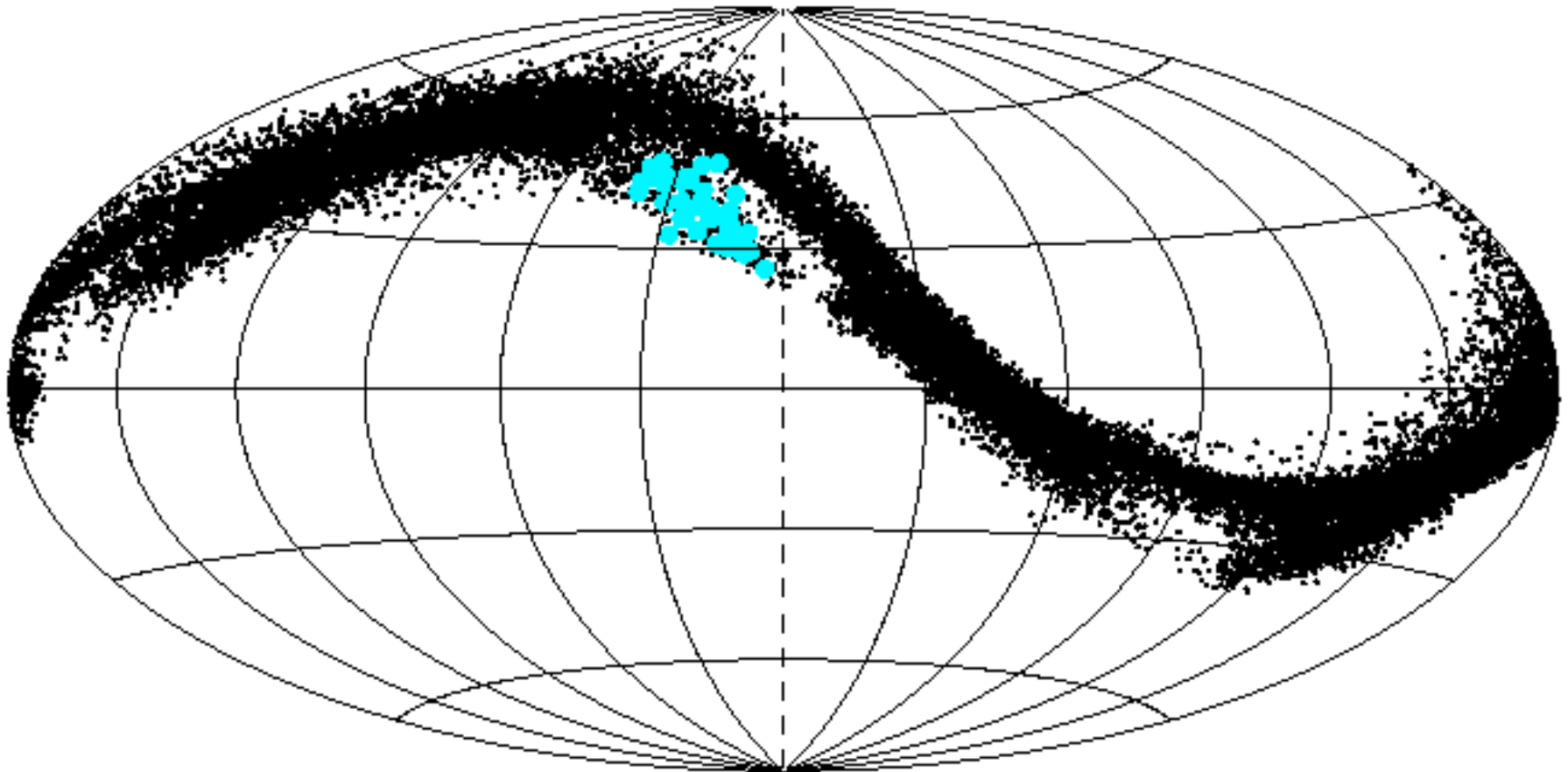
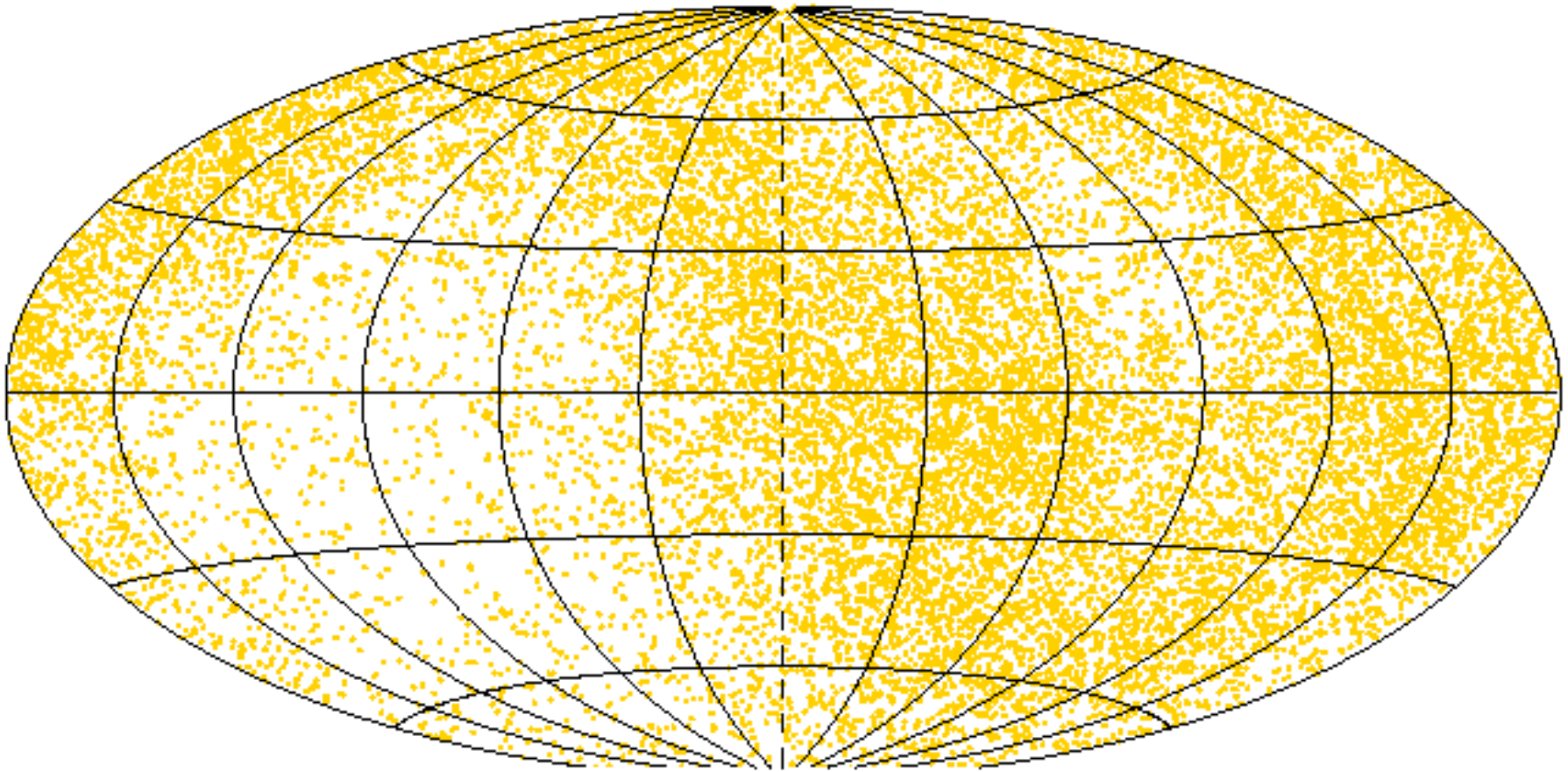


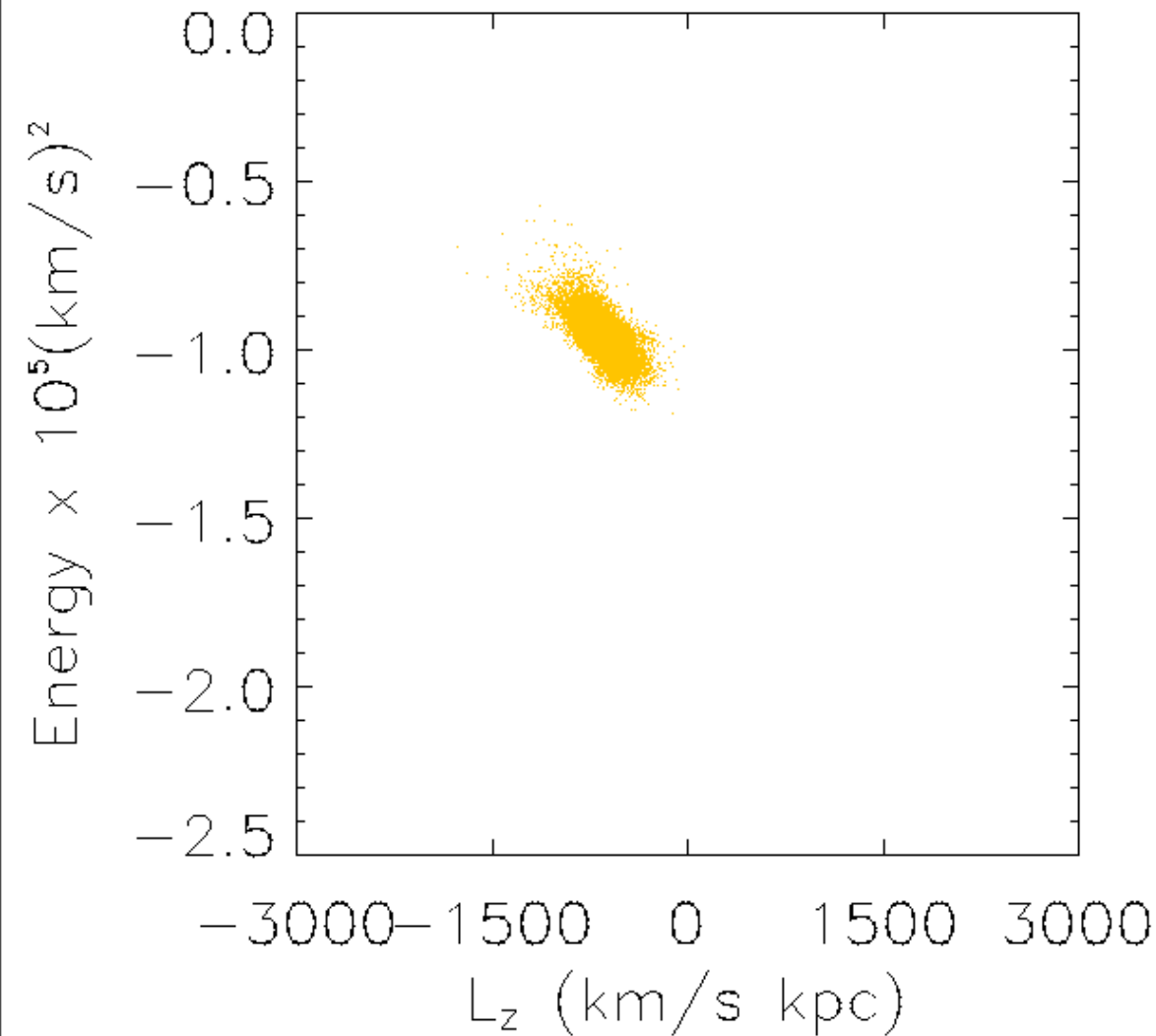
Detecting satellites using integrals of motion



