Galactic Dynamics and the hunt for dark matter

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Outline

- The collisionless Boltzmann equation
- Jeans equations
- Integrals of motion & Jeans Theorems
  - Application: Tidal streams
- Distribution functions
  - Application: Direct detection predictions
- The Virial Theorem
- Mass estimators - why you should be skeptical
- Distribution functions
  - Application - dark matter in the Milky Way
- Controversial cases: Segue 1, Hercules, Canis Major

Dynamical timescales (when is a sytem "collisionless"?)

Consider a star moving through a galaxy of N stars each of mass m. The deflection force due to a star at perpendicular distance b is given by

$$F_\perp = \frac{Gm^2}{b^2+x^2}\cos\theta \quad = \quad \frac{Gm^2 b}{(b^2+x^2)^{3/2}}$$

$$\simeq \quad \frac{Gm^2}{b^2}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-3/2}$$

Newton's Laws imply

$$\begin{aligned} m\dot{\mathbf{v}}_\perp &= \mathbf{F}_\perp \\ \Longrightarrow \quad |\delta\mathbf{v}_\perp| &\simeq \frac{Gm}{bv}\int_{-\infty}^{\infty}\left(1+s^2\right)^{-3/2}\mathrm{d}s = \frac{2Gm}{bv} \end{aligned}$$

Mean surface density of galaxy is $N/\pi R^2$ so number of perturbations with impact parameter $b \rightarrow b + \delta b$ per crossing is given by

$$\delta n = \frac{N}{\pi R^2} 2\pi \, b \, \mathrm{d}b = \frac{2N}{R^2} \, b \, \mathrm{d}b$$

Mean square velocity change per crossing:

$$\delta v_\perp^2 \simeq \left(\frac{2Gm}{bv} \right)^2 \frac{2N}{R^2} b \, \mathrm{d}b$$

Total velocity change per crossing is

$$\Delta v_\perp^2 \quad \equiv \int_{b_{\min}}^{R} \delta v_\perp^2 \simeq 8N \left(\frac{Gm}{Rv} \right)^2 \ln \Lambda$$

$$\text{where} \qquad \ln \Lambda \quad \equiv \ln \left(\frac{R}{b_{\min}} \right)$$

For self-gravitating system, typical velocity is given by (see later)

$$v^2 \approx \frac{GNm}{R} \implies \frac{\Delta v_\perp^2}{v^2} = \frac{8 \ln \Lambda}{N}$$

So, the number of crossings required to change velocity by of order itself is

$$n_{\mathrm{relax}} = \frac{N}{8 \ln \Lambda}$$

and the time required for this to happen is

$$t_{\mathrm{relax}} = n_{\mathrm{relax}} \, t_{\mathrm{cross}}$$

where $\quad t_{\mathrm{cross}} = \dfrac{R}{v}$

t_{cross} is the **crossing time** and t_{relax} is the **relaxation time**

A system is called **collisionless** if $t_{\rm relax}$ is large (e.g. larger than a Hubble time.

Some systems may be approximated as collisionless if we are only studying them for a time that is much less then their relaxation time.

The evolution of **collisional** systems (e.g. star clusters) is dominated by encounters between the stars

In these two talks, I will be focussing on systems which can be well-approximated as collisionless

The Collisionless Boltzmann equation

Consider a large number of stars moving in a smooth potential $\Phi(\mathbf{x}, t)$. The **distribution function (DF)** or phase-space density gives a complete description of the system at time t.

The number of stars in the volume $\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{v}$ is $f(\mathbf{x}, \mathbf{v}, t)\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{v}$
If the phase-space coordinates are $\mathbf{w} = (\mathbf{x}, \mathbf{v})$ then we have

$$\dot{\mathbf{w}} = (\mathbf{v}, -\nabla\Phi)$$

If we assume a collisionless flow in phase space, this implies a continuity equation

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial (f \dot{w}_\alpha)}{\partial w_\alpha} = 0$$

We also have

$$\sum_{\alpha=1}^{6} \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} = \sum_{i=1}^{3} \left(\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right) = -\sum_{i=1}^{3} \frac{\partial}{\partial v_i} \left(\frac{\partial \Phi}{\partial x_i} \right) = 0$$

This leads directly to the **Collisionless Boltzmann Equation** (CBE; a.k.a. Liouville's Theorem, Vlasov equation)

$$\boxed{\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0}$$

i.e.

