Few-Body Theory
Universal aspects in low-energy few-body systems

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Efimov Physics: Halo Nuclei and Cold Atoms

Universal aspects in low-energy few-body systems

- Efimov Physics - Brazilian few-body group contributions
  - Scaling properties of few-body systems (bound to scattering)
  - Nuclear Physics: halo-nuclei $n-n-$core systems
  - Atomic Physics: Cold-atom studies with two-kind of atoms
- Recent studies on halo-nuclei scattering: The neutron$-{}^{19}$C
  - Motivation based on related works in scattering with three-body systems and the Efimov states
  - Scattering observables: $k\cot\delta_0$ and cross-section σ.
  - Pole-positions of $k\cot\delta_0$ as a function of the E_{nc} bound-states.

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Universal scaling of scattering observables for mass-imbalanced three-body systems

Main motivations for such theoretical studies are the present experimental possibilities
(i) in nuclear physics, by considering halo-nuclei systems; and
(ii) in cold-atom laboratories when considering mixed atomic species.

In both the cases, our interest is to verify universal properties of scattering observables for three-body systems near the unitary limit.

Examples we are considering:

Halo-nuclei: $n \to (n^{18}\mathrm{C})$
Ultracold atoms: The $\alpha \to (\alpha\beta)$ **system, when** $m_\alpha \gg m_\beta$

Ultracold atoms: The $\alpha - (\alpha\beta)$ system, where $m_\alpha \gg m_\beta$

- The Efimov scaling factor of three-body system near the unitary limit, can be well identified in scattering observables of one atomic species α when colliding with a two-body $\alpha\beta$ bound-state system.
- In the unitary limit, two levels are related by an exponential scaling factor $\exp(2\pi/s_0)$, where s_0 is a constant that varies according to the mass-ratio m_H/m_L. For $m_H = m_L$, the energy-ratio is predicted to be ~ 515, such that it will be quite difficult for an experimental verification.
- As the scaling behavior is better verified for large imbalance-mass systems, the results can be relevant for the going-on experimental observations of Efimov physics in cold-atom laboratories.
- In case of $m_H = 100 m_L$, the ratio between consecutive levels of the bound-state energy spectrum is given by $\exp(2\pi/s_0) \sim 4.7$.

Motivations from ultracold atom laboratories

- Going to scattering region, the discrete Efimov scaling factor can also be well identified in scattering observables of one atomic species α when colliding with a two-body $\alpha\beta$ bound-state.
- Heidelberg group is studying the extreme mass-imbalance mixtures composed by ^{133}Cs and ^{6}Li atomic species [J. Ulmanis et al, PRL **117** (2016) 153201].
- Ultracold degenerate mixtures of alkali-metal-rare-earth molecules, ^{174,173}Yb$-$^{6}Li have also been considered by H. Hara et al [PRL **106** (2011) 205304] and Hansen et al [PRA **84** (2011) 011606(R)].
- Therefore, we understand that more favorable conditions are accessible to probe the rich Efimov physics in cold-atom laboratories. with low-energy collision of a heavy atom in a weakly-bound molecule as LiCs or LiYb.
- Note that, for the above mentioned examples, we have mass-ratios as m_L/m_H =0.034 for LiYb and 0.045 for LiCs

Universality in weakly-bound 3-body systems: Efimov physics

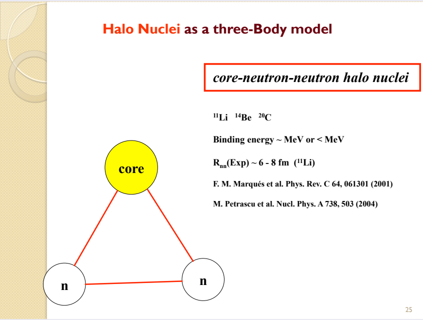
Efimov Effect: First predicted in 1970 by Vitaly Efimov (then at the Ioffe Physico-Technical Institute, St. Petersburg, Russia), when solving the quantum mechanics three-boson equation, Efimov effect is an apparent counter-intuitive phenomenon.

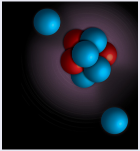
If two bosons interact in such a way that a two-body bound state is exactly on the verge of being formed, then in a three-boson system one should observe an infinite number of bound states. This phenomenon was shown that does not occur for less than three dimensions.

If one would be able to change the interaction strength, by making it either weaker or stronger, the number of three-body bound states would become finite.

The phenomenon is part of some general universal behavior of quantum few-body systems.

Why the actual interest?
Atomic physicists learned how to manipulate the interaction strength between atoms. Following that, several experimental groups obtain strong indications of three-body states as predicted by Efimov.





Properties of halo nuclei

- Weak interaction between the core and the halo-nucleons, such that we can consider a halo nuclei as three non-identical particle (n — n—core) system, and neglect the structure of the core.
- This three-body system has large two-body scattering lengths in comparison of the range of the interactions, suggesting the three-body energies are Efimov states.
- The radius of this kind of nuclei is much more than the expected value: $R = r_0 A^{1/3}$

Efimov states in Halo Nuclei

Fedorov and Jensen, Phys. Rev. Lett. 25 (1993) 4103

Fedorov, Jensen, and Riisager, Phys. Rev. Lett. 73 (1994) 2817

Dasgupta, Mazumdar, and Bhasin, Phys. Rev. C 50 (1994) R550

VOLUME 73, NUMBER 21 PHYSICAL REVIEW LETTERS 21 NOVEMBER 1994

Efimov States in Halo Nuclei

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(Received 13 April 1994)

We investigate conditions for the occurrence of Efimov states in the recently discovered halo nuclei. These states could appear in systems where one neutron is added to a pronounced one-neutron halo nucleus. Detailed calculations of the properties of the states are presented and promising candidates are discussed.