Detecting satellites using integrals of motion



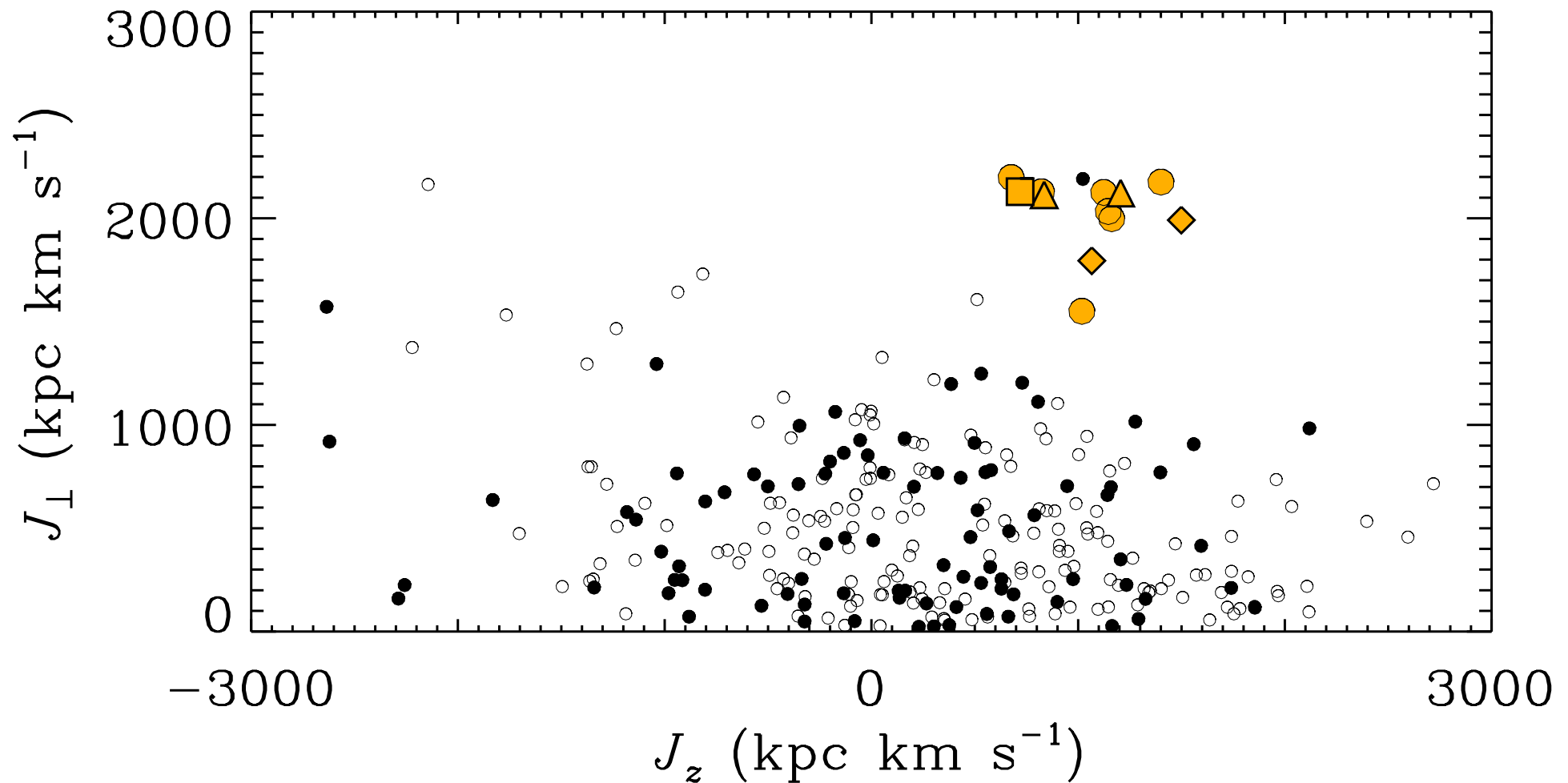
Detecting satellites using integrals of motion



Using integrals of motion we can detect satellites that are no longer visible in position-velocity configuration space

Wilkinson,..., Helmi,..., et al. (2005)

Detection of substructure in Hipparcos data set



Helmi et al. (1999)

Dynamical friction

Dynamical process which removes angular momentum from orbit of a satellite system leading to orbiting decay and merging

A satellite of mass M moving through a background of smaller masses m (e.g. stars, dark matter) experiences a net deceleration parallel to its velocity \mathbf{v} , due to wake of smaller masses which forms behind it

If the DF of the background is Maxwellian, ie.

$$f(\mathbf{v}) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$$

and we define two dimensionless quantities

$$\Lambda \equiv \frac{b_{\max} v_0^2}{G(M + m)} \qquad X \equiv \frac{v_M}{\sqrt{2}\sigma}$$

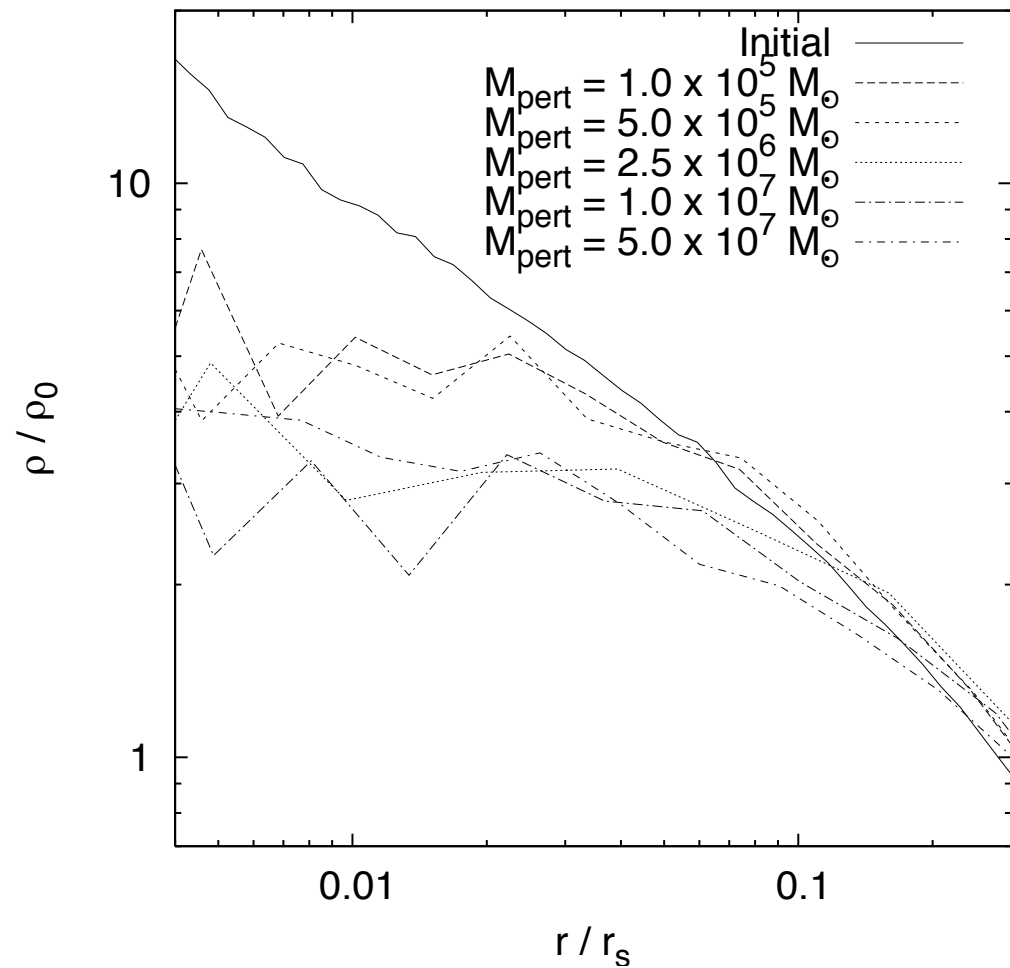
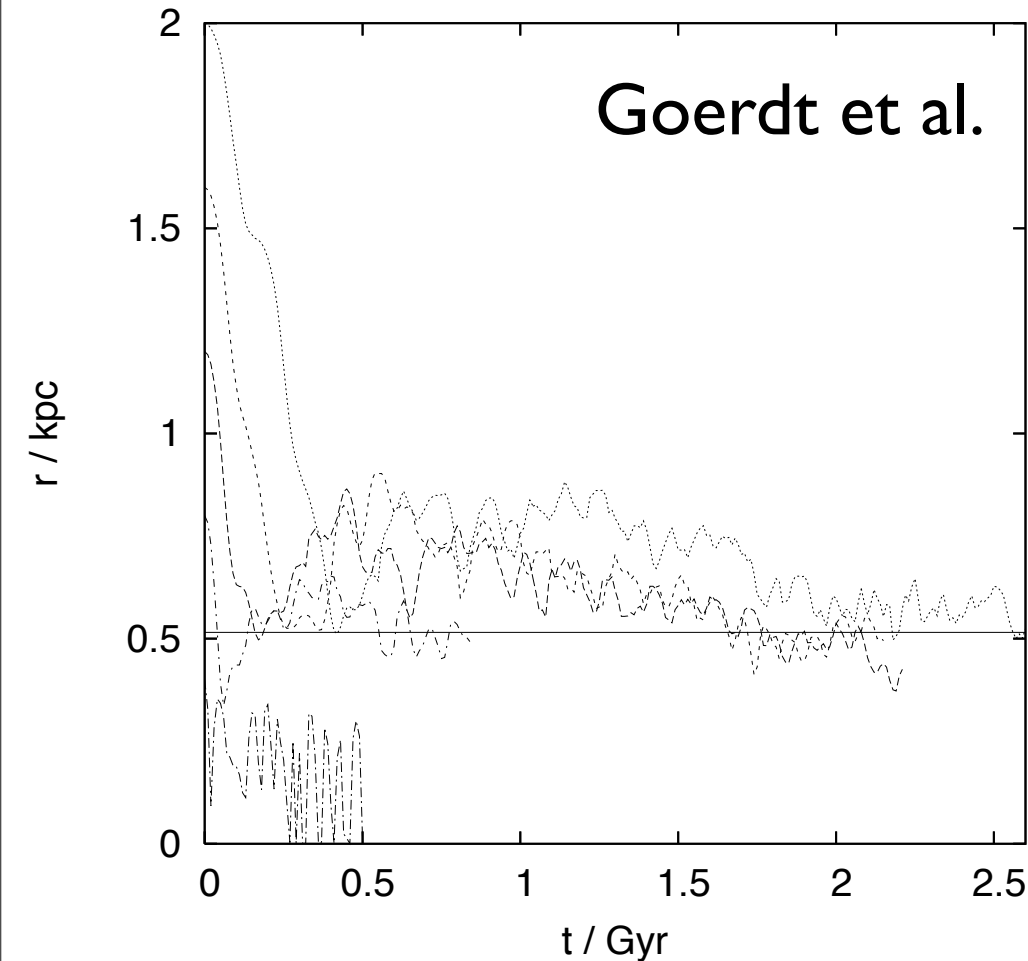
then the deceleration is given by

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi \ln \Lambda G^2 \rho M}{v_M^3} \left[\text{erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2} \right] \mathbf{v}_M$$

Notes:

- The drag acceleration $\propto M \implies$ drag force $\propto M^2$
- Drag force $\propto 1/v_M^2$
- Dynamical friction leads to observable decay of satellite galaxies (e.g. the Magellanic Clouds)
- This formulation (due to Schwarzschild) makes many simplifying assumptions, but nevertheless performs well for most applications

Effects of dynamical friction



- Dynamical friction drags satellites into higher density regions
- Can also lead to modification of parent halo profiles

Jeans Theorems

By definition, $I(\mathbf{x}, \mathbf{v})$ is an integral if and only if

$$\begin{aligned}\frac{d}{dt} I [\mathbf{x}(t), \mathbf{v}(t)] = 0 &\implies \nabla I \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial I}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = 0 \\ &\implies \mathbf{v} \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial \mathbf{v}} = 0\end{aligned}$$

$\implies I$ is a solution of the CBE

Leads to two theorems:

Jeans Theorem: *Any steady-state solution of the CBE depends on the phase space coordinates only through integrals of motion in the galactic potential, and any function of the integrals yields a steady-state solution of the CBE.*

Proof: First part follows from the above. Now, if f is a function of the integrals $(I_1 \dots I_n)$ then

$$\frac{d}{dt} f[I_1(\mathbf{x}, \mathbf{v}), \dots, I_n(\mathbf{x}, \mathbf{v})] = \sum_{m=1}^n \frac{\partial f}{\partial I_m} \frac{dI_m}{dt} = 0$$

as required.