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) = 0$$

or

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla \mathbf{f} - \nabla \Phi . \frac{\partial f}{\partial \mathbf{v}} = 0$$

or

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} \implies \frac{\mathrm{d}f}{\mathrm{d}t} = 0$$

Interpretation of the distribution function:

- Think of DF as a probability density, rather than a phase space density.

- To get an observable we integrate over a volume of phase space.

- For example, the z-velocity dispersion of M dwarf stars within 1pc of the sun is

$$\sigma_z^2 = P^{-1} \int \mathrm{d}^3\mathbf{v} \int v_z^2 Q_1(\mathbf{x},\mathbf{v}) f_M(\mathbf{x},\mathbf{v}) \mathrm{d}^3\mathbf{x}$$

where $P = \int \mathrm{d}^3\mathbf{v} \int Q_1(\mathbf{x},\mathbf{v}) f_M(\mathbf{x},\mathbf{v}) \mathrm{d}^3\mathbf{x}$

and $Q_1 = 1$ within 1pc of the Sun, and zero outside.

Jeans equations

Take the zeroth moment of the CBE by integrating over all $\mathrm{d}^3\mathbf{v}$

$$\int \frac{\partial f}{\partial t}\mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i}\mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i}\mathrm{d}^3\mathbf{v} = 0$$

Defining $\nu \equiv \int f \mathrm{d}^3\mathbf{v}$ $\quad \overline{v_i} \equiv \frac{1}{\nu}\int f v_i \mathrm{d}^3\mathbf{v}$

we find $\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \overline{v_i})}{\partial x_i} = 0$

Now take first moment of CBE

$$\frac{\partial}{\partial t}\int f v_j \mathrm{d}^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i}\mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i}\mathrm{d}^3\mathbf{v} = 0$$

from which we obtain $\frac{\partial(\nu \overline{v_j})}{\partial t} + \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} + \nu \frac{\partial \Phi}{\partial x_j} = 0$

where the last term was obtained using

$$\int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = -\int \frac{\partial v_j}{\partial v_i} f \mathrm{d}^3\mathbf{v} = -\int \delta_{ij} f \mathrm{d}^3\mathbf{v} = -\delta_{ij}\nu$$

Combining the first and second equations, we obtain the Jeans equations

$$\boxed{\nu \frac{\partial \overline{v_j}}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i}}$$

where $\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v_i})(v_j - \overline{v_j})} = \overline{v_i v_j} - \overline{v_i}\,\overline{v_i}$

(c.f. Euler equation of fluid flow)

Note: σ_{ij}^2 is symmetric - the principle axes are axes of the velocity ellipsoid

Jeans equations for spherical systems

In spherical polar coordinates, and assuming a steady-state system with $\overline{v_r} = \overline{v_\theta} = 0$, Jeans equations become

$$\frac{\mathrm{d}(\nu \overline{v_r^2})}{\mathrm{d}r} + \frac{\nu}{r}\left[2\overline{v_r^2} - \left(\overline{v_\theta^2} + \overline{v_\phi^2}\right)\right] = -\nu \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

If the density and velocity structures are both invariant under rotation about the centre, we must have