RAPID COMMUNICATIONS

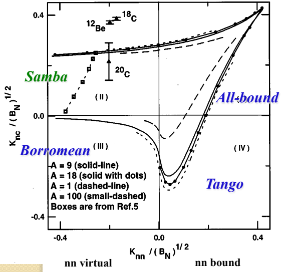
PHYSICAL REVIEW C VOLUME 56, NUMBER 5 NOVEMBER 1997

Universal aspects of Efimov states and light halo nuclei (*)

A. E. A. Amorim,[1,2] T. Frederico,[3] and Lauro Tomio[1]

The parametric region in the plane defined by the ratios of the energies of the subsystems and the three-body ground state, in which Efimov states can exist, is determined. We use a renormalizable model that guarantees the general validity of our results in the context of short-range interactions. The experimental data for one- and two-neutron separation energies, implies that among the halo nuclei candidates, only ^{20}C has a possible Efimov state, with an estimated energy less than 14 KeV below the scattering threshold. [S0556-2813(97)50611-X]

Threshold conditions for an excited N+1 Efimov state

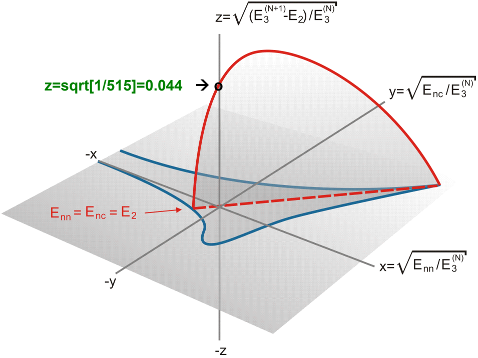


$K_2 = (E_2)^{1/2}$ (where 2 refers to n-n or n-c) and B_N is the N-th 3body bound state.
Here, it is shown the boundary of the region where the binding energy of the (N+1)th Efimov state is zero for different core masses (A).
Negative values for the two-body observables correspond to virtual states.
It is also shown three experimental data, corresponding to the halo nuclei ^{20}C, ^{18}C, ^{12}Be [Audi and Wapstra, NPA595(1995)409].
The squares, connected with dashed lines, are obtained from Fig. 2 of Fedorov et al. PRL73(1994)2817.

See also Canham & Hammer "Universal properties and structure of halo nuclei", Eur. Phys. J. A **37**, 367 (2008)

(*) Work done as part of the PhD thesis of Amorim, from 1995 and 1996, submitted for publ. in 1996.

Threshold conditions and Scaling function



$$\frac{B_3^{(N+1)}}{B_3^{(N)}} = F\left(\frac{K_{\alpha\alpha}}{\sqrt{B_3^{(N)}}};\frac{K_{\alpha\beta}}{\sqrt{B_3^{(N)}}};A\right),$$

$$A \equiv M_\beta / M_\alpha$$

Exploring the Efimov physics: From bound to scattering

- Consider the general case of three-boson system with non-identical masses such that $m_\alpha \gg m_\beta$, when the two-body scattering length is close to infinite.
- Two levels of the 3-body spectrum are related by a scaling factor: $\exp{(2\pi/s_0)}$, where s_0 is a constant that varies according to the mass-ratio [See Braaten and Hammer, Phys. Rep. 428 (2006) 259].
- The maximum energy-ratio occurs for identical masses, predicted to be ~ 515.
- When $m_\alpha = 100 m_\beta$, $\exp{(2\pi/s_0)} \sim 4.7$, with particular interest for experiments with cold atoms.
  (We have also defined $m_H \equiv m_\alpha$ and $m_L \equiv m_\beta$, for atomic systems.)

Equal-mass case: Pole in $k\cot\delta$ for the $n-d$ system

- W.T.H. van Oers and J.D. Seagrave, Phys. Lett. B **24** (1967) 562.
  By analyzing data for $n - d$ scattering, they pointed out a pole in $k\cot\delta$.
- A.S. Reiner, Phys. Lett. B **28** (1969) 387.
  The anomalous effective range expansion of the doublet $n - d$ elastic scattering is due to a pole just below the threshold.
- J.S. Whiting and M.G. Fuda, Phys. Rev. C **14** (1976) 18.
  The pole position and residue was obtained from dispersion relation and exact solution of 3B equations.
- B.A. Girard and M.G. Fuda, Phys. Rev. C **19** (1979) 579.
  The existence of the triton virtual state was found on the basis of the effective range expansion.

Pole (singularity) in kcot(δ)

PHYSICAL REVIEW VOLUME 118, NUMBER 5 JUNE 1, 1960

Low-Energy Photodisintegration of H^3 and He^3 and Neutron-Deuteron Scattering*

L. M. Delves†
Clarendon Laboratory, Oxford, England
(Received December 11, 1959; revised manuscript received January 4, 1960)

the experimental data agree well with a scattering length of 8.3 f, but are consistent also with $a_0=0.8$ f, if, as is plausible in this case, $k\cot\delta$ has a pole between zero energy and the triton bound state. Moreover, assuming $a_0=8.3$ leads to an effective range of 2.3

W. T. H. van Oers, J. D. Seagrave, Phys.Lett. 24B(1967)*562*

The neutron-deuteron effective range expansion - pole in kcot(δ)

For the n-d ^2S phase shifts it is found that the effective range expansion has a pole at a negative energy (E_{lab}= −0.1 MeV).

