Introduction to Lattice QCD

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Overview of lattice QCD

- Why lattice QCD ?
- How lattice QCD works
- Limitations of lattice QCD
- Example of observables accessible from lattice QCD
- Masses, decay constant, form factors . . .
- I will not give too many details about algorithmic aspects

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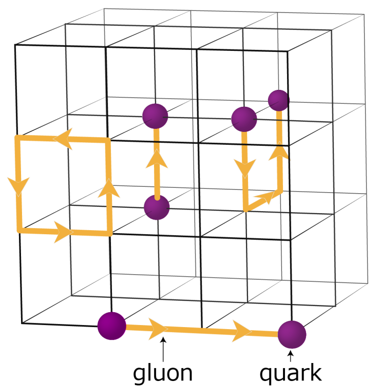
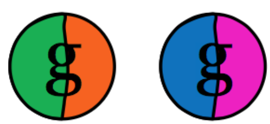


Color charge

- QCD is based on the gauge group SU(3) : three colors (red, green, blue)
- **Quarks** carry a SU(3) color

- **Anti-quarks** also carry SU(3) (anti)-colors

- **Gluons** carry a color and a anticolor



→ Gluons carry a color charge : different from QED (photon electrically neutral)

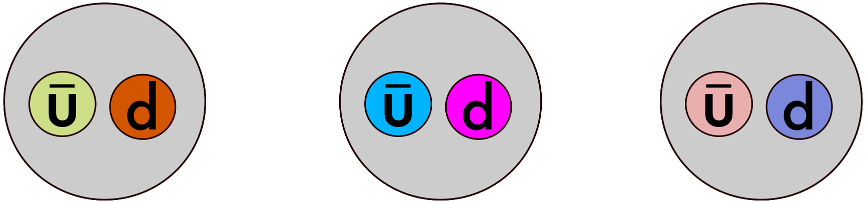
→ Gluons interact with themself via QCD !

Mesons and baryons

- Quarks and gluons are not directly observed in detectors
- We observe only hadrons (bound states, colorless particles)

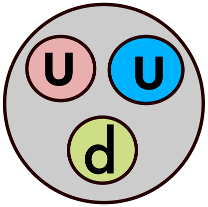
  - Mesons (quark + anti-quark)

green+anti-green blue+anti-blue red+anti-red

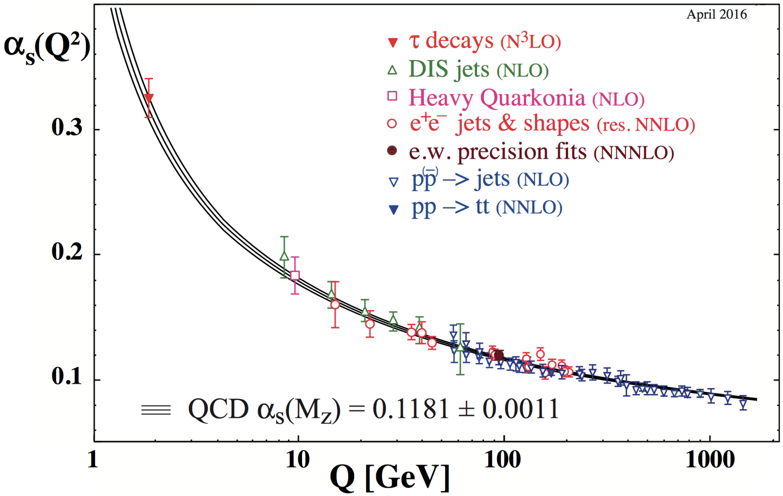


  - Baryons (three quarks or three anti-quarks)

red + blue + green