$$\overline{v_\theta^2} = \overline{v_\phi^2}$$

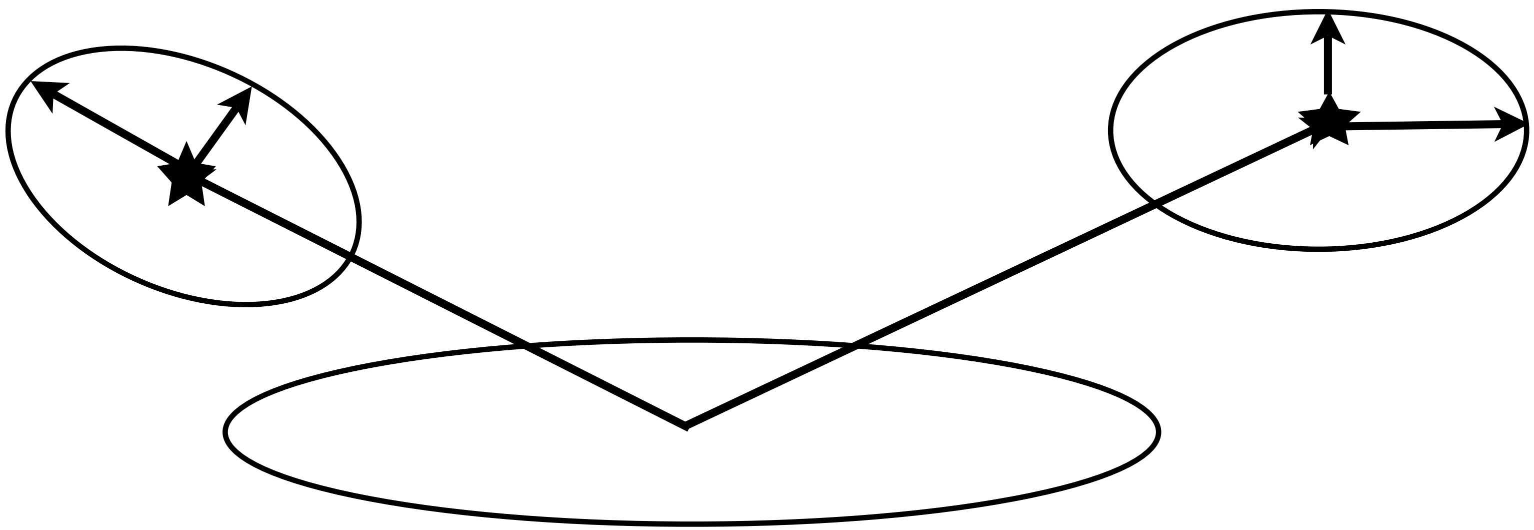
Define the Binney anisotropy parameter: $\beta(r) \equiv 1 - \dfrac{\overline{v_\theta^2(r)}}{\overline{v_r^2(r)}}$

Hence, Jeans equations become

$$\boxed{\frac{1}{\nu(r)} \frac{\mathrm{d}(\nu(r)\overline{v_r^2(r)})}{\mathrm{d}r} + 2\frac{\beta(r)\overline{v_r^2(r)}}{r} = -\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}}$$

Velocity ellipsoid in the Galactic disk

Need to know the orientation of the velocity ellipsoid as a function of position in the disk of the Milky Way in order to measure the mass of the disk, and amount of dark matter in Solar neighbourhood.



Depending on shape of gravitational potential, the velocity ellipsoid can be aligned either in spherical coordinates or cylindrical coordinates.

Assuming that we know the gravitational force $\mathbf{F}$, we can obtain the mass density ρ from Poisson's equation

$$\nabla.\mathbf{F} = -4\pi G\rho$$

Assuming an axisymmetric system this becomes

$$\frac{1}{R}\frac{\partial}{\partial R}(RF_R) + \frac{\partial F_z}{\partial z} = -4\pi G\rho \implies \rho = -\frac{1}{4\pi G}\left(\frac{\partial F_z}{\partial z} - \frac{1}{R}\frac{\partial v_c^2}{\partial R}\right)$$

To proceed we need to know vertical force F_z (a.k.a. K_z).

Jeans equations in cylindrical coordinates tell us that

$$\begin{aligned}\frac{\partial(\nu\overline{v}_z)}{\partial t} + \frac{\partial(\nu\overline{v_R v_z})}{\partial R} \quad &+ \quad \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \frac{\nu\overline{v_R v_z}}{R} + \nu\frac{\partial\Phi}{\partial z} = 0 \\ \implies \nu F_z \quad &= \quad \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \frac{1}{R}\frac{\partial(R\nu\overline{v_R v_z})}{\partial R}\end{aligned}$$

Assuming that we know the gravitational force $\mathbf{F}$, we can obtain the mass density ρ from Poisson's equation

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$$\implies \nu F_z = \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \frac{1}{R}\frac{\partial(R\nu\overline{v_R v_z})}{\partial R}$$

i.e. F_z depends on the "tilt" of the velocity ellipsoid

Integrals of motion

- An integral of motion is a function $I(\mathbf{x}, \mathbf{v})$ which is constant along any orbit, i.e.
$$I[\mathbf{x}(t_1), \mathbf{v}(t_1)] = I[\mathbf{x}(t_2), \mathbf{v}(t_2)]$$
- Examples:
  - Energy is an integral in a static potential
  - L_z is an integral in an axisymmetric potential with symmetry axis z.
  - Three components of angular momentum are integrals in a spherical potential.
- Integrals can be **isolating** or **non-isolating**. Only isolating integrals are useful - they allow orbits to be classified according to the values of their integrals.

Tidal Radii

Consider a satellite of mass m moving on a circular orbit about a galaxy of mass M with constant angular speed Ω. In this steadily rotating potential, the Jacobi integral E_J is an integral of the motion

$$E_J = \frac{1}{2}v^2 + \Phi(\mathbf{x}) - \frac{1}{2}\left|\boldsymbol{\Omega} \times \mathbf{x}\right|^2 = \frac{1}{2}v^2 + \Phi_{\text{eff}}(\mathbf{x})$$

At low E_J, zero velocity surfaces $\Phi(\mathbf{x}) = E_J$ are closed around one or other body. Larger E_J surfaces surround both bodies.

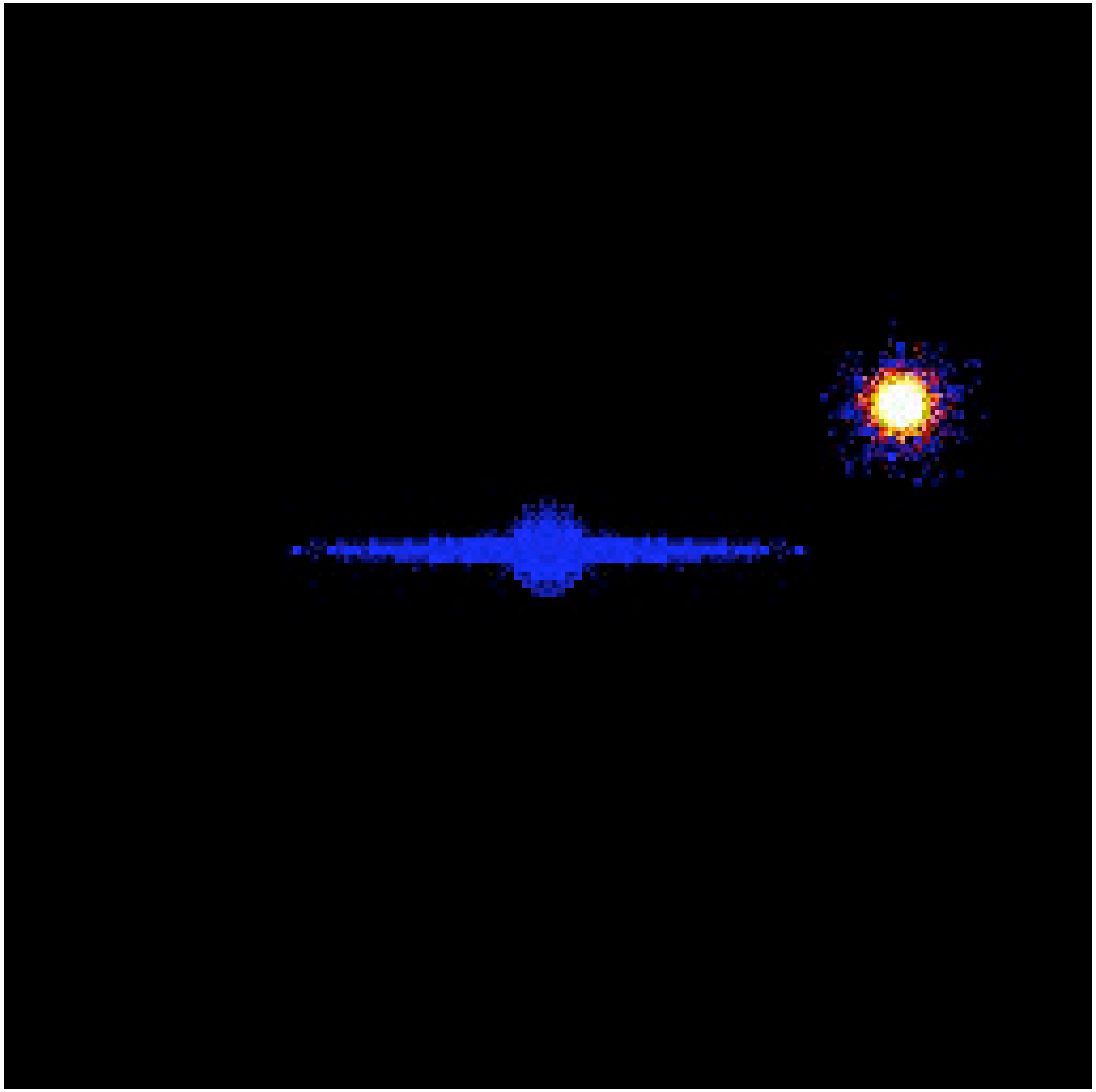
Last closed surface around the satellite is called the **tidal radius**, r_t.

For two point masses a distance D apart, r_t is given by

$$r_t \simeq \left(\frac{m}{3M}\right)^{1/3} D$$

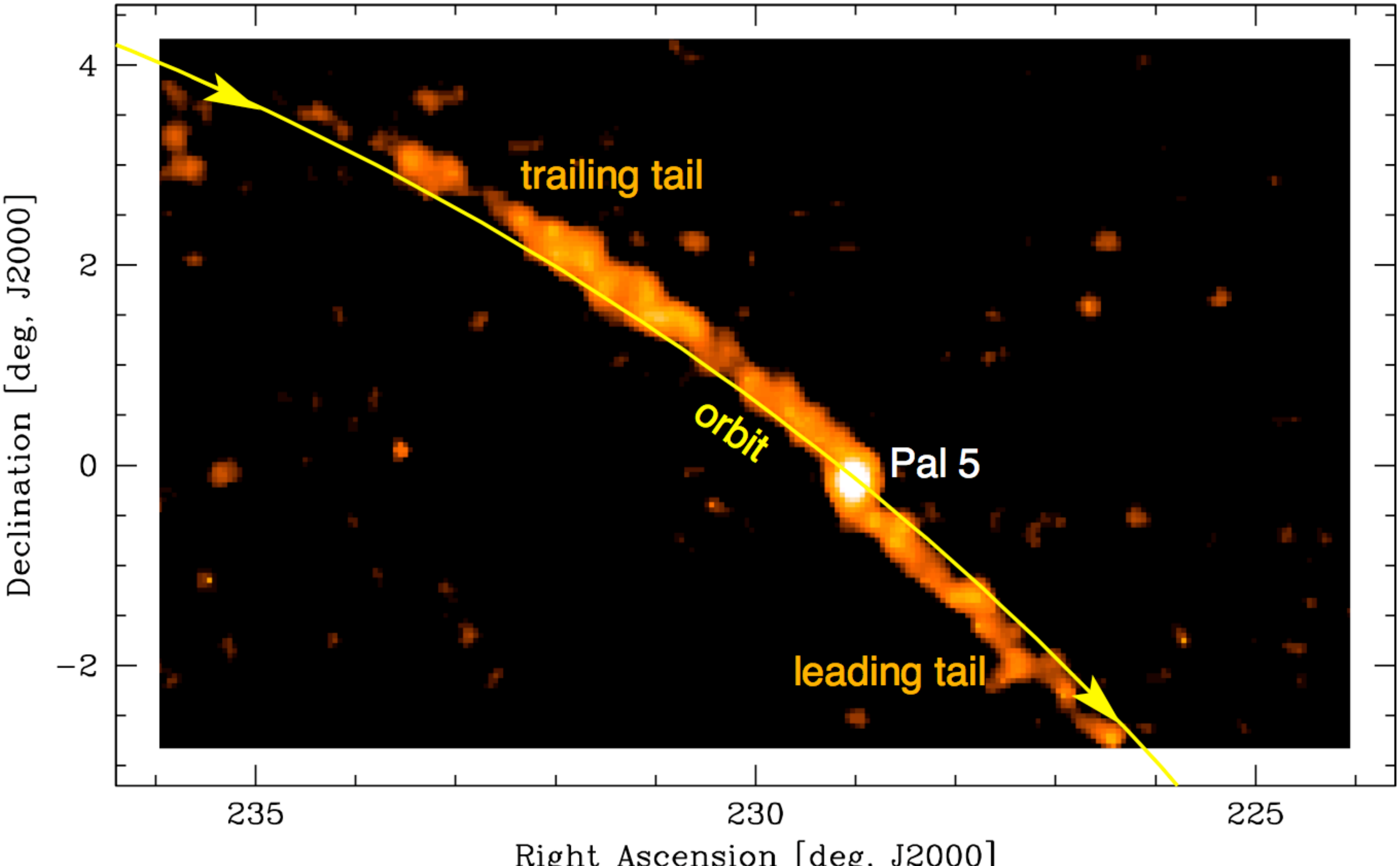
To first order, stars outside this radius will eventually be stripped.

Stellar streams



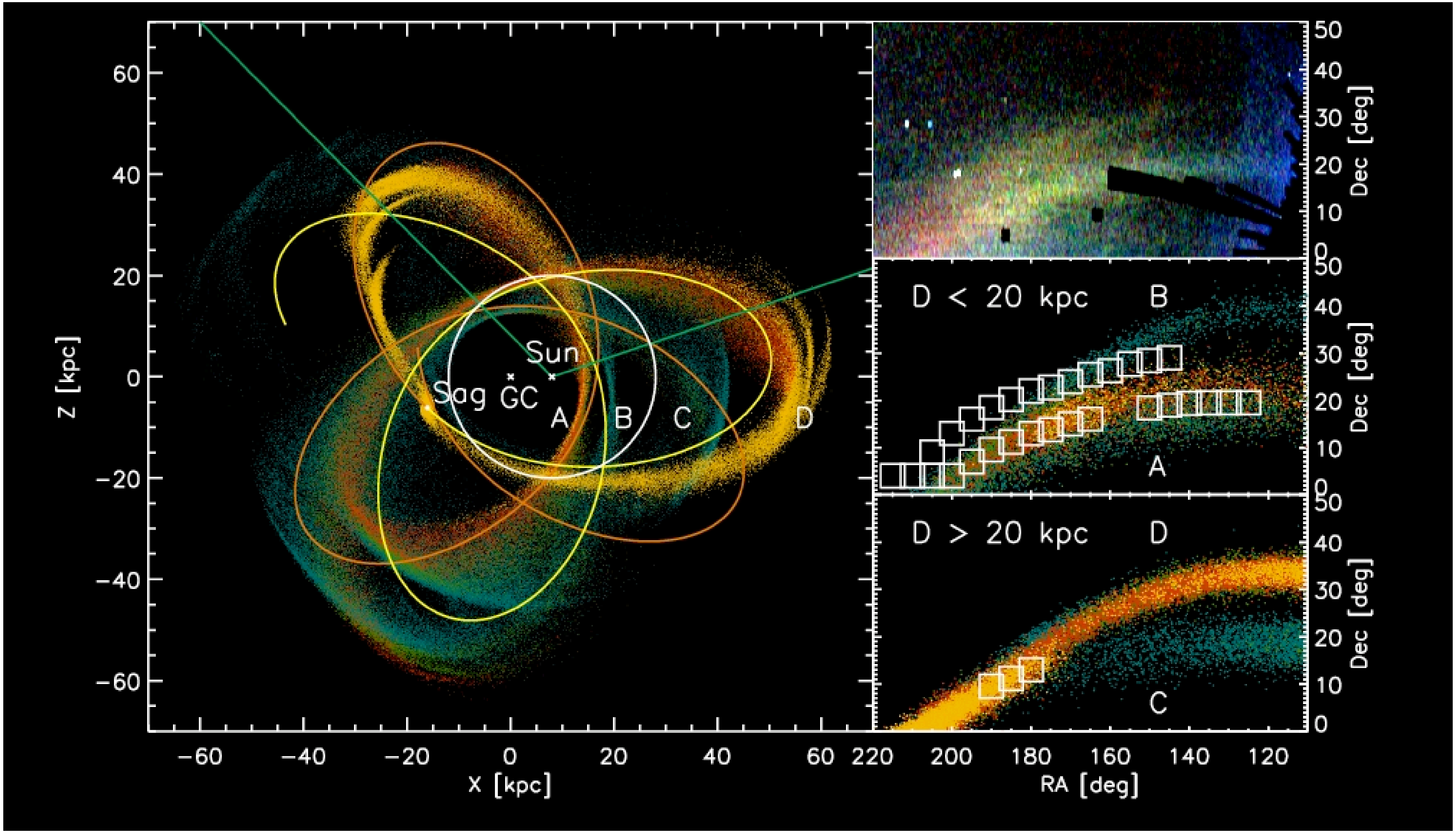
Johnston

Tidal tails of Pal 5



SDSS, 2002

The tails of Sagittarius



Fellhauer et al. (2006)