In the n-d doublet s-wave elastic scattering, the energy dependence of the phase-shift has to be changed from a simple effective range formula to a more complex one.

$$k\cot\delta_0 = -A + Bk^2 - \frac{C}{1+Dk^2},$$

PHYSICAL REVIEW C VOLUME 14, NUMBER 1 JULY 1976

Pole in $k\cot\delta$ for doublet, s-wave, n-d scattering

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(Received 23 March 1976)

The position of the pole in $k\cot\delta$, for doublet, s-wave, n-d scattering, and its residue are shown to be correlated with the doublet scattering length. An approximate, analytic solution of the N/D equations of Barton and Phillips indicates a linear dependence on the doublet scattering length for the pole position, and a ... solutions of ... nd to be ... for $k\cot\delta$, ... formalism is ... nd cut in an

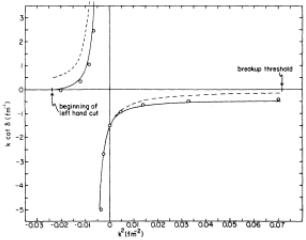


FIG. 3. $k\cot\delta$ for the doublet, s state as a function of k^2, where k is the relative wave number. Solid lines are from the exact N/D calculations. Dashed lines are from Eq. (25). Circles are for either separable potential. In all cases the doublet scattering length is 0.652 fm.

Unbalanced-mass case: Pole in $kcot\delta$ for the halo-nuclei system

- M.T. Yamashita, T. Frederico and LT, Phys. Rev. Lett. **99** (2007) 269201; Phys. Lett. B **660** (2008) 339.
  Shows the migration of the excited Efimov state of $n - n -^{18} C$ to the virtual state, as well as performed calculations on the $n -^{19} C$ scattering within zero-range interaction.
- M.A. Shalchi, M.T. Yamashita, M.R. Hadizadeh, T. Frederico, LT, Phys. Lett. B **764** (2017) 196. Neutron$-^{19}$C scattering: Emergence of universal properties in a finite-range potential; Phys. Lett. B **771** (2017) 635 (Erratum).
- A. Deltuva, Phys. Lett. B **772** (2017) 657 Neutron$-^{19}$C scattering: Towards including realistic interactions.

Formalism

The amplitude of on–shell scattering of *n* from *nc* target;

$$h_n(q,E) = \mathcal{V}(q,k,E) + \frac{2}{\pi}\int dq'\, q'^2\, \frac{\mathcal{V}(q,q',E)h_n(q',E)}{q'^2 - k^2 - \mathrm{i}\epsilon}$$

- q: the momentum of the spectator particle (n) with respect to the CM of the $(n-c)$ subsystem.
- the on-energy-shell: $k \equiv |\vec{k}_i| = |\vec{k}_f| = \sqrt{[2(A+1)m/(A+2)]\,(E - E_{nc})}$

where

$$\mathcal{V}(q,q',E) = \frac{\pi}{2}\bar{\tau}_{nc}(q) \times \left[K_2(q,q',E) + \int dk\, k^2\, K_1(q,k,E)\tau_{nn}(k)K_1(q',k,E)\right]$$

For both $n-n$ and $n-c$ two-body interactions, we use one–term separable Yamaguchi-type potentials:

$$V(p,p') = \lambda \left(\frac{1}{p^2+\beta^2} \right) \left(\frac{1}{p'^2+\beta^2} \right),$$

where

$$\lambda = \frac{-2\pi\mu}{\beta(\beta \pm \kappa)},$$

and the range of the interaction

$$r_0 = \frac{1}{\beta} + \frac{2\beta}{(\beta \pm \kappa)^2},$$

$\bar{\tau}_{nc}$ and τ_{nn} (reflecting 2B t–matrices):

$$\bar{\tau}_{nc}(q) = \frac{-\beta_{nc}(\beta_{nc}+\kappa_{nc})^2(\beta_{nc}+\kappa_{3nc})^2(\kappa_{nc}+\kappa_{3nc})(A+1)^2}{\mu_{nc}\pi(2\beta_{nc}+\kappa_{3nc}+\kappa_{nc})A(A+2)}$$

$$\tau_{nn}(q) = \frac{2\beta_{nn}}{\mu_{nn}\pi}\frac{(\beta_{nn}+\kappa_{nn})^2(\beta_{nn}+\kappa_{3nn})^2}{(-2\beta_{nn}-\kappa_{3nn}+\kappa_{nn})(\kappa_{nn}+\kappa_{3nn})},$$

where:

$$\begin{aligned}
\kappa_{nn} &= \sqrt{-mE_{nn}} \\
\kappa_{nc} &= \sqrt{-\frac{2mA}{A+1}E_{nc}} \\
\kappa_{3nn} &= \sqrt{-m(E-\frac{(A+2)q^2}{4Am})} \\
\kappa_{3nc} &= \sqrt{-\frac{2mA}{A+1}(E-\frac{(A+2)q^2}{2(A+1)m})}.
\end{aligned}$$

