Non-parametric Characterization of Gravitational-Wave Polarizations

Journée GDR ISIS / GDR OG, Paris
Oct, 8th 2018

J. Flamant[1], P. Chainais [1], E. Chassande-Mottin[2], F. Feng[2], N. Le Bihan[3]

[1] CRIStAL, CNRS, Univ. Lille and Centrale Lille

[2] APC, Université Paris Diderot, CNRS/IN2P3, France

[3] GIPSA-Lab, CNRS, Université Grenoble-Alpes

Context

GW are polarized: 2 orthogonal polarizations h_+ and $h_\times$

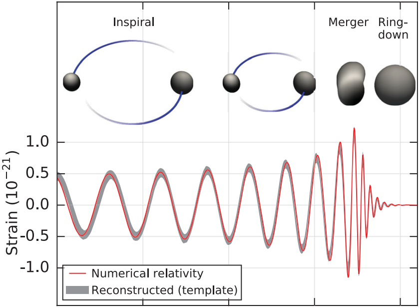
1 detector

$$\alpha_+ h_+(t) + \alpha_\times h_\times(t)$$

≥ 2 GW detectors:

$h_+, h_\times$ reconstruction feasible

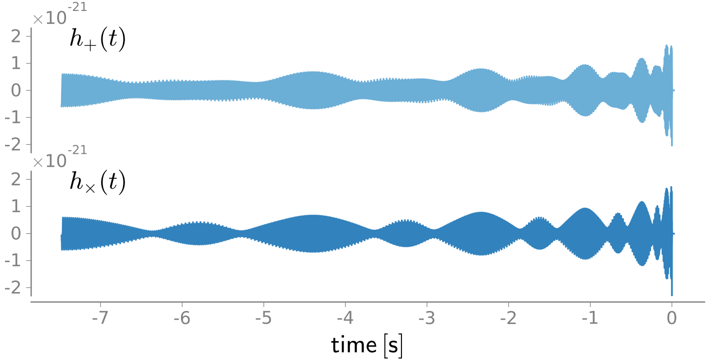
with e.g. sparsity-based algorithms (Feng *et al.*, EUSIPCO 2018)



Assumption: polarizations h_+ and $h_\times$ are available

$\Longrightarrow$ physical/dynamical properties of the GW source

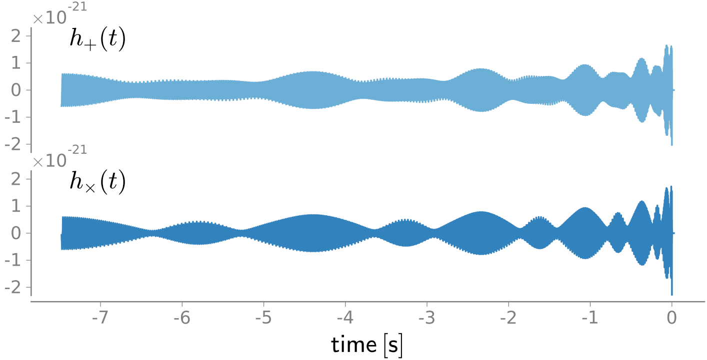
Precessing binary systems



black hole - neutron star binary with misaligned spin

$$M_1 = 10M_\odot \quad M_2 = M_\odot \quad \vec{S}_1 = (-0.7, -0.7, 0) \quad \vec{S}_2 = (0, 0, 0)$$

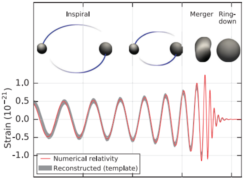
Precessing binary systems

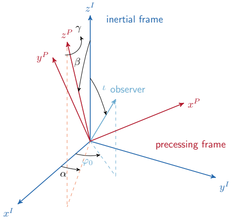


GW strain $h(t) = h_+(t) - \boldsymbol{i} h_\times(t) \equiv$ bivariate signal

Nonparametric GW precession characterization
using recent results on time-frequency analysis of bivariate signals

Modelling GW from precessing systems (I)





(Babak *et al.*, 2016)

observer/inertial I-frame

$$h^I(t), h^I_{\ell,m}(t)$$

precessing P-frame

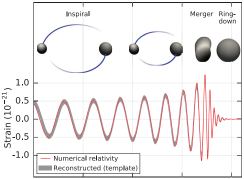
$$h^P(t), h^P_{\ell,m}(t)$$

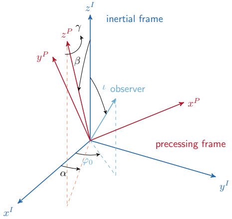
Euler angles α, β, γ Observer (ι, φ_0)