A more useful theorem is:

Strong Jeans Theorem: *The distribution function of a steady-state galaxy in which almost all orbits are regular with incommensurable frequencies may be presumed to be a function only of three independent isolating integrals.*

For steady-state spherical systems, this implies that the DF is a function $f(E, \mathbf{L})$ and if the system is spherically symmetric in all properties, then $f = f(E, L)$

Distribution functions of spherical systems

For a self-consistent system we have

$$\begin{aligned}\nabla^2\Phi &= 4\pi G \int f(E, L) d^3\mathbf{v} \\ &= 4\pi G \int f\left(\frac{1}{2}v^2 + \Phi, |\mathbf{r} \times \mathbf{v}|\right) d^3\mathbf{v}\end{aligned}$$

Let relative potential Ψ and relative energy ε be given by

$$\Psi \equiv -\Phi + \Phi_0 \quad \varepsilon \equiv -E + \Phi_0 = \Psi - \frac{1}{2}v^2$$

NB: In terms of Ψ , Poisson's equation is

$$\nabla^2\Psi = -4\pi G\rho$$

with boundary condition $\Psi \rightarrow \Phi_0$ as $|\mathbf{x}| \rightarrow \infty$

If $f \equiv f(\varepsilon)$ then

$$\overline{v_r^2} = \frac{1}{\rho} \int dv_r dv_\theta dv_\phi v_r^2 f\left[\Psi - \frac{1}{2}(v_r^2 + v_\theta^2 + v_\phi^2)\right]$$

$$\overline{v_\theta^2} = \frac{1}{\rho} \int dv_r dv_\theta dv_\phi v_\theta^2 f\left[\Psi - \frac{1}{2}(v_r^2 + v_\theta^2 + v_\phi^2)\right]$$

Hence $\overline{v_r^2} = \overline{v_\theta^2} = \overline{v_\phi^2}$, i.e. the velocity dispersion tensor is isotropic.

Poisson equation becomes (choosing Φ_0 such that $f(\varepsilon) = 0$ for $\varepsilon < 0$)

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi}{dr} \right) &= -16\pi^2 G \int_0^{\sqrt{2\Psi}} f\left(\Psi - \frac{1}{2}v^2\right) v^2 dv \\ &= -16\pi^2 G \int_0^\Psi f(\varepsilon) \sqrt{2(\Psi - \varepsilon)} d\varepsilon \end{aligned}$$

Constructing DFs

One approach is to assume a “reasonable” form for the DF in terms of integrals of motion.

For example, consider DF of form

$$f_K(\varepsilon) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-3/2} \left(e^{\varepsilon/\sigma^2} - 1 \right) & \varepsilon > 0 \\ 0 & \varepsilon \leq 0 \end{cases}$$

Poisson equation becomes

$$\frac{d}{dr} \left(r^2 \frac{d\Psi}{dr} \right) = -4\pi G \rho_1 r^2 \left\{ e^{\Psi/\sigma^2} \operatorname{erf} \left(\frac{\sqrt{\Psi}}{\sigma} \right) - \sqrt{\frac{4\Psi}{\pi\sigma^2}} \left(1 + \frac{2\Psi}{3\sigma^2} \right) \right\}$$

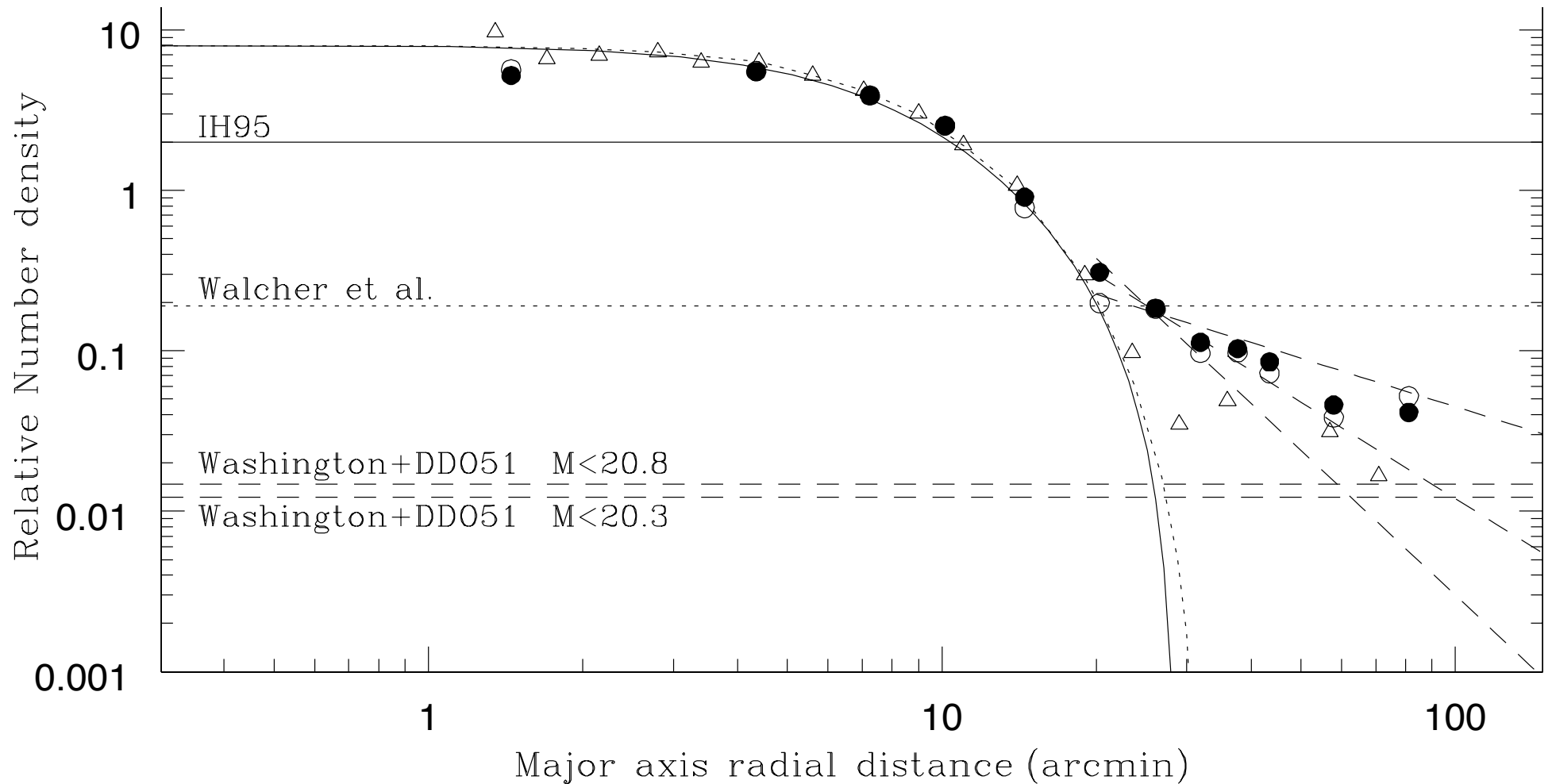
This DF leads to the King models which are characterised either by their concentration

$$c \equiv \log_{10}(r_t/r_0)$$

or by the ratio $\Psi(0)/\sigma^2$.

- King models are good representations of globular clusters where interactions lead to Maxwellian energy distribution
- Surface brightness profile has a core+halo structure which falls to zero at a finite radius r_t - the “King tidal radius”.
- When applied to collisionless systems, they are just fitting formulae - “tidal radius” is just a parameter

Beware over-interpretation of King models



Majewski et al., 2005

Building isotropic DF from $\rho(r)$

Density is related to DF by

$$\rho(r) = 4\pi \int_0^\Psi f(\varepsilon) \sqrt{2(\Psi - \varepsilon)} d\varepsilon$$
$$\Rightarrow \frac{1}{\sqrt{8\pi}} \rho(\Psi) = 2 \int_0^\Psi f(\varepsilon) \sqrt{\Psi - \varepsilon} d\varepsilon$$

Differentiating with respect to Ψ we obtain

$$\frac{1}{\sqrt{8\pi}} \frac{d\rho(\Psi)}{d\Psi} = \int_0^\Psi \frac{f(\varepsilon)}{\sqrt{\Psi - \varepsilon}} d\varepsilon$$

which is an Abel equation for $f(\varepsilon)$ with solution (Eddington's Formula)

$$f(\varepsilon) = \frac{1}{\sqrt{8\pi^2}} \frac{d}{d\varepsilon} \int_0^\varepsilon \frac{d\rho}{d\Psi} \frac{d\Psi}{\sqrt{\varepsilon - \Psi}}$$

or, alternatively

$$f(\varepsilon) = \frac{1}{\sqrt{8}\pi^2} \left[\int_0^\varepsilon \frac{d^2\rho}{d\Psi^2} \frac{d\Psi}{\sqrt{\varepsilon - \Psi}} + \frac{1}{\sqrt{\varepsilon}} \left(\frac{d\rho}{d\Psi} \right)_{\Psi=0} \right]$$

NB: Since we assumed that DF depends only on energy, this distribution function has an isotropic velocity distribution

If $f(\varepsilon)$ is non-negative everywhere, then this is the *unique* isotropic DF corresponding to the density profile $\rho(r)$

Models with anisotropic velocity dispersion tensors

DFs of the form $f(E, L)$ have anisotropic velocity dispersion tensors.