K_1 and K_2 functions:

$$K_1(q,q',E) = \int dx \frac{1}{E - \frac{q^2}{m} + \frac{q'^2(A+1)}{2Am} + \frac{qq'x}{m}}$$
$$\times \left(q^2 + \frac{q'^2}{4} + qq'x + \beta_{nn}^2\right)^{-1} \left(q'^2 + \frac{q^2A^2}{(A+1)^2} + \frac{2qq'Ax}{(A+1)} + \beta_{nc}^2\right)^{-1}$$

$$K_2(q,q',E) = \int dx \frac{1}{E - \frac{q^2(A+1)}{2Am} - \frac{q'^2(A+1)}{2Am} + \frac{qq'x}{Am}}$$
$$\times \left(q'^2 + \frac{q^2}{(A+1)^2} + \frac{2qq'x}{(A+1)} + \beta_{nc}^2\right)^{-1} \left(q^2 + \frac{q'^2}{(A+1)^2} + \frac{2qq'x}{(A+1)} + \beta_{nc}^2\right)^{-1}$$

Handling the singularities (in case of zero-range interactions) by a subtraction renormalization approach:

$$\Gamma_n(q,k;E) = \mathcal{V}(q,k;E) + \frac{2}{\pi}\int_0^\infty dp \left[p^2\mathcal{V}(q,p;E) - k^2\mathcal{V}(q,k;E)\right]\frac{\Gamma_n(p,k;E)}{p^2-k^2},$$

$$h_n(q;E) = \frac{\Gamma_n(q,k;E)}{1 - \frac{2}{\pi}k^2\int_0^\infty dp\,\frac{\Gamma_n(p,k;E)}{p^2-k^2-\mathrm{i}\epsilon}}.$$

On–shell scattering amplitude:

$$h_n(k;E) = [k\cot\delta_0 - \mathrm{i}k]^{-1},$$

where

$$k\cot\delta_0 = \frac{1}{\Gamma_n(k,k;E)}\left[1 - \frac{2}{\pi}k^2\int_0^\infty dp\,\frac{\Gamma_n(p,k;E)-\Gamma_n(k,k;E)}{p^2-k^2}\right].$$

Scattering differential cross section:

$$\frac{d\sigma}{d\Omega} = |h_n(k;E)|^2$$

Calculation of bound and virtual-states:

$$h_{nc}(q) = (q^2 - k_i^2)\chi_n(q) \quad , \quad h_{nn}(q) = \chi_c(q)$$

We have:

$$\begin{aligned} h_{nc}(q) &= \bar{\tau}_{nc}(q) \int dq' \, q'^2 \left(k_2 (q, q', E) \frac{h_{nc}(q')}{q'^2 - k_i^2 + i\epsilon} + k_1 (q, q', E) h_{nn}(q') \right) \\ h_{nn}(q) &= \tau_{nn}(q) \int dq' \, q'^2 \, k_1 (q', q, E) \frac{h_{nc}(q')}{q'^2 - k_i^2 + i\epsilon}, \end{aligned}$$

By going to the second sheet of the complex energy:

$$\begin{aligned} h_{nc}(q) &= \bar{\tau}_{nc}(q)[\pi k_v k_2 (q, -ik_v, E) h_{nc}(-ik_v) \\ &+ \int dq' \, q'^2 \left(k_2 (q, q', E) \frac{h_{nc}(q')}{q'^2 + k_v^2} + k_1 (q, q', E) h_{nn}(q') \right)] \\ h_{nn}(q) &= \tau_{nn}(q) \left[\pi k_v k_1 (-ik_v, q, E) h_{nc}(-ik_v) + \int dq' \, q'^2 \, k_1 (q', q, E) \frac{h_{nc}(q')}{q'^2 + k_v^2} \right], \end{aligned}$$

single equation for both bound and virtual states (l=b, v):

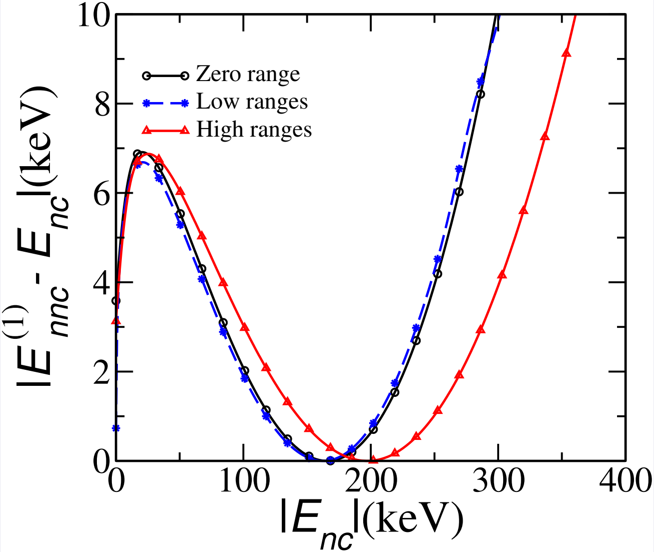
$$h_{nc}(q) = 2k_v \mathcal{V} (q, -ik_v, E) h_{nc}(-ik_v) \delta_{l,v} + \frac{2}{\pi} \int dq' \, q'^2 \, \mathcal{V} (q, q', E) \frac{h_{nc}(q')}{q'^2 + k_l^2}$$

Typical values of Yamaguchi potential parameters to reproduce

- ground state binding energy of ^{20}C with $E = -3.5$ MeV
- nn virtual state energy with $E_{nn} = -143$ keV

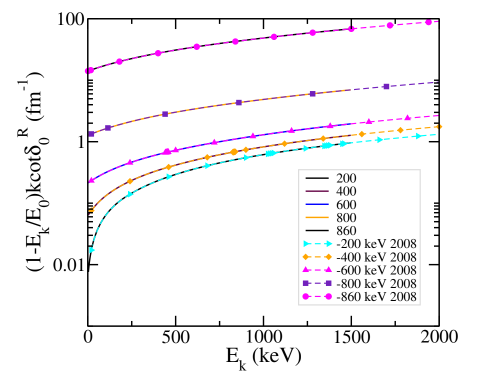
$\lvert E_{^{19}\mathrm{C}}\rvert$(keV)	$\beta_{nn} = 1.34\ \mathrm{fm}^{-1}$ $\beta_{nc}\ (\mathrm{fm}^{-1})$	r_{nc} (fm)	$\beta_{nn} = 24.5\ \mathrm{fm}^{-1}$ $\beta_{nc}\ (\mathrm{fm}^{-1})$	r_{nc} (fm)
200	0.971	2.736	18.970	0.157
400	0.754	3.233	17.036	0.174
600	0.598	3.720	15.592	0.190
800	0.477	4.255	14.395	0.205

Trajectory of 3B bound and virtual states



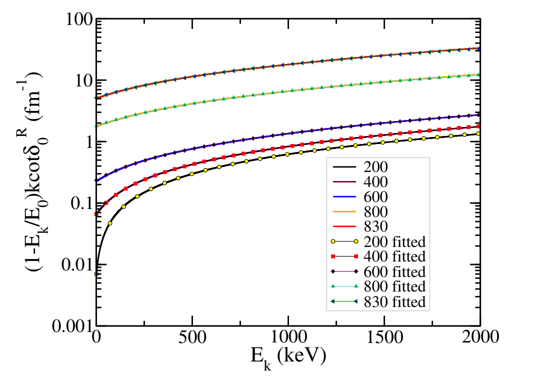
$k \cot \delta_0^R$ - Renormalized zero-range potential

- Results for $(1 - E_K/E_0) k \cot \delta_0^R$ as a function of the colliding neutron energy E_K, obtained with the renormalized zero–range potential. with corresponding fitting, where E_0 is the energy corresponding to the pole position, and $E_K \equiv k^2/(2\mu_{n,nc})$.
  The $n - c$ binding energies ($|E_{^{19}C}|$), are given inside the panel.



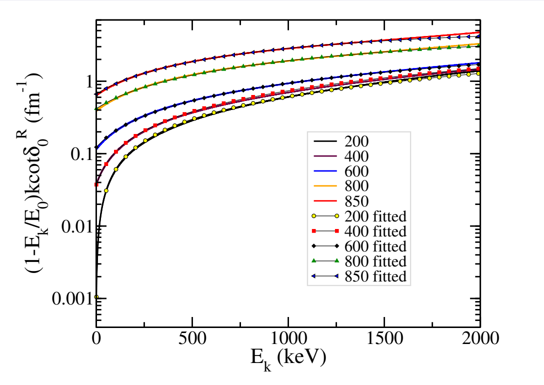
$k\cot\delta_0^R$, with finite low-range interactions (high values of β)

- Results for $(1 - E_K/E_0)k\cot\delta_0^R$ as a function of the colliding neutron energy E_K, considering a few values of $|E_{^{19}\mathrm{C}}|$, given inside the panel, obtained with the finite low-range potential (large βs), with corresponding fitting. E_0 is the energy corresponding to the pole position, and $E_K \equiv k^2/(2\mu_{n,nc})$.



$k \cot \delta_0^R$, with finite high-range interactions (low values of β

- Results for $(1 - E_K/E_0) k \cot \delta_0^R$ as a function of the colliding neutron energy E_K, obtained with the finite high-range potential (low βs), with corresponding fitting. E_0 is the energy corresponding to the pole position, and $E_K \equiv k^2/(2\mu_{n,nc})$.



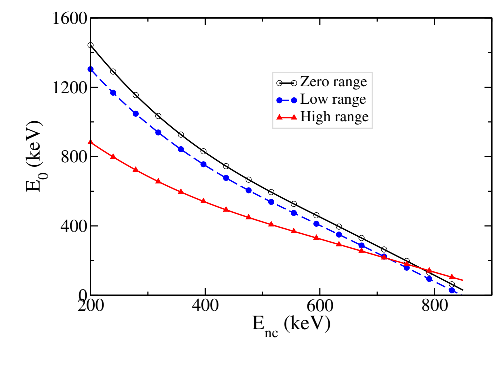
Pole positions in $k \cot \delta_0$ as a function of E_{nc}

$$a_{AD} = (1.46 - 2.15 \tan[s_0 \ln(a\Lambda_*) + 0.09])a.$$

$$a_0/a_B = \exp\left(\frac{\pi/2 - 0.59654}{s_0}\right)$$

a_B where one Efimov state is at the threshold, and a_0 where the atom-dimer scattering length is zero, or the pole in $k \cot \delta_0$ is at zero scattering energy, can be extended to the case of the $n - n - {}^{18}C$ system, when $a_{nn}^{-1} = 0$. For this mass imbalanced case $s_0 = 1.12$ with $A = 18$, and the analogous of the ratio a_B/a_0 is $\sqrt{E^0_{^{19}C}/E^B_{^{19}C}}$,

$$\sqrt{E^B_{^{19}C}/E^0_{^{19}C}} \approx \exp\left(-\frac{\pi/2 - 0.59654}{1.12}\right) = 0.419$$



For the low-range potential, $E^0_{^{19}C} = 850$ keV and $E^B_{^{19}C} = 167$ keV r esulting $\sqrt{E^B_{^{19}C}/E^0_{^{19}C}} = 0.44$ and for high-range potential $E^0_{^{19}C} = 940$ keV and $E^B_{^{19}C} = 190$ keV resulting $\sqrt{E^B_{^{19}C}/E^0_{^{19}C}} = 0.45$,

Effective-range expression

$$k \cot \delta_0^R = \frac{-a^{-1} + b\, E_K + c\, E_K^2}{1 - E_K / E_0} = \frac{d}{1 - \frac{E_K}{E_0}} + e + f\, \frac{E_K}{E_0},$$

- residue: $d = -\frac{1}{a} + bE_0 + cE_0^2$
- $e = -bE_0 - cE_0^2$
- $f = -cE_0^2$

Parameters for the effective-range fitting – zero-range results.

Table: Effective-range parameters, obtained by fitting the effective-range expression to the results shown for the case that we have zero-range interactons, for a few values of $|E_{^{19}\mathrm{C}}|$ (first column).

Adjusting the table given in Yamashita et al. PLB 670, 49 (2008).

$\|E_{^{19}\mathrm{C}}\|$ (keV)	$-1/a$ (fm^{-1})	b (fm.keV)$^{-1}$	c (fm.keV^2)$^{-1}$	E_0 (keV)
200	$5.155 \cdot 10^{-3}$	$5.498 \cdot 10^{-4}$	$5.995 \cdot 10^{-8}$	1442.6
400	$6.280 \cdot 10^{-2}$	$6.593 \cdot 10^{-4}$	$1.004 \cdot 10^{-7}$	823.89
600	0.220	$9.284 \cdot 10^{-4}$	$1.508 \cdot 10^{-7}$	451.40
800	1.299	$3.242 \cdot 10^{-3}$	$3.447 \cdot 10^{-7}$	114.981
850	5.624	$1.260 \cdot 10^{-2}$	$1.1921 \cdot 10^{-6}$	28.851

Parameters for the effective-range fitting − low-range, r_{nc}.

Table: Effective-range parameters, obtained by fitting Eq. (12) to Fig. 2, when considering different values of $|E_{^{19}\mathrm{C}}|$ (first column) with short-range Yamaguchi potentials.

$\|E_{^{19}\mathrm{C}}\|$ (keV)	$-1/a$ (fm^{-1})	b (fm.keV)$^{-1}$	c (fm.keV^2)$^{-1}$	E_0 (keV)	d (fm^{-1})	r_{nc} (fm)
200	$6.028 \cdot 10^{-3}$	$5.579 \cdot 10^{-4}$	$5.717 \cdot 10^{-8}$	1304	0.831	0.157
400	$6.555 \cdot 10^{-2}$	$6.742 \cdot 10^{-4}$	$9.144 \cdot 10^{-8}$	749.0	0.622	0.174
600	0.234	$9.840 \cdot 10^{-4}$	$1.316 \cdot 10^{-7}$	402.9	0.652	0.190
800	1.798	$4.467 \cdot 10^{-3}$	$3.519 \cdot 10^{-7}$	78.86	2.153	0.205
830	5.149	$1.198 \cdot 10^{-2}$	$8.578 \cdot 10^{-7}$	28.98	5.497	0.208

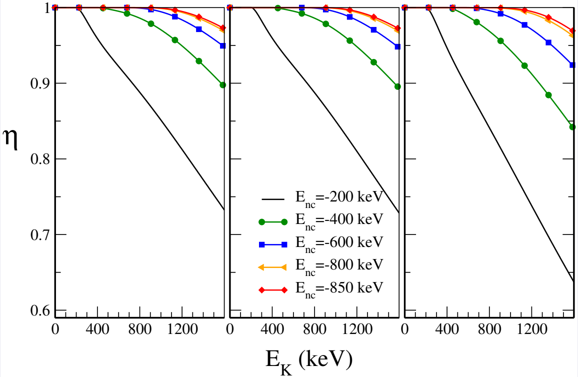
Parameters for the effective-range fitting − high-range, r_{nc}.

Table: Effective-range parameters, obtained by fitting Eq. (12) to Fig. 2, when considering different values of $|E_{^{19}\mathrm{C}}|$ (first column) with high-range Yamaguchi potentials.

$\lvert E_{^{19}\mathrm{C}}\rvert$ (keV)	$-1/a$ (fm^{-1})	b $(\mathrm{fm.keV})^{-1}$	c $(\mathrm{fm.keV}^2)^{-1}$	E_0 (keV)	d (fm^{-1})	r_{nc} (fm)
200	$5.020 \cdot 10^{-3}$	$5.267 \cdot 10^{-4}$	$7.580 \cdot 10^{-8}$	881.9	0.528	2.736
400	$4.216 \cdot 10^{-2}$	$6.319 \cdot 10^{-4}$	$2.806 \cdot 10^{-8}$	537.7	0.390	3.233
600	0.122	$8.395 \cdot 10^{-4}$	$-2.372 \cdot 10^{-8}$	324.8	0.392	3.720
800	0.405	$1.746 \cdot 10^{-3}$	$-2.332 \cdot 10^{-7}$	132.9	0.633	4.255
850	0.661	$2.603 \cdot 10^{-3}$	$-4.228 \cdot 10^{-7}$	85.60	0.880	4.403

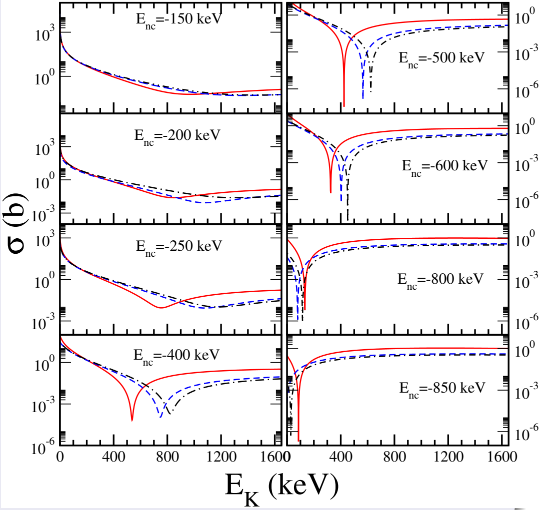
The $s-$wave absorption parameter $\eta = |e^{2i\delta_0}|$ as a function of projectile neutron energy

- left frame: zero–range potential
- middle frame: low–range Yamaguchi potential (with large β)
- right frame: high–range Yamaguchi potential (with small β)



The $s-$wave elastic $n-nc$ cross section as a function of projectile neutron energy

- red solid lines: Yamaguchi potential with low β
- blue dashed-lines: Yamaguchi potential with high β
- black dash-dotted lines: zero-range



Resume for the case of $n-^{20}$C scattering

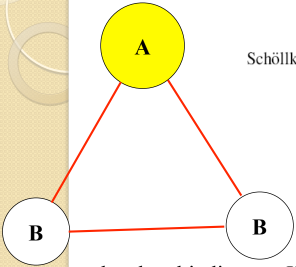
- We investigated the low-energy properties of the elastic $s-$wave scattering for the neutron$-^{19}$C near the critical condition for the occurrence of an excited Efimov state.
- Our calculations extends a zero-range approach to finite-range two-body interactions, where it was shown that the real part of the elastic $s-$wave phase shift (δ_0^R) reveals a zero when the $n-n-c$ system is close to an excited Efimov state (bound or virtual).
- We verified that by considering a finite-range potential, the results for the $s-$wave scattering amplitude present universal scaling features, with the variation of the ^{19}C binding energy for fixed ^{20}C and neutron-neutron singlet virtual state energies.
- The scaling of the effective-range parameters and the pole position of $k \cot \delta_0^R$, are in general consistent with the scaling of the zero-range potential, but the variation of this parameters shows less sensitivity to the variation of $n-^{18}C$ subsystem energy for higher range values.
- The ratio $\sqrt{E^B_{^{19}C}/E^0_{^{19}C}}$ obtained for finite range potentials changing from 0.44 to 0.45 are close to the universal ratio ≈ 0.419.

- The excited three-body ^{20}C state turns into a virtual state for a large ^{19}C binding, the threshold moves from 167 keV to 190 keV when the effective ranges are increased to reasonable physical values.
- We have also clarified that the analytical structure of the unitary cut is not affected by the potential range or mass asymmetry of the three-body system.
- We move to atoms this approach with mass-imbalanced three-particle systems, in view of the actual interest in verifying Efimov physics with different mixtures of atomic species.

Efimov Physics in weakly-bound atomic systems

- Investigations of Efimov physics in atomic systems have being studied by several groups. One of the focus have been the ^{4}He trimer, which was suggested having an excited Efimov state by Cornelius and Gloeckle [J. Chem.Phys. 85, 3906 (1986)].
- Observation of such Efimov state was recently reported by Kunitski et al., Science 348, 551 (2015).
- The studies of Helium trimer have a long story, with realistic calculations been performed by Kolganova, Motovilov and Sofianos [Phys.Rev.A 56, R1686 (1997)]. See also, Kolganova, Phys. of Part. Nucl., 1108 (2015); and Kolganova, E. A.; Motovilov, A. K.; Sandhas, Few-Body Syst. 51, 249 (1989).

Atomic weakly bound three-body systems



Schöllkopf, W., Toennies, J. P.: Science **266**, 1345 (1994)

dimer $R_{^4He-^4He} \sim 50$ A

A-B-B weakly bound molecules

ultra-low binding ~ mK or < mK

$^{133}Cs_3$ (trapped ultracold gas near a Feshbach resonance)

$^4He_3$ $\quad$ $^4He_2 - {}^7Li$ $\quad$ $^4He_2 - {}^6Li$ $\quad$ $^4He_2 - {}^{23}Na$

Delfino, Frederico and L.T., "Prediction of a weakly bound excited state in the $^4He_2$-^7Li Molecule", J. of Chem. Phys. 113 (2000) 7874.

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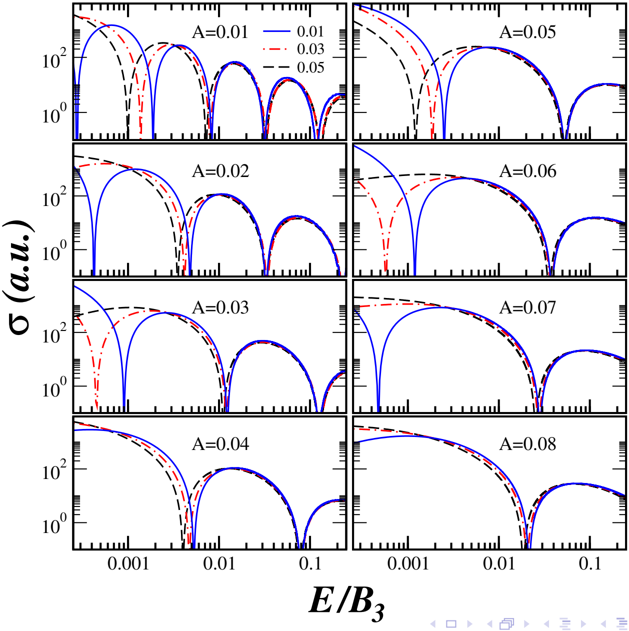
Motivations from ultracold atom laboratories

- In the unitary limit, two levels are related by an exponential scaling factor $\exp(2\pi/s_0)$, where s_0 is a constant that varies according to the mass-ratio m_H/m_L. For $m_H = m_L$, the energy-ratio is predicted to be ~ 515, such that it will be quite difficult for an experimental verification.
- Optimal situations can occur for $m_H \gg m_L$.
- In case of $m_H = 100 m_L$, the ratio between consecutive levels of the bound-state energy spectrum is given by $\exp(2\pi/s_0) \sim 4.7$.
- We show that the discrete Efimov scaling factor can be well identified in the corresponding scattering observables of one atomic species α when colliding with a two-body $\alpha\beta$ bound-state.
- Our present results can be quite relevant for the going-on experimental observations of Efimov physics in cold-atom laboratories.

Motivations from ultracold atom laboratories

- Heidelberg group is studying the extreme mass-imbalance mixtures composed by ^{133}Cs and ^{6}Li atomic species [J. Ulmanis et al, PRL **117** (2016) 153201].
- Ultracold degenerate mixtures of alkali-metal-rare-earth molecules, ^{174,173}Yb$-^{6}$Li have also been considered by H. Hara et al [PRL **106** (2011) 205304] and Hansen et al [PRA **84** (2011) 011606(R)].
- Therefore, we understand that more favorable conditions are accessible to probe the rich Efimov physics in cold-atom laboratories. with low-energy collision of a heavy atom in a weakly-bound molecule as LiCs or LiYb.
- Note that, for the above mentioned examples, we have mass-ratios as $m_L/m_H =$0.034 for LiYb and 0.045 for LiCs


A=0.01
0.01
0.03
0.05
A=0.05
A=0.02
A=0.06
A=0.03
A=0.07
A=0.04
A=0.08
10^2
10^0
0.001
0.01
0.1
σ (a.u.)
E/B_3

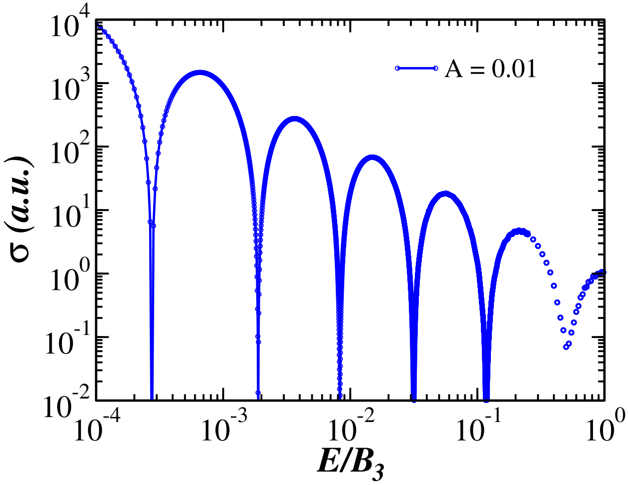


Figure: The case that $m_\beta / m_\alpha = 0.01$, with $B_{\alpha\beta} / \mathcal{B}_3 = 0.01$.

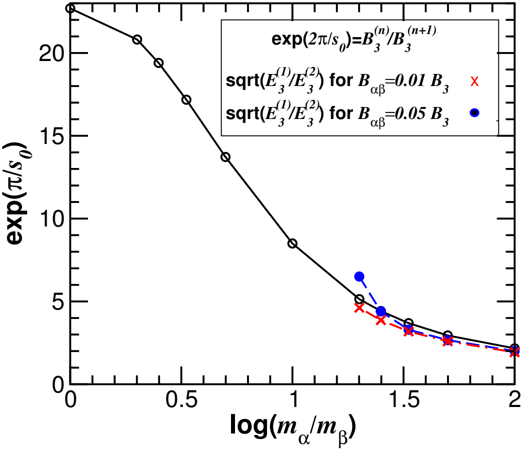


Figure: The ratios between the scattering energies corresponding to the positions of the zeros for the cross-section are shown with blue bullets, in the same plot already verified for the Efimov spectrum, when varying the mass ratio.

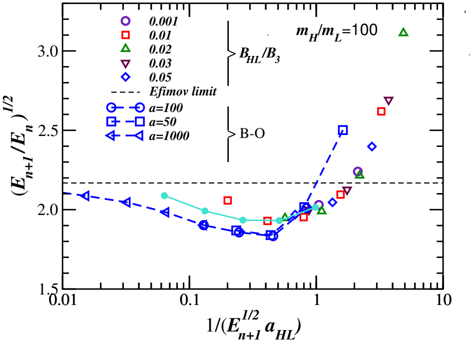


Figure: Adiabatic (blue-dashed curve) and zero-range Faddeev calculations (symbols shown inside) for $m_H/m_L = 100$. Scaling plot, with the ratios between consecutive energy-poles of $k \cot \delta_0$, where $k^2 = (m_H/\hbar^2)E_k$) and $1/(k_n a_{HL}) \approx \sqrt{\frac{2B_{HL}}{100E_n}}$.

THANKS!

[1]

[1] Work supported by the Brazilian agencies CAPES, CNPq and FAPESP