Spherical harmonics expansion

$$h^I(t) = h^I_+(t) - \boldsymbol{i} h^I_\times(t) = \sum_{m=-2}^{2} h^I_{2,m}(t)\,{}_{-2}Y_{2,m}(\iota, \varphi_0)$$

Modelling GW from precessing systems (I)





(Babak *et al.*, 2016)

observer/inertial I-frame

$$h^I(t), h^I_{\ell,m}(t)$$

precessing P-frame

$$h^P(t), h^P_{\ell,m}(t)$$

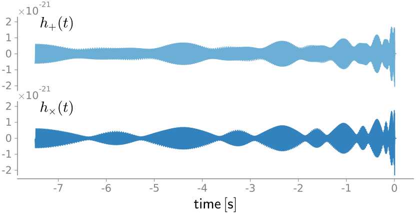
key point: P-frame waveform $\simeq$ non-precessing-like GW

I-frame spherical harmonic coefficients

$$h^I_{\ell,m} = \sum_{m'=-\ell}^{\ell} h^P_{\ell,m'} \underbrace{D^{\ell,*}_{m'm}(-\gamma,-\beta,-\alpha)}_{\text{Wigner-D functions}}$$

Modelling GW from precessing systems (II)

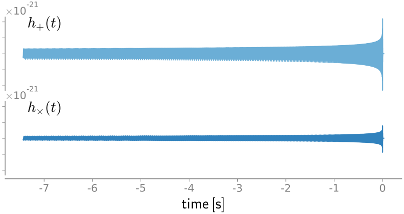
precession $\alpha(t), \beta(t), \gamma(t)$ arbitrary



$$h^I(t) = h^I(t, \alpha, \beta, \gamma, \iota, \varphi_0)$$

precession dynamics + observer

no precession $\alpha(t) = \beta(t) = \gamma(t) = 0$



$$h^I(t) = h^I(t, \iota, \varphi_0)$$

observer only

Quaternion embedding of bivariate signals

GW strain $h(t) = h_+(t) - \boldsymbol{i}h_\times(t) \equiv$ bivariate signal

generalization of the analytic signal to bivariate signals[1]

Quaternion embedding of a bivariate signal

$$h_{\mathbb{H}}(t) = \underbrace{(\text{original signal } h(t))}_{\in\text{span}\{1,\boldsymbol{i}\}} + \underbrace{(\text{Hilbert transform})\boldsymbol{j}}_{\in\text{span}\{\boldsymbol{j},\boldsymbol{k}\}}$$

4D algebra $\mathbb{H}$ $\quad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1 \quad$ ⚠ $\boldsymbol{ij} = \boldsymbol{k},\ \boldsymbol{ij} = -\boldsymbol{ji}$ ⚠

relies on a tailored quaternion Fourier transform (QFT)

One-to-one correspondence

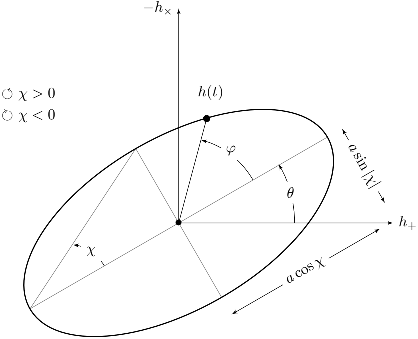
$$h(t) \in \mathbb{C} \longleftrightarrow h_{\mathbb{H}}(t) \in \mathbb{H}$$

[1] J. Flamant, N. Le Bihan, P. Chainais (2017). Time–frequency analysis of bivariate signals. *Appl. Comp. Harm. Anal.*

Quaternion embedding and interpretation

Euler polar form

$$h_{\mathbb{H}}(t) = a(t)e^{\boldsymbol{i}\theta(t)}e^{-\boldsymbol{k}\chi(t)}e^{\boldsymbol{j}\varphi(t)}$$



usual instantaneous features

$a(t) \geq 0$ amplitude

$\varphi(t) \in [0, 2\pi)$ phase

$+$

instantaneous polarization

$\theta(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ orientation

$\chi(t) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ ellipticity

From quaternion embedding to precession

Identification using $h_{\mathbb{H}}(t)$

$$\begin{bmatrix} a(t) \\ \theta(t) \\ \chi(t) \\ \varphi(t) \end{bmatrix} = f\left(\begin{bmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \\ \iota \\ \varphi_0 \end{bmatrix}\right)$$

$$\theta(t), \chi(t) \Longleftrightarrow \alpha(t), \beta(t), \gamma(t)$$

polarization modulation $\Longleftrightarrow$ precession

f mapping is explicit (yet tedious)

