \beta = \gamma = 1$ $\quad \lambda \ll 1$ $\quad \lambda = c_1 \exp(-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}) + c_2 \exp(i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x})$

Hamiltonian formalism

Normal variables: $$\lambda_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a^*_{-\mathbf{k}}}{\sqrt{2k}}, \quad \dot{\lambda}_{\mathbf{k}} = \frac{\sqrt{k}(a_{\mathbf{k}} - a^*_{-\mathbf{k}})}{i\sqrt{2}},$$ Fourier space

Hamiltonian equation: $$i\dot{a}_{\mathbf{k}} = \frac{\partial H}{\partial a^*_{\mathbf{k}}} \quad \text{where} \quad H = H_{\text{free}} + H_{\text{int}}$$

$$H_{\text{free}} = \sum_{\mathbf{k}} k|a_{\mathbf{k}}|^2$$

4 wave processes

$$\begin{aligned} H_{\text{int}} = \frac{1}{4} \sum_{1,2,3,4,5} \frac{\delta_{123}\delta^1_{45}}{\sqrt{k_2k_3k_4k_5}} & \left\{ \left[\left(\frac{p_5}{p_1} + \frac{q_5}{q_1}\right) k_4 \left(-\frac{a_4a_5 + a^*_{-4}a^*_{-5}}{k_5 + k_4} + \frac{a^*_{-4}a_5 + a_4a^*_{-5}}{k_5 - k_4} \right) + \frac{p_4q_5}{p_1q_1}(a_4 + a^*_{-4})(a_5 + a^*_{-5}) \right] \right. \\ & k_2k_3(a_2 - a^*_{-2})(a_3 - a^*_{-3}) + \left[-\left(\frac{p_5}{p_1} - \frac{q_5}{q_1}\right)(p_2p_3 - q_2q_3)k_4 \left(-\frac{a_4a_5 + a^*_{-4}a^*_{-5}}{k_5 + k_4} + \frac{a^*_{-4}a_5 + a_4a^*_{-5}}{k_5 - k_4} \right) \right. \\ & \left. \left. + \frac{p_4q_5}{p_1q_1}(\mathbf{k}_2 \cdot \mathbf{k}_3)(a_4 + a^*_{-4})(a_5 + a^*_{-5}) \right] (a_2 + a^*_{-2})(a_3 + a^*_{-3}) \right\} \\ & + \frac{1}{2} \sum_{\mathbf{k},1,2,3,4} \frac{\delta^{\mathbf{k}}_{12}\delta^{\mathbf{k}}_{34}}{\sqrt{k_1k_2k_3k_4}} \left\{ \frac{(\mathbf{k} \cdot \mathbf{k}_2)k_1p_3q_4}{pq} \left(-\frac{a_1a_2 + a^*_{-1}a^*_{-2}}{k_2 + k_1} + \frac{a^*_{-1}a_2 + a_1a^*_{-2}}{k_2 - k_1} \right) (a^*_3 + a_{-3})(a^*_4 + a_{-4}) \right. \\ & \left. + \frac{k_1k_3p_2q_4}{pq} \left(a_1a_2 + a_1a^*_{-2} - a^*_{-1}a_2 - a^*_{-1}a^*_{-2}\right) \left(a^*_3a^*_4 + a^*_3a_{-4} - a_{-3}a^*_4 - a_{-3}a_{-4}\right) \right\}. \end{aligned} \qquad (6)$$

Kinetic equation of GW turbulence

$$H_{3\to 1} = 0$$

$$\dot{n}_{\mathbf{k}} = 4\pi \int |T^{\mathbf{k}\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}|^2 n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}}\right] \delta(\mathbf{k}+\mathbf{k}_3-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\omega_{\mathbf{k}}+\omega_{\mathbf{k}_3}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2})\,d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

$$n_{\mathbf{k}} = \langle |a_{\mathbf{k}}|^2 \rangle$$

with $T^{12}_{34} = \frac{1}{4}(W^{12}_{34} + W^{21}_{34} + W^{12}_{43} + W^{21}_{43})$, $W^{12}_{34} = Q^{12}_{34} + Q^{34}_{12}$

$$Q^{12}_{34} = \frac{1}{4\sqrt{k_1k_2k_3k_4}} \left\{ 2\left(\frac{p_4}{p_1-p_3} - \frac{q_4}{q_1-q_3}\right) \frac{k_2(p_1p_3 - q_1q_3)}{k_1-k_3} - 2\left(\frac{p_4}{p_1-p_3} + \frac{q_4}{q_1-q_3}\right)\frac{k_1k_2k_3}{k_1-k_3} \right. \tag{12}$$
$$\left. + \left(\frac{p_2}{p_1+p_2} - \frac{q_2}{q_1+q_2}\right)\frac{k_1(p_3p_4 - q_3q_4)}{k_1+k_2} - \left(\frac{p_2}{p_1+p_2} + \frac{q_2}{q_1+q_2}\right)\frac{k_1k_3k_4}{k_1+k_2} + \frac{2k_1k_3p_2q_4}{(p_1+p_2)(q_1+q_2)} + \frac{2k_1p_3(q_2k_4 + k_2q_4)}{(p_1-p_3)(q_1-q_3)} \right\}.$$