Let (v, η, ψ) be coordinates in velocity space oriented such that

$$v_r = v \cos \eta, \quad v_\theta = v \sin \eta \cos \psi, \quad v_\phi = v \sin \eta \sin \psi$$

We then have

$$\begin{aligned} \rho(r) &= \int f(\varepsilon, L) d^3\mathbf{v} \\ &= 2\pi \int_0^\pi \sin \eta d\eta \int_0^\infty f\left(\Psi - \frac{1}{2}v^2, |rv \sin \eta|\right) v^2 dv \end{aligned}$$

To proceed we must choose the form of the L -dependence

One common choice is $F(\varepsilon, L) = L^{-2\beta} f(\varepsilon)$ which leads to models with $\beta(r) \equiv \text{const}$ (i.e. constant anisotropy)

Another family of models are known as Osipkov-Merritt models:

$$\begin{aligned} f(\varepsilon, L) &\equiv f(Q) \\ Q \equiv \varepsilon - \frac{L^2}{2r_a^2} &= \Psi - \frac{1}{2}v^2 \left(1 + \frac{r^2}{r_a^2} \sin^2 \eta \right) \end{aligned}$$

where r_a is the anisotropy radius. If we assume that $f(Q) = 0$ for $Q \leq 0$ then

$$\begin{aligned} \rho(r) &= 2\pi \int_0^\pi \sin \eta d\eta \int_0^\Psi f(Q) \frac{\sqrt{2(\Psi - Q)} dQ}{\left[1 + \left(\frac{r}{r_a} \right)^2 \sin^2 \eta \right]^{3/2}} \\ \Rightarrow \left(1 + \frac{r^2}{r_a^2} \right) \rho(r) &= 4\pi \int_0^\Psi f(Q) \sqrt{2(\Psi - Q)} dQ \end{aligned}$$

Comparing this with the derivation of the Eddington formula, we conclude that

$$f(Q) = \frac{1}{\sqrt{8\pi^2}} \left[\int_0^Q \frac{d^2 \rho_Q}{d\Psi^2} \frac{d\Psi}{\sqrt{Q - \Psi}} + \frac{1}{\sqrt{Q}} \left(\frac{d\rho_Q}{d\Psi} \right)_{Q=0} \right]$$

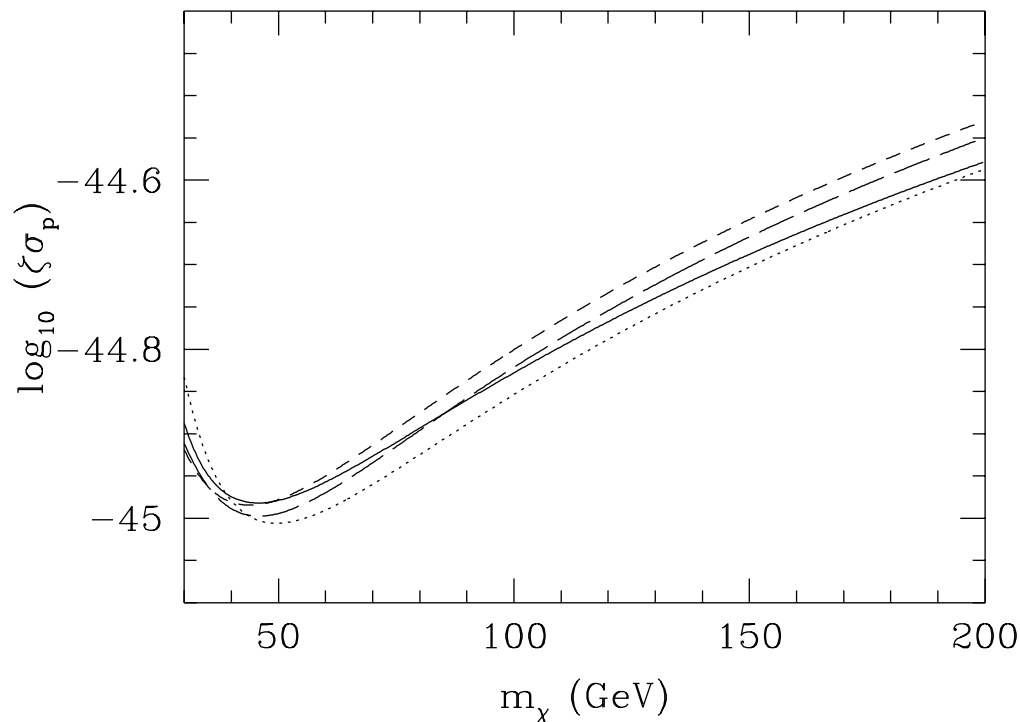
where

$$\rho_Q(r) \equiv \left(1 + \frac{r^2}{r_a^2} \right) \rho(r)$$

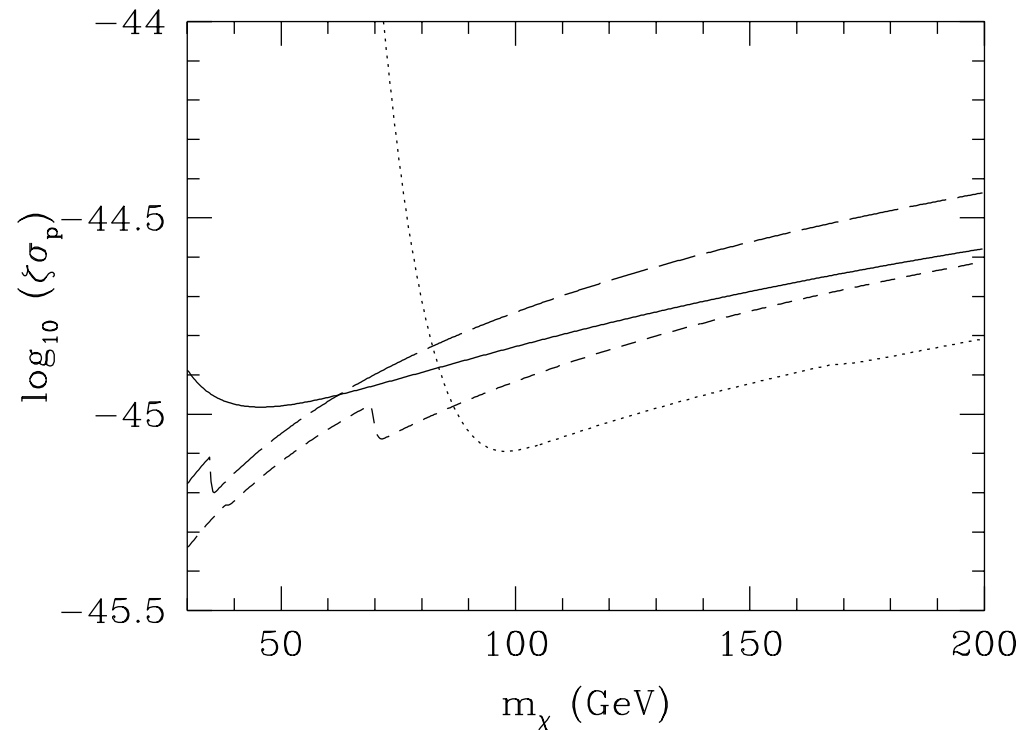
The models are useful because they are isotropic at small radii $r \ll r_a$ but become radially anisotropic at large radii $r \gg r_a$

They are widely used because they are one of the simplest anisotropic models to construct.

Impact of Milky Way models on DM direct detection rates



Changing from isotropy to
Osipkov-Merritt anisotropy



Including halo substructure

Green (2002)

The Tensor Virial Theorem

Multiply the CBE by x_k and integrate over *spatial* variables

$$\int x_k \frac{\partial(\rho \bar{v}_j)}{\partial t} d^3 \mathbf{x} = - \int x_k \frac{\partial(\rho \bar{v}_i \bar{v}_j)}{\partial x_i} d^3 \mathbf{x} - \int \rho x_k \frac{\partial \Phi}{\partial x_j} d^3 \mathbf{x}$$

Define potential energy tensor **W** and kinetic energy tensor **K**

$$W_{jk} \equiv - \int \rho(\mathbf{x}) x_j \frac{\partial \Phi}{\partial x_k} d^3 \mathbf{x}$$

$$K_{jk} \equiv \frac{1}{2} \int \rho \bar{v}_j \bar{v}_k d^3 \mathbf{x}$$

$$K_{jk} = T_{jk} + \frac{1}{2} \Pi_{jk}$$

where

$$T_{jk} \equiv \frac{1}{2} \int \rho \bar{v}_j \bar{v}_k d^3 \mathbf{x} \quad \Pi_{jk} \equiv \int \rho \sigma_{jk}^2 d^3 \mathbf{x}$$

Hence we obtain

$$\frac{1}{2} \frac{d}{dt} \int \rho (x_k \overline{v_j} + x_j \overline{v_k}) d^3 \mathbf{x} = 2T_{jk} + \Pi_{jk} + W_{jk}$$

Define the moment of inertia tensor \mathbf{I}

$$I_{jk} \equiv \int \rho x_j x_k d^3 \mathbf{x}$$

Hence, obtain the **tensor virial theorem**

$$\frac{1}{2} \frac{d^2 I_{jk}}{dt^2} = 2T_{jk} + \Pi_{jk} + W_{jk}$$

Taking the trace, in steady state, yields the scalar virial theorem

$$2K + W = 0$$

Tensor virial theorem for steady-state system:

$$2T_{jk} + \Pi_{jk} + W_{jk} = 0$$

\implies velocities of stars are related directly to shape of gravitational potential well

Define the “gravitational radius” via $r_g \equiv \frac{GM^2}{|W|}$

Scalar virial theorem implies $\langle v^2 \rangle = \frac{|W|}{M} = \frac{GM}{r_g} \simeq 0.4 \frac{GM}{r_h}$

Tensor virial theorem for steady-state system:

$$\text{Streaming motion} \rightarrow 2T_{jk} + \Pi_{jk} + W_{jk} = 0$$

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Tensor virial theorem for steady-state system:

$$\begin{array}{ccc} & \textcircled{2T_{jk}} & + \textcircled{\Pi_{jk}} + W_{jk} = 0 \\ \nearrow & & \uparrow \\ \text{Streaming} & & \text{Random} \\ \text{motion} & & \text{motion} \end{array}$$

\Rightarrow velocities of stars are related directly to shape of gravitational potential well

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Tensor virial theorem for steady-state system:

$$\text{Streaming motion} \rightarrow \textcircled{2T_{jk}} + \textcircled{\Pi_{jk}} + \textcircled{W_{jk}} = 0$$

Random motion Potential energy

\Rightarrow velocities of stars are related directly to shape of gravitational potential well

Define the “gravitational radius” via $r_g \equiv \frac{GM^2}{|W|}$

Scalar virial theorem implies $\langle v^2 \rangle = \frac{|W|}{M} = \frac{GM}{r_g} \simeq 0.4 \frac{GM}{r_h}$

Mass Estimation

- Mass estimates for point masses using Keplerian tracers are relatively straightforward
- Estimating the mass of an extended system, is more difficult
- Simplifying assumptions often required
- If tracers are pressure-supported (ie kinematically “hot”) then significantly more complicated
- Robust mass estimation is crucial for dark matter studies as it provides density distributions

Simple Mass Estimators I - Virial estimator

Consider a *self-gravitating* system of N galaxies, each of mass m .

The moment of inertia I of the system is given by

$$I = \frac{1}{2}m \sum_i \mathbf{R}_i^2$$

Differentiate: $\ddot{I} = m \sum_i \ddot{\mathbf{R}}_i \cdot \mathbf{R}_i + m \sum_i \dot{\mathbf{R}}_i^2$

Now $m \sum_i \ddot{\mathbf{R}}_i \cdot \mathbf{R}_i = -Gm^2 \sum_i \sum_{i \neq j} \frac{\mathbf{R}_{ij} \cdot \mathbf{R}_i}{R_{ij}^3}$

$$= -Gm^2 \sum_i \sum_{j < i} \frac{1}{R_{ij}}$$
$$= W$$

Hence we obtain

$$-Gm^2 \sum_{i < j} \left\langle \frac{1}{R_{ij}} \right\rangle_t + m \sum_i \langle V_i^2 \rangle_t = 0$$

Now, assuming an isotropic velocity distribution and spherical symmetry, we have

$$\langle V_i^2 \rangle = 3 \langle V_{z,i}^2 \rangle \qquad \langle R_{ij}^{-1} \rangle = \frac{2}{\pi} \langle R_{\perp,ij}^{-1} \rangle$$

From this we obtain the **Virial Mass Estimator**

$$M = \frac{3\pi N}{2G} \frac{\sum_i \langle \langle V_{z,i}^2 \rangle_t \rangle}{\sum_{i < j} \langle \langle R_{\perp,ij}^{-1} \rangle_t \rangle}$$

Simple Mass Estimators II - Projected Mass Estimator (Bahcall & Tremaine, 1981)

Mass estimator for system with tracer particles moving around a point mass M

Consider the quantity $q \equiv \frac{v_z^2 R}{G}$

Assume a velocity distribution $F(E, L^2)$ which depends on two integrals of motion, and integrate over all (\mathbf{r}, \mathbf{v}) .

Find that $\langle q \rangle = \frac{\pi M}{32} (3 - 2\langle e^2 \rangle)$

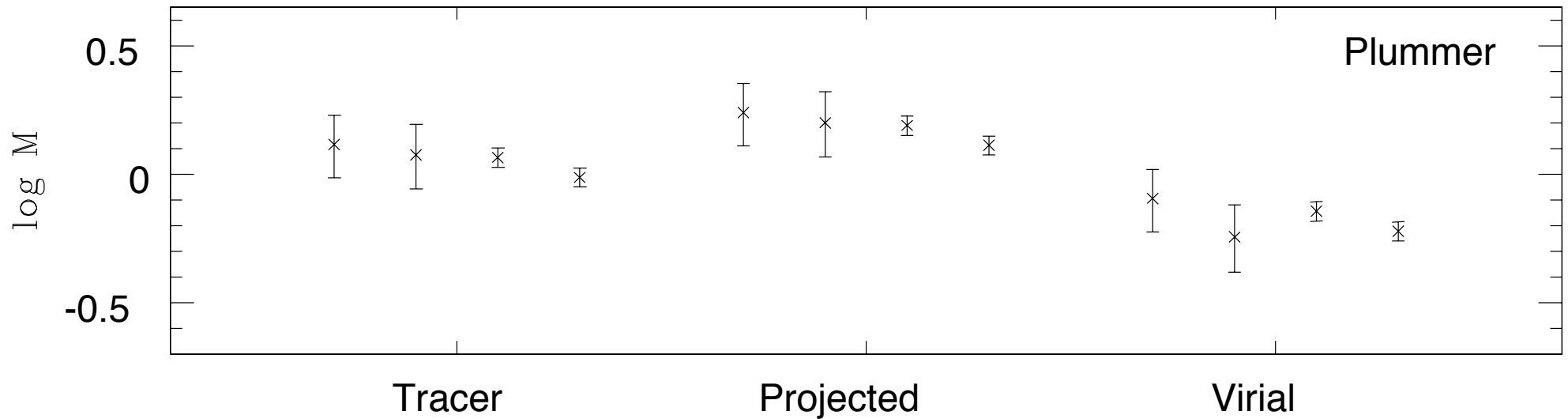
Hence, the **projected mass estimator** is

$$M = \frac{f}{\pi G N} \sum_{i=1}^N v_{z,i}^2 R_i$$

For isotropic orbits: $\langle e^2 \rangle = 1/2 \implies f = 16$

For radial orbits: $\langle e^2 \rangle = 1 \implies f = 32$

Comparison of estimators



Evans et al. (2003)

- All simple estimators have systematic uncertainties
- Very sensitive to assumptions about (an)isotropy of velocity distribution

Mass estimation with Jeans equations

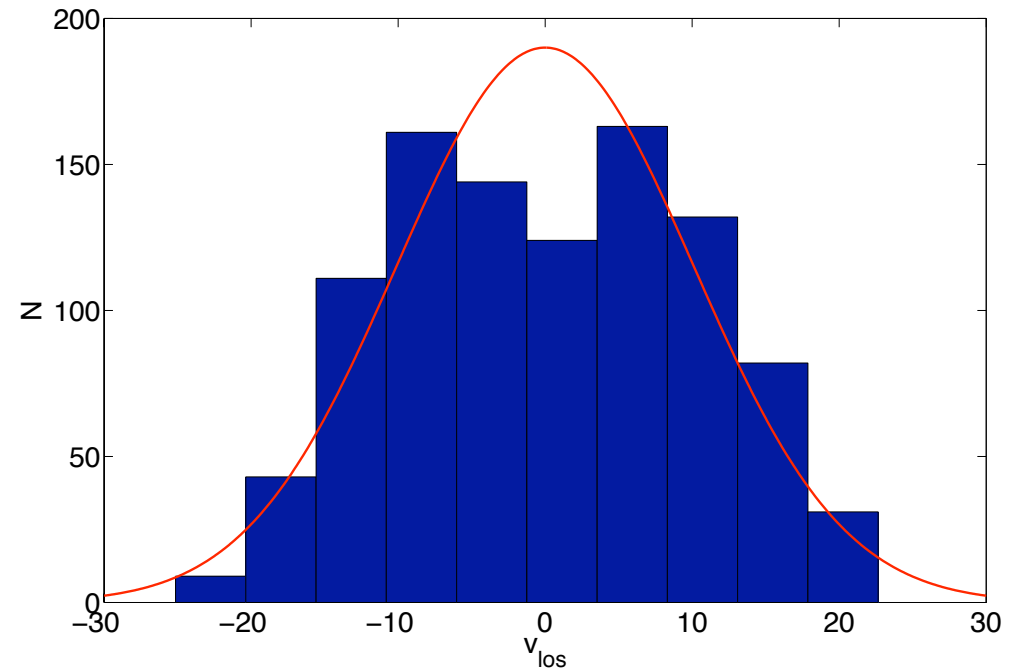
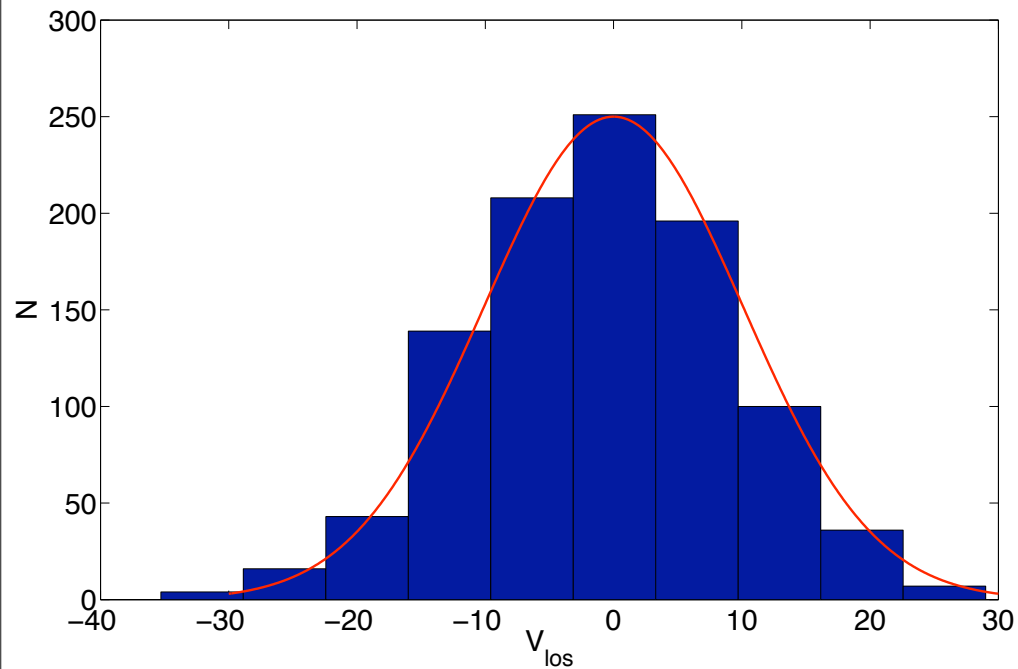
Jeans equations give simple relation between kinematics, the light distribution and the underlying mass distribution

$$M(r) = -\frac{r^2}{G} \left(\frac{1}{\nu} \frac{d \nu \sigma_r^2}{d r} + 2 \frac{\beta \sigma_r^2}{r} \right) \quad \beta(r) = 1 - \frac{\langle v_t^2 \rangle}{2 \langle v_r^2 \rangle}$$

We can either:

1. Assume a parameterised mass model $M(r)$ and velocity anisotropy $\beta(r)$ and fit kinematic data (e.g. dispersion profile)
or
2. Use Jeans equations to determine mass profile from projected kinematics and a fit to the light distribution

Degeneracies in mass models - why dispersion profiles aren't enough

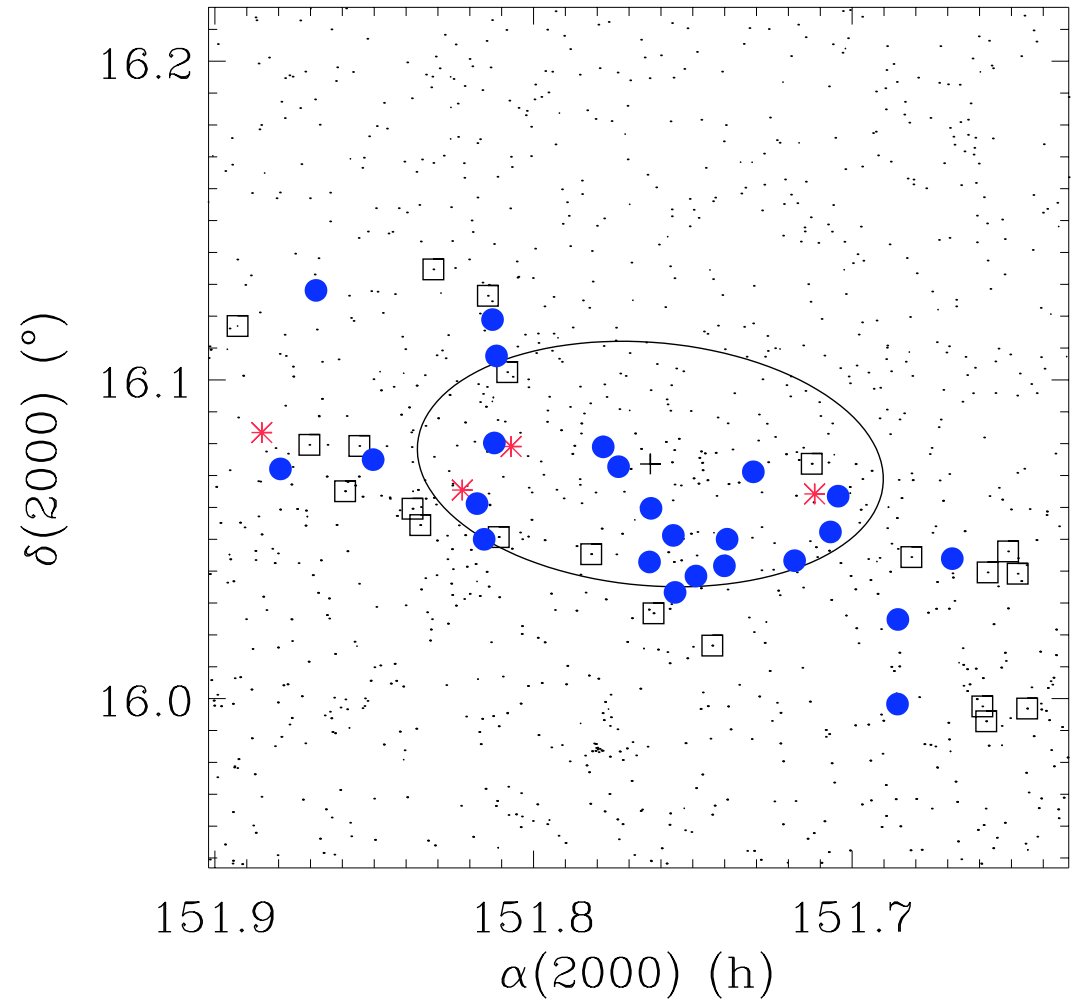
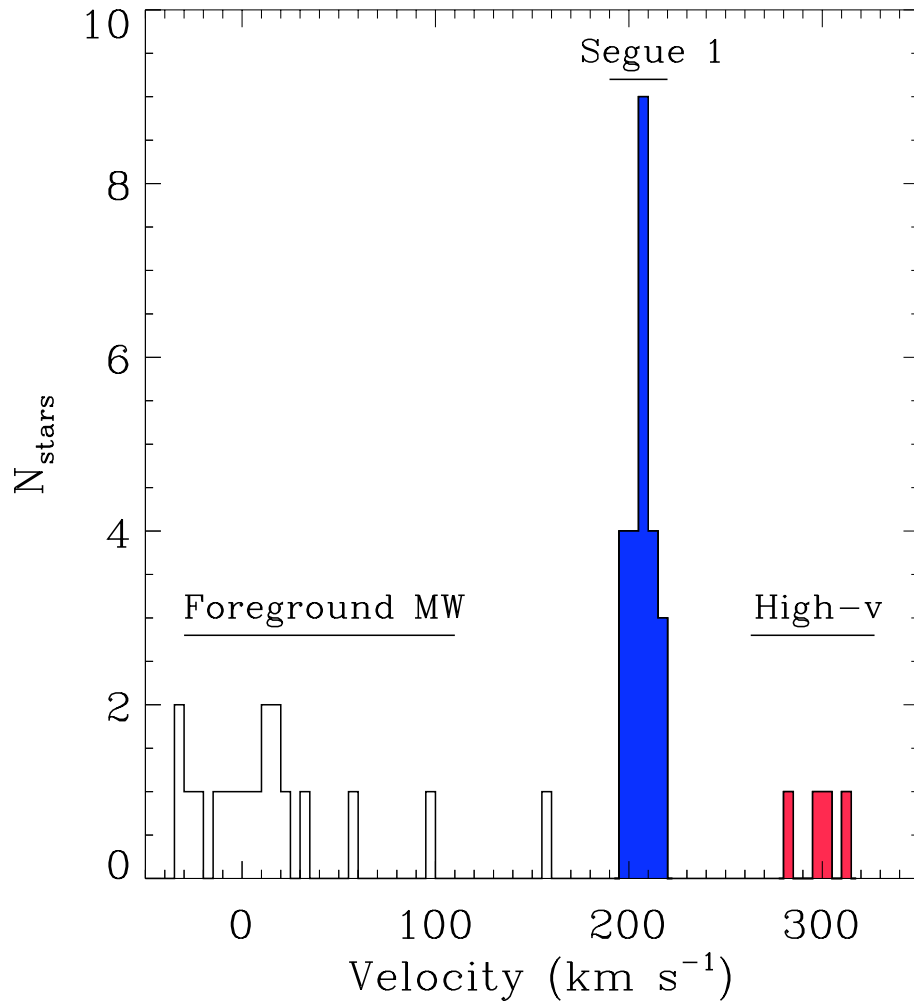


Tangential anisotropy (circular motions) can lead to inflation of observed line of sight velocity dispersion

Mass estimates with limited data

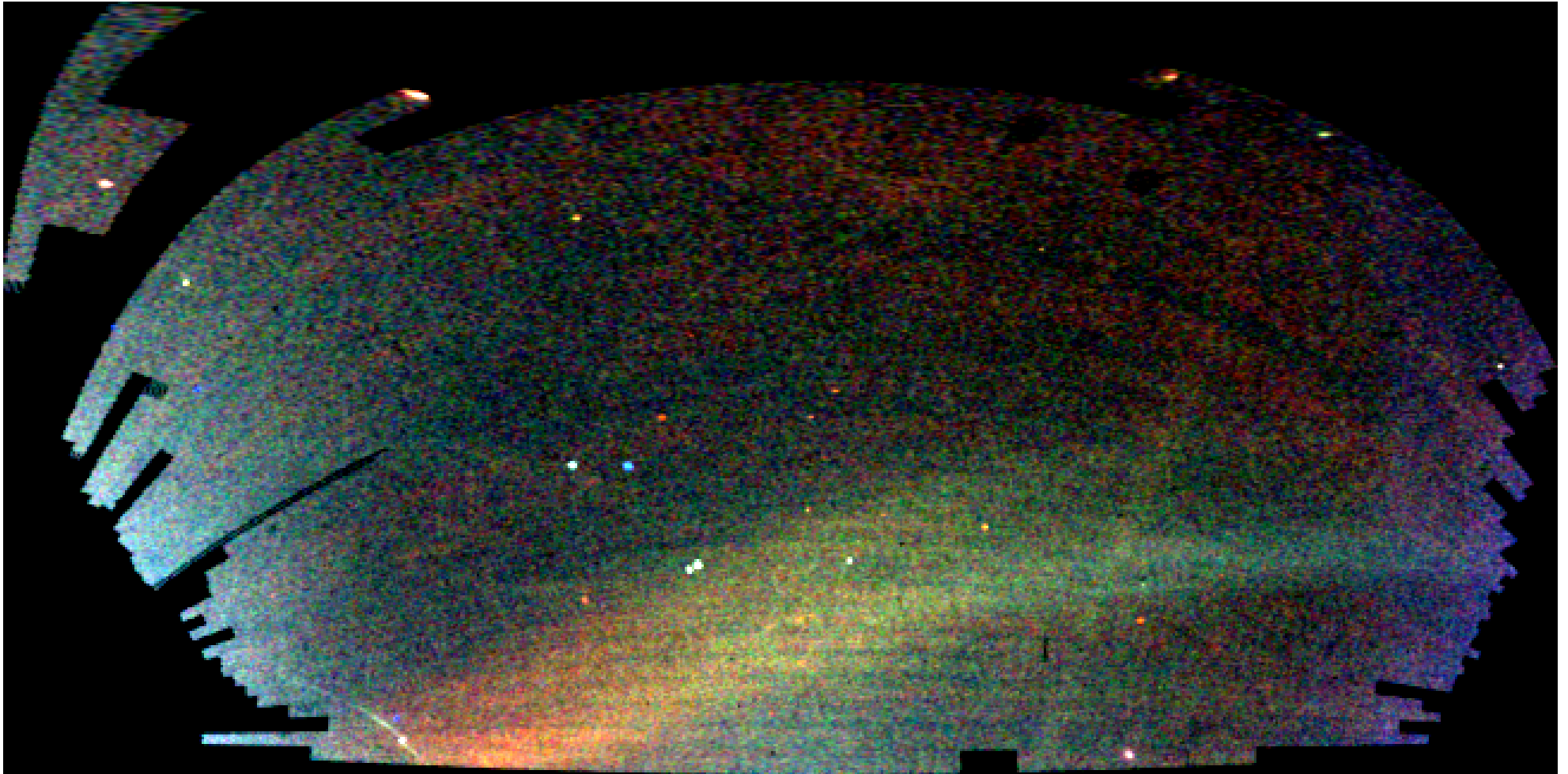
- We need surface brightness profile $\Sigma(R)$ of tracer population and information about tracer velocity distribution
- Problems with $\Sigma(R)$:
 - For faint systems profile may be very noisy
 - Tracer profile may not match that of main galaxy population (e.g. if observing metal-rich stars in a dSph)
- In absence of a velocity dispersion profile $\sigma(R)$, need to make strong assumptions about the shape of the profile
 - e.g. for faint dSphs might assume isotropy and a flat dispersion profile since this is seen in brighter dSphs
 - mass-follows-light is often assumed although not justified in brighter dSphs

Segue I - the least-luminous dSph?



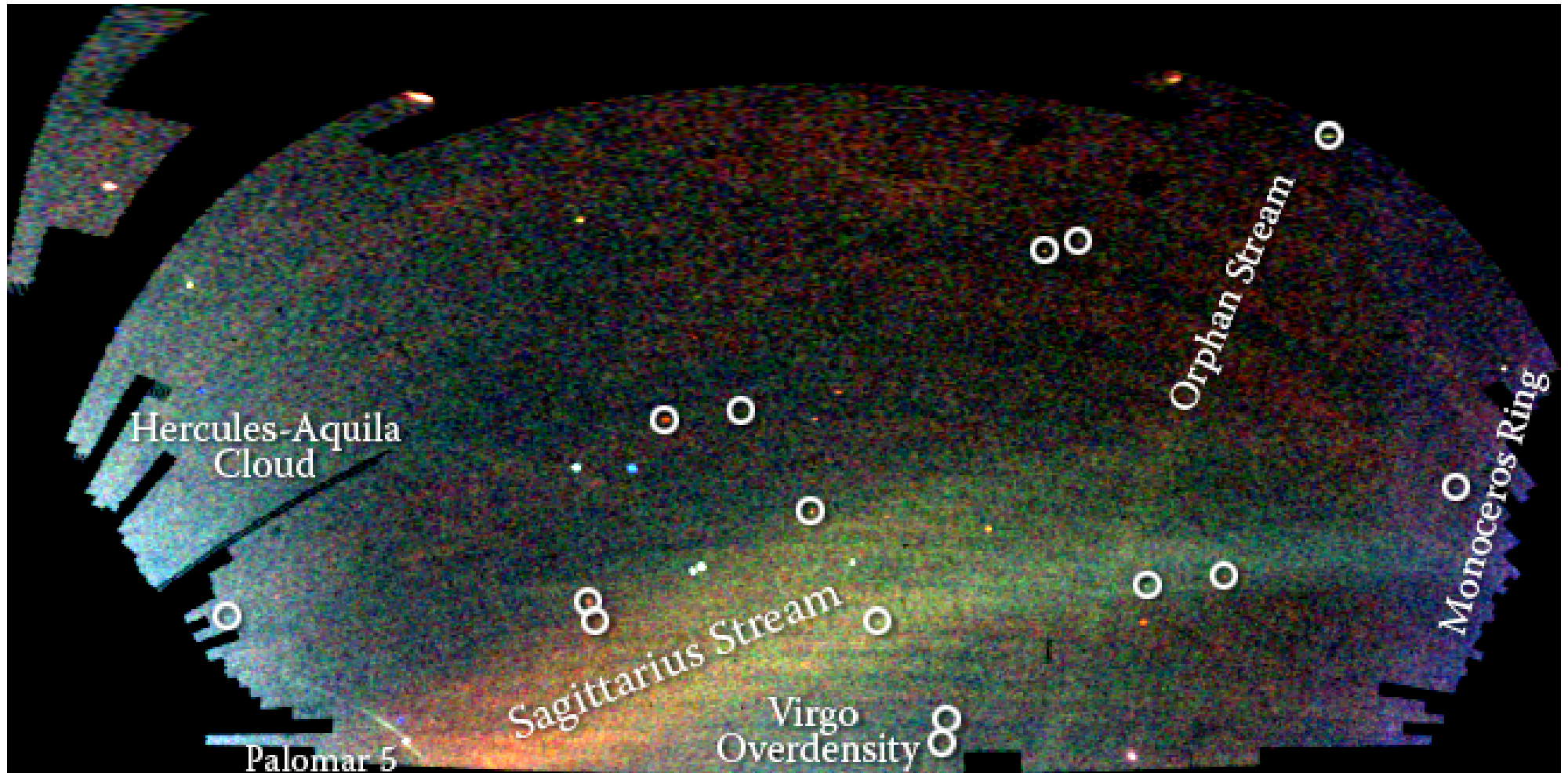
Geha et al., (2008)

The halo around Segue I



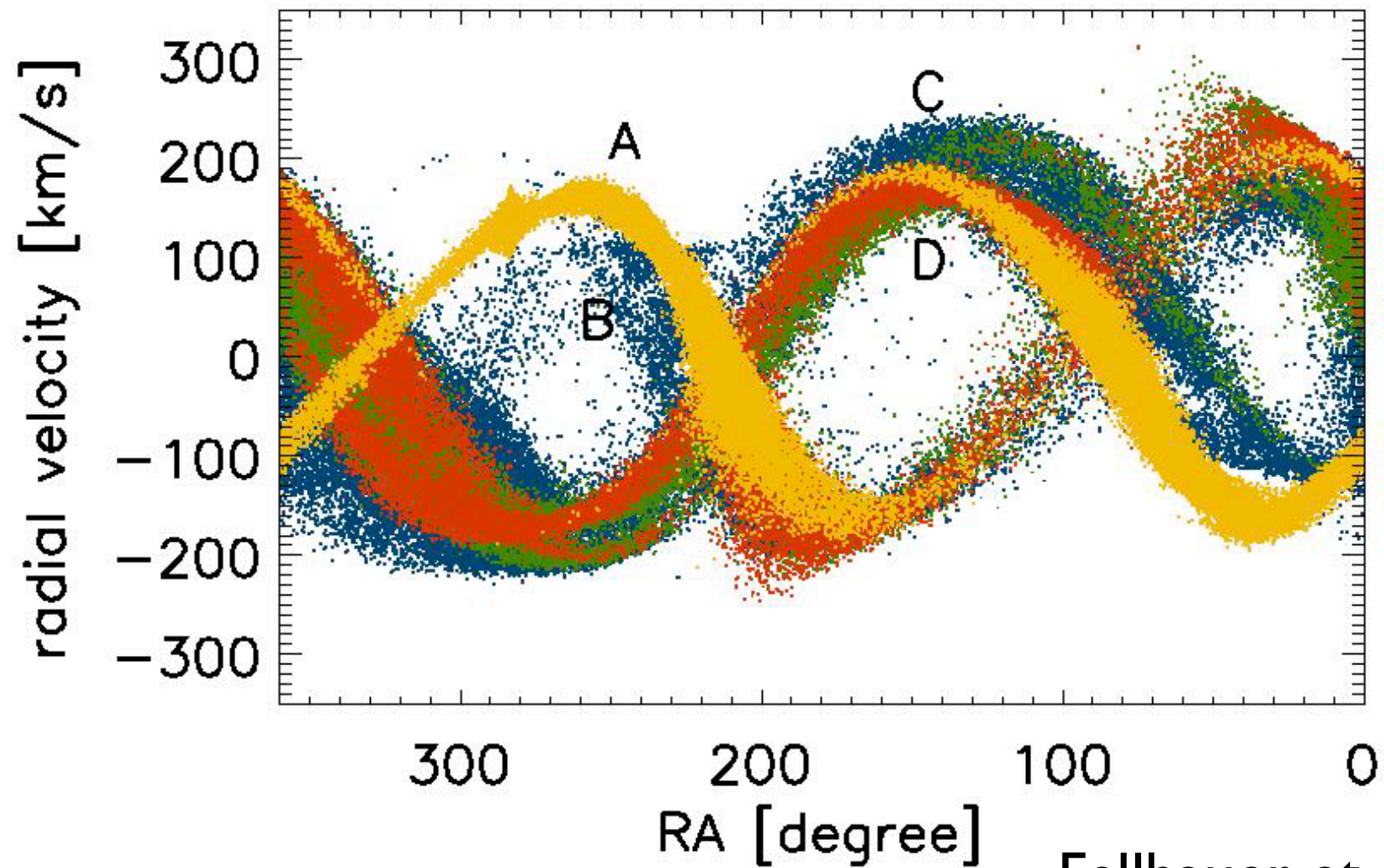
Belokurov et al. (2006)

The halo around Segue I



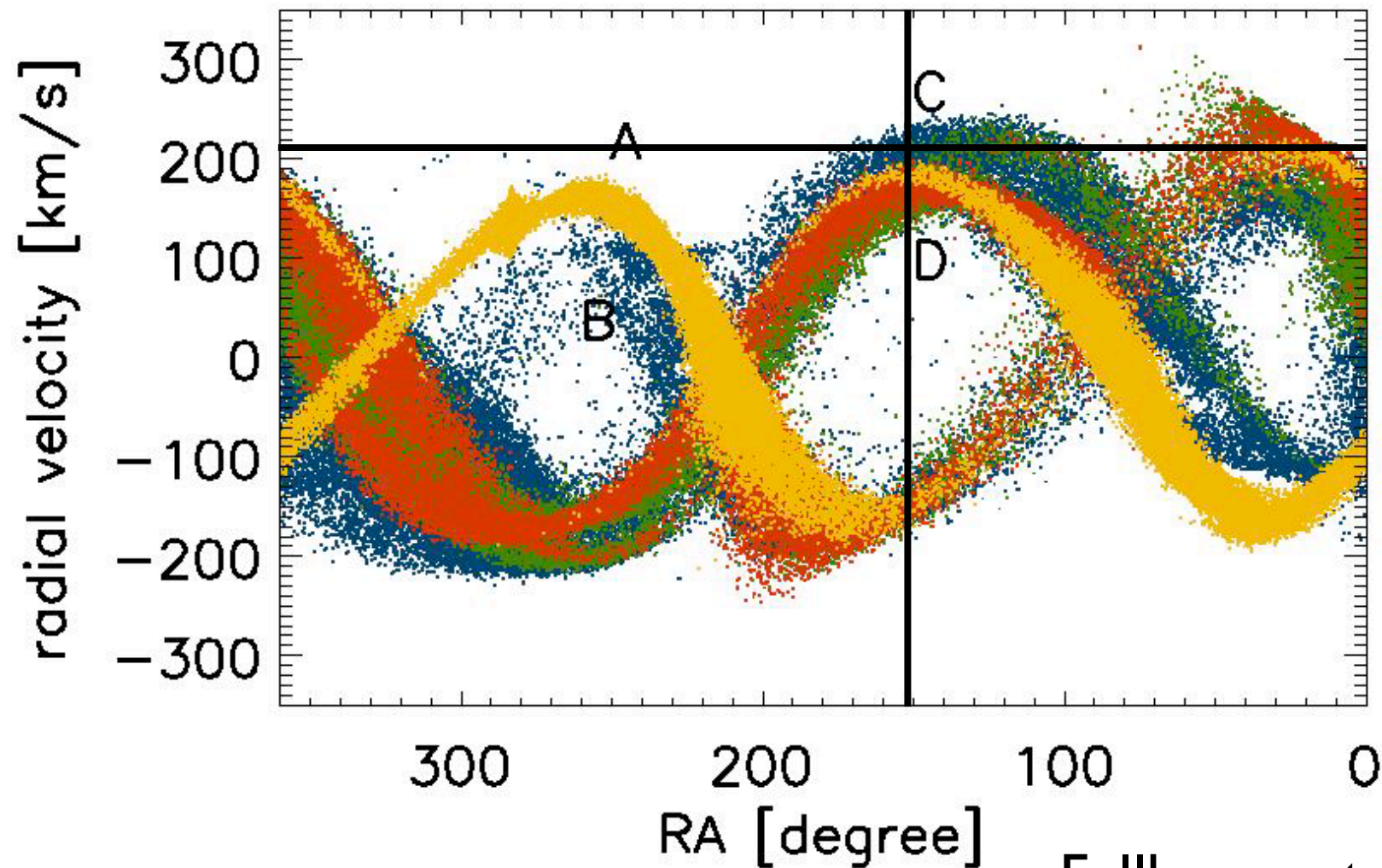
Belokurov et al. (2006)

Velocity of Sagittarius tails



Fellhauer et al. (2006)

Velocity of Sagittarius tails



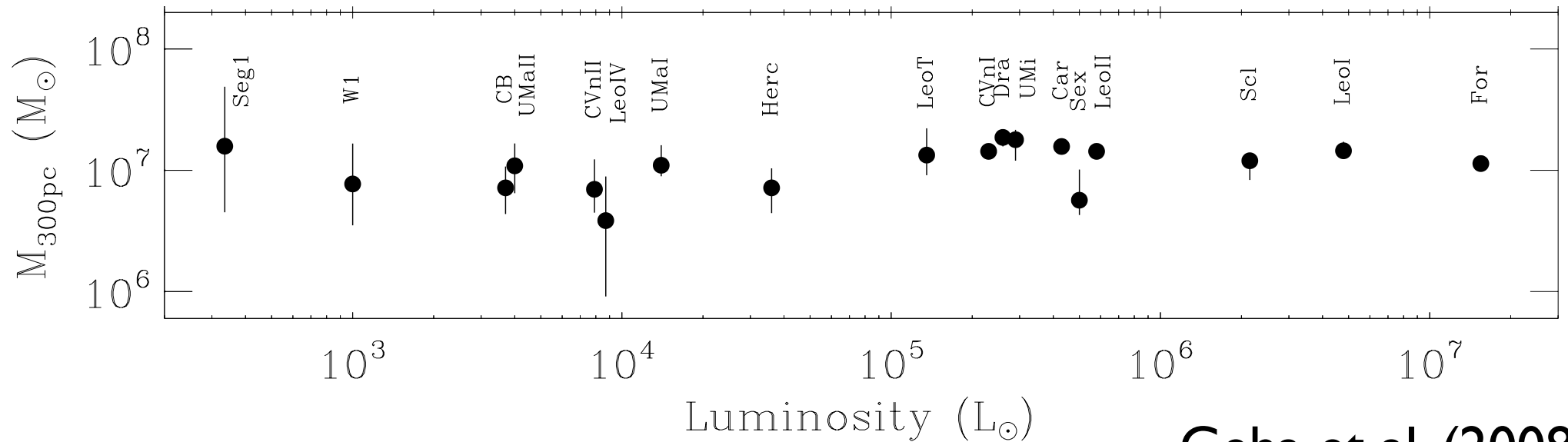
Fellhauer et al. (2006)

Is Segue sample contaminated by Sgr tails?

Could this inflate dispersion from expected value ($\sim 0.4 \text{ km s}^{-1}$)?

See paper by Niederte-Ostholt (2009, subm.)

Seg1 - a constant halo mass scale?

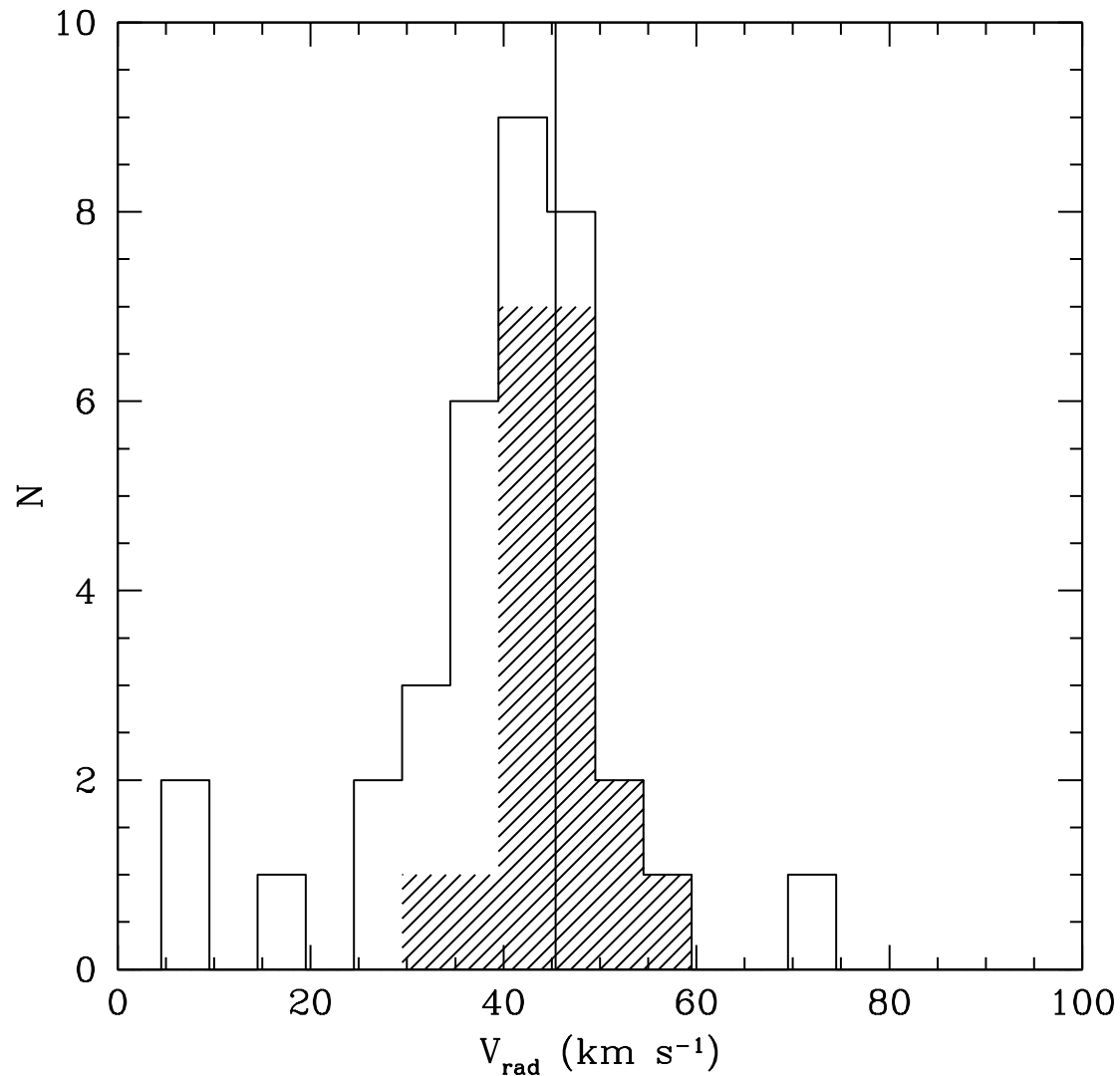


Geha et al. (2008)

NB: Segue I mass in this plot is *extrapolated* from $5 \times 10^5 M_\odot$ within 50pc

Density within 50pc still high enough to produce interesting gamma ray flux if central mass estimate is correct

Hercules - another case for caution



Galactic foreground
interlopers removed
using Stromgren
photometry

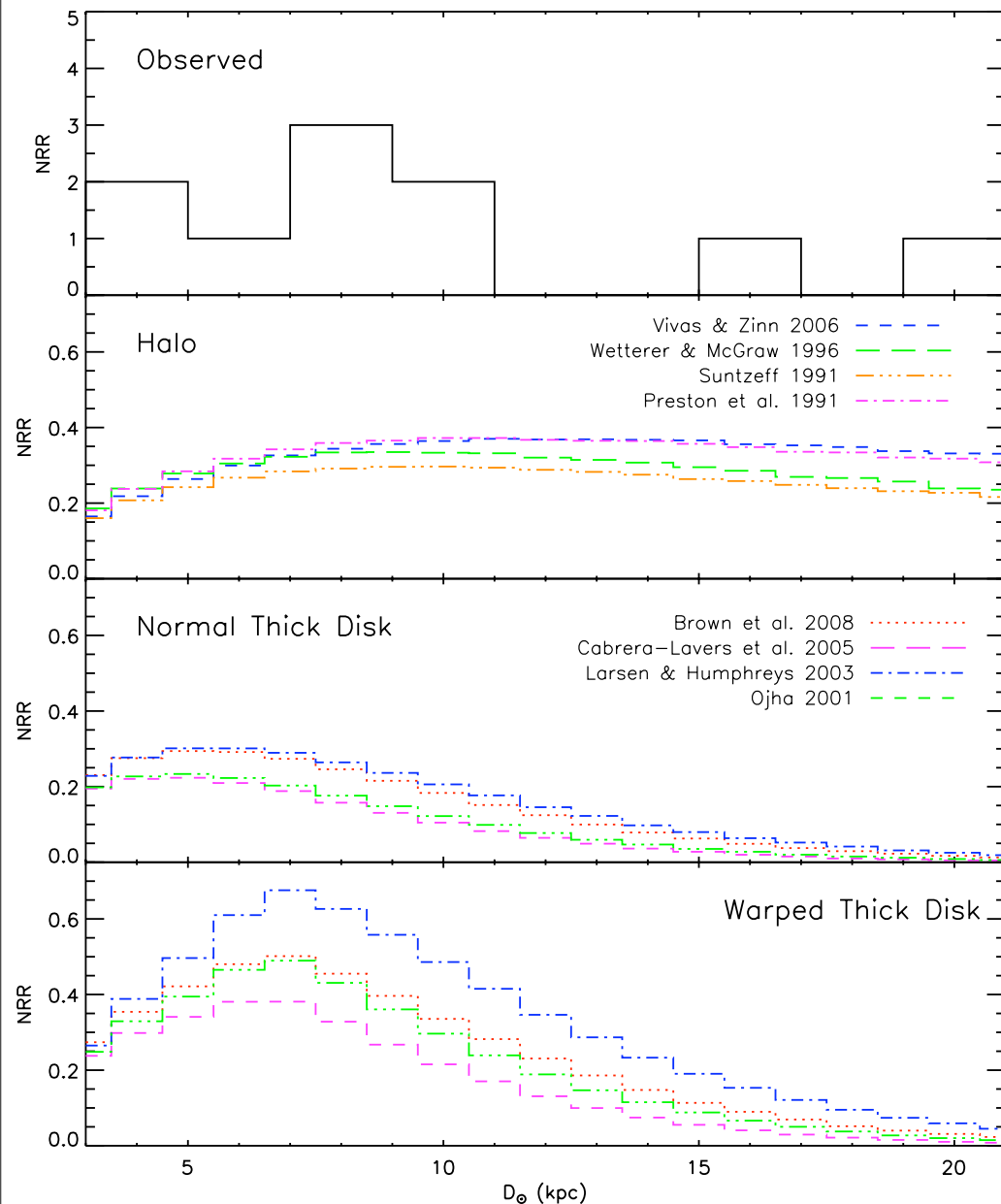
Velocity dispersion
reduced to

$$3.7 \pm 0.9 \text{ km s}^{-1}$$

\Rightarrow factor 2 reduction
in estimated mass

Aden et al. (2009)

The Canis Major controversy



Mateu et al, (2009)

On-going debate about whether this is a disrupted dSph or merely the flared and warped outer disk of the Milky Way

Outer disk is very complicated, therefore it's hard to determine which fields are “symmetric” and hence whether the over-density is real

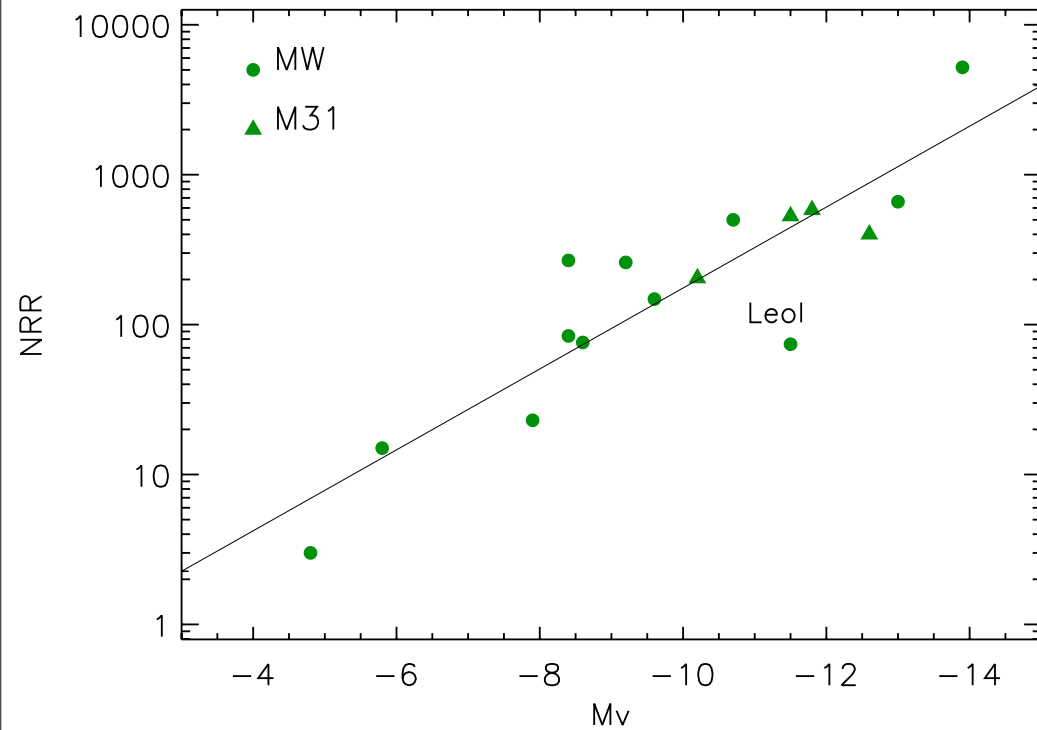
Recent work by Mateu et al. (2009) found no excess of RR Lyrae stars in the over-density \Rightarrow unlike any known dSph

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⇒ unlike any known dSph



Mateu et al, (2009)

Conclusions

Questions to ask *before* believing a mass model

- Was equilibrium assumed? Is this reasonable?
- What size of kinematic sample was used? Sufficient?
- What magnitude are the individual velocity errors - would any claimed kinematic features be resolvable?
- What foreground/background interlopers are in the field? (Remember that halo is a mess)
- Were Jeans equations used? Does the resulting model have a DF that is non-negative everywhere?
- What has been assumed about velocity anisotropy?