Instantaneous Stokes parameters

✗ numerical instability of θ, χ, φ direct estimates

Instantaneous Stokes parameters

Common description of polarization properties in optics

$$\begin{aligned} S_0(t) &= |a(t)|^2 \\ S_1(t) &= |a(t)|^2 \cos 2\chi(t) \cos 2\theta(t), \\ S_2(t) &= |a(t)|^2 \cos 2\chi(t) \sin 2\theta(t), \\ S_3(t) &= |a(t)|^2 \sin 2\chi(t). \end{aligned}$$

Basic quaternion calculus shows that:

$$|h_{\mathbb{H}}(t)|^2 = S_0(t), \quad h_{\mathbb{H}}(t)\boldsymbol{j}\overline{h_{\mathbb{H}}(t)} = \boldsymbol{i}S_3(t) + \boldsymbol{j}S_1(t) + \boldsymbol{k}S_2(t)$$

From quaternion embedding to precession

Identification using $h_{\mathbb{H}}(t) = h(t) + \mathcal{H}h(t)\boldsymbol{j}$

$$\begin{bmatrix} S_0(t) \\ S_1(t) \\ S_2(t) \\ S_3(t) \end{bmatrix} = g\left(\begin{bmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \\ \iota \\ \varphi_0 \end{bmatrix}\right)$$

$$\frac{S_1(t)}{S_0(t)}, \frac{S_2(t)}{S_0(t)}, \frac{S_3(t)}{S_0(t)} \Longleftrightarrow \alpha(t), \beta(t), \gamma(t)$$

polarization modulation $\Longleftrightarrow$ precession

g mapping is explicit (yet still tedious)

From quaternion embedding to precession

Identification using $h_{\mathbb{H}}(t) = h(t) + \mathcal{H}h(t)\boldsymbol{j}$

$$\begin{bmatrix} S_0(t) \\ S_1(t) \\ S_2(t) \\ S_3(t) \end{bmatrix} = g\left(\begin{bmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \\ \iota \\ \varphi_0 \end{bmatrix}\right)$$

$$\frac{S_1(t)}{S_0(t)}, \frac{S_2(t)}{S_0(t)}, \frac{S_3(t)}{S_0(t)} \Longleftrightarrow \alpha(t), \beta(t), \gamma(t)$$

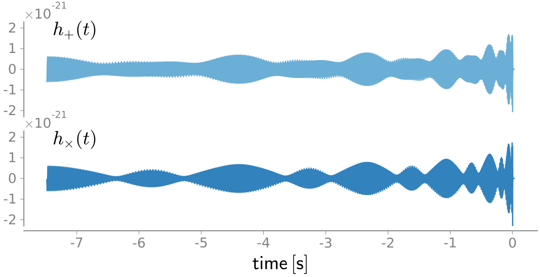
polarization modulation $\Longleftrightarrow$ precession

g mapping is explicit (yet still tedious)

$$\begin{aligned} S_3 = a_0^2 \frac{1}{64}\frac{5}{\pi}\Big\{ & 2\left(-2\left(\cos^2(\beta)+1\right)\sin(2\alpha-2\varphi_0)\cos(\iota) - 2\sin(2\beta)\sin(\iota)\sin(\alpha-\varphi_0)\right) \\ & \times\left(2\left(\cos^2(\iota)+1\right)\sin(2\alpha-2\varphi_0)\cos(\beta) + 2\sin(\beta)\sin(2\iota)\sin(\alpha-\varphi_0)\right) \\ & -2\left(4\sin(\beta)\sin(\iota)\cos(\alpha-\varphi_0) + 4\cos(\beta)\cos(\iota)\cos(2\alpha-2\varphi_0)\right) \\ & \times\left(\left(\cos^2(\beta)+1\right)\left(\cos^2(\iota)+1\right)\cos(2\alpha-2\varphi_0) + 3\sin^2(\beta)\sin^2(\iota) + \sin(2\beta)\sin(2\iota)\cos(\alpha-\varphi_0)\right)\Big\} \end{aligned}$$

Precession diagnostic – simulated data

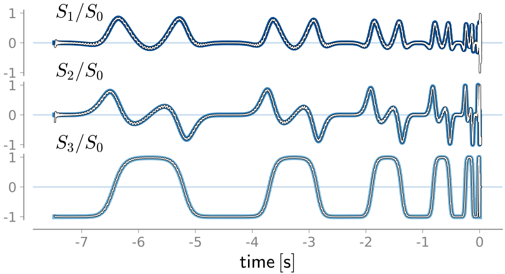
SEOBNRv3 simulation of strongly precessing BH-NS binary



precession dynamics

$\updownarrow$

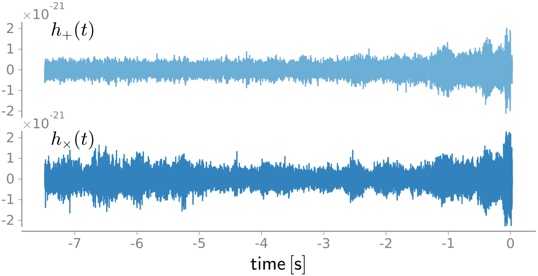
polarization modulation



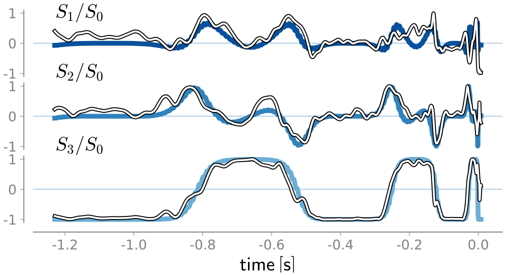
Stokes parameters estimated using $h_{\mathbb{H}}(t)$

Precession diagnostic – realistic simulated noise

reconstruction from realistic noisy LIGO-Virgo observations



h_+, $h_\times$ reconstructed using a sparsity promoting algorithm (LASSO)
(Feng *et al.*, 2018)



residual noise
↓
Stokes parameters estimated from the ridge of a quaternion CWT
(Flamant *et al.*, 2017)

Summary

1. precessing compact binary system
2. observation using ≥ 2 detectors e.g. h_1, h_2
3. reconstruction of $(h_+, h_\times) \leftarrow (h_1, h_2)$ (Feng *et al.*, 2018)
4. emission model using spherical harmonics I-frame $\overset{\alpha,\beta,\gamma}{\longleftrightarrow}$ P-frame
5. analysis of the bivariate signal $h(t) = h_+(t) - \boldsymbol{i} h_\times(t)$

   quaternion embedding $h_{\mathbb{H}}(t)$

   $\Downarrow$

   $(a, \varphi, \theta, \chi)$ and Stokes parameters
6. **Stokes parameters $\longleftrightarrow$ decipher binary precession**

Conclusion

GW strain $h(t) = h_+(t) - \boldsymbol{i}h_\times(t) \equiv$ bivariate signal

Nonparametric GW precession characterization
using recent results on time-frequency analysis of bivariate signals

Features

- instantaneous Stokes parameters as robust observables
- explicit relations with precession parameters (Euler angles)
- no hypothesis on precession dynamics
- can be refined (e.g., higher ℓ modes)
- deciphering of any effect impacting polarization

next: analysis of a real event

Time-frequency representations of bivariate signals

generalization for multicomponent/noisy bivariate signals

Example: Quaternion Short-Term Fourier Transform

Extend the STFT to the QFT setting

$$Sh(t,\nu) = \int h(u) \underbrace{g(u-t)}_{\text{window}} \underbrace{\exp(-\boldsymbol{j}2\pi\nu u)}_{\text{QFT kernel}} \mathrm{d}u$$

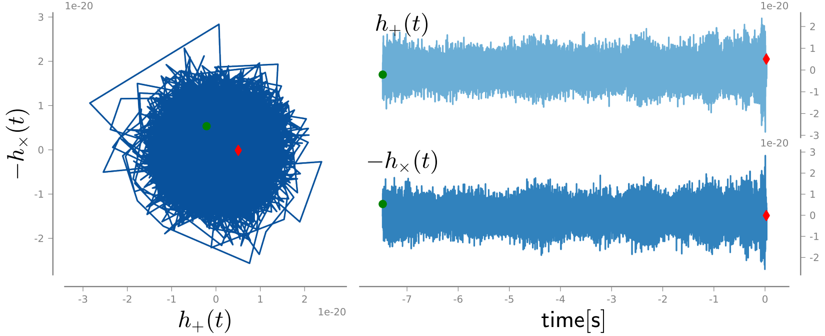
$$|Sh(t,\nu)|^2 \rightarrow \text{Time-frequency energy density}$$

$$Sh(t,\nu)\boldsymbol{j}\overline{Sh(t,\nu)} \rightarrow \text{Time-frequency Stokes parameters } S_1, S_2, S_3$$

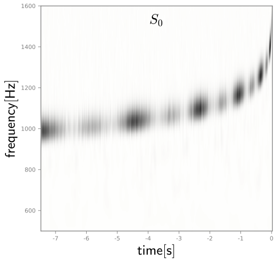
Theorems
- inversion
- conservation: energy polarization

Generic framework for TF analysis with the QFT: spectrograms, scalograms, ...

Time-frequency analysis of precessing GW



TF energy density



Time-frequency analysis of precessing GW