[Zakharov & Filonenko, SPD, 1967]

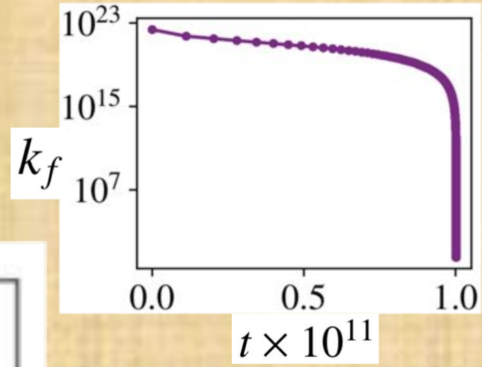
[SG & Nazarenko, PRL, 2017]

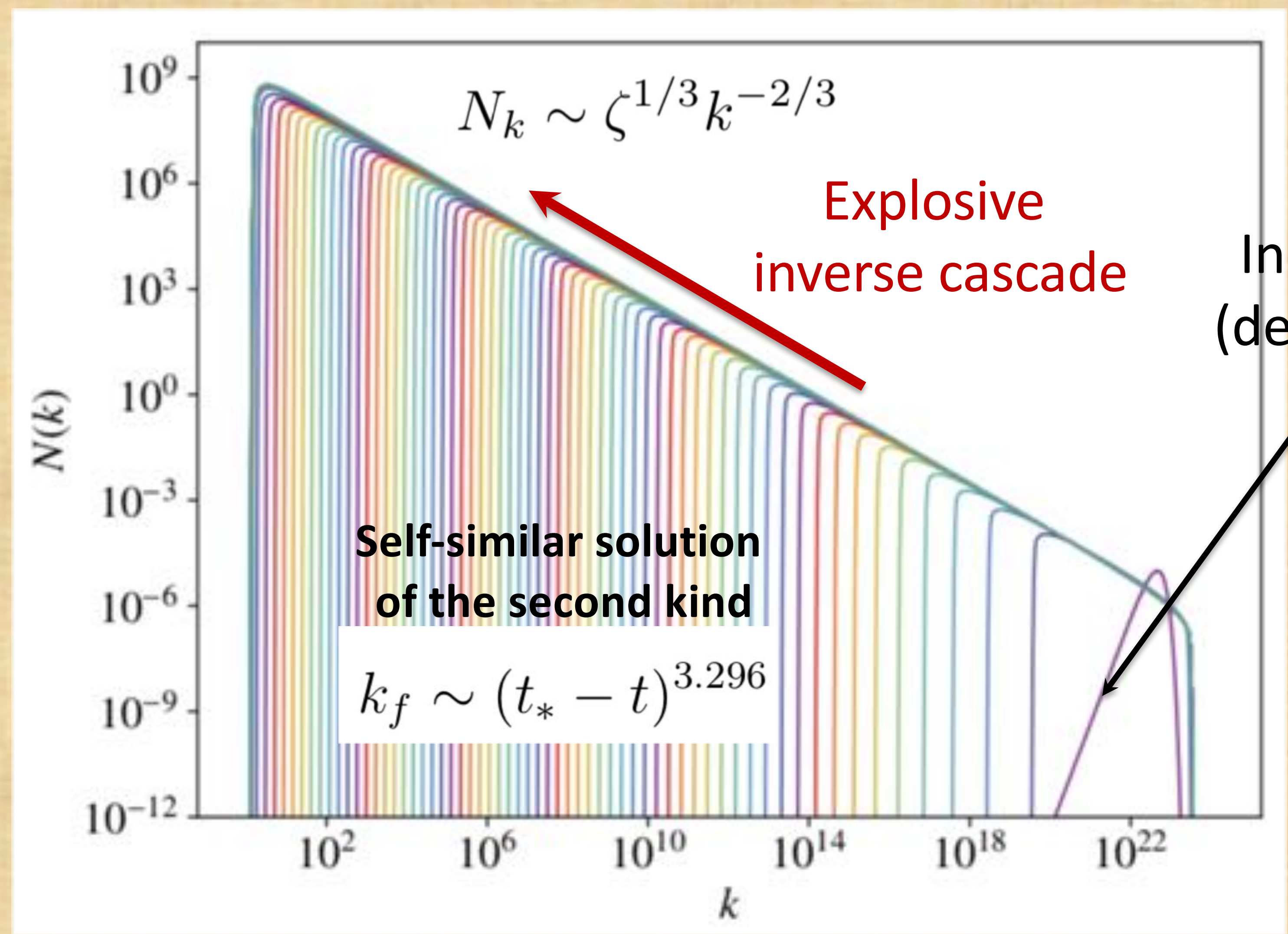
Constant flux (stationary) solutions:

Energy		Wave action
$\mathcal{E} = \iint \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} = \text{const}$		$\mathcal{N} = \iint n_{\mathbf{k}} d\mathbf{k} = \text{const}$
$E_k^{(1D)} \sim \varepsilon^{1/3} k^0$	Isotropic spectra	$N_k^{(1D)} \sim \zeta^{1/3} k^{-2/3}$
Direct cascade		**Inverse cascade**

Numerical simulation in the local approximation

$$\frac{\partial N(k)}{\partial t} = \frac{\partial}{\partial k}\left[k^2 N^2(k)\frac{\partial(kN(k))}{\partial k}\right] - \nu k^4 N(k) - \eta \frac{N(k)}{k^4}$$

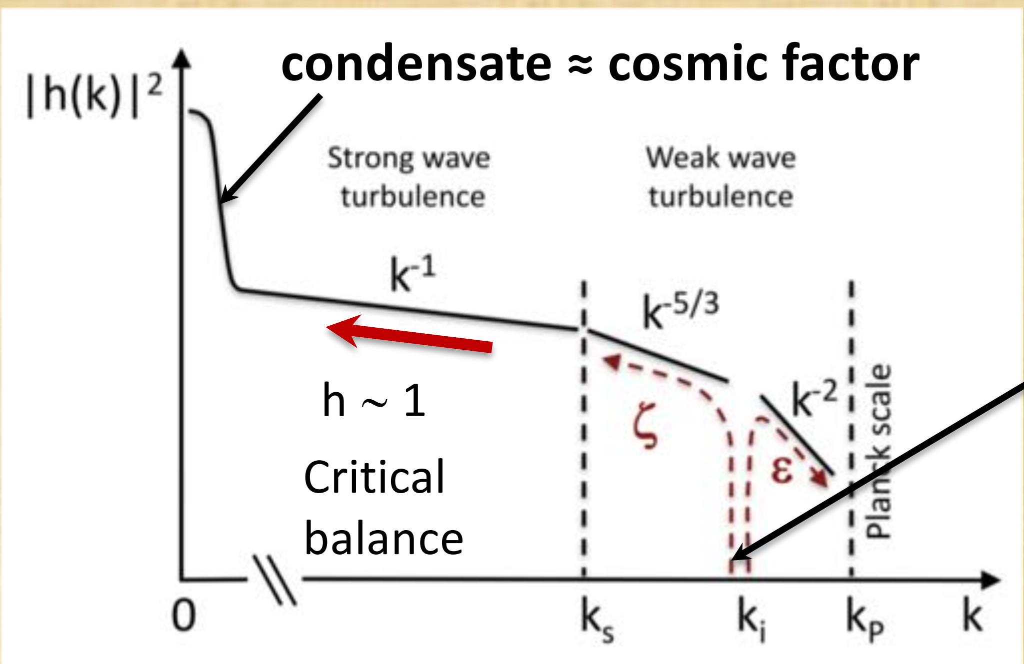




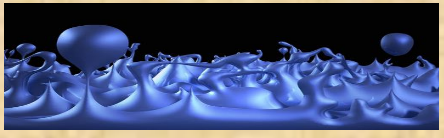
[SG, Nazarenko, Buchlin & Thalabard, 2018]

Big-Bang scenario driven by space-time turbulence

Conjecture



Quantum foam at Planck time



$h \leq 0.1$

$h \sim |h_{\mu\nu}|$

Formation of a condensate in a finite time $(10^{-39}\text{s} - 10^{-37}\text{s})$
(compatible with the causal principle)

Explosive growth of the condensate as **a nonlinear mechanism of accelerated expansion** ('turbulent inflation')

[SG, Laurie & Nazarenko, 2018]

Conclusion 1

- First analytical theory of **space-time turbulence**
- No tuning parameters
- The Riemann (4^{th} order) curvature tensor is non-trivial
  the Kretschmann scalar is non trivial

- Application to the **early universe** (peace of the story; conjecture)
- Strong wave turbulence as the main driver for the formation of a **condensate** in a finite time (compatible with the causal principle)
- A rapid growth of the condensate is interpreted as an **inflation**
- $|h(k)|^2 \sim k^{-1}$ is compatible with the Harrison-Zeldovich spectrum

- Can we perform **numerical simulations** of space-time turbulence?
- What is the origin of the inverse cascade in turbulent black holes?

Conclusion 2

Some comments:

- There is one Killing vector in this problem; this not a cylindrical GW
- Although an Alfvén wave is a nonlinear solution of incompressible MHD, weak Alfvén wave turbulence exists because the problem deals with a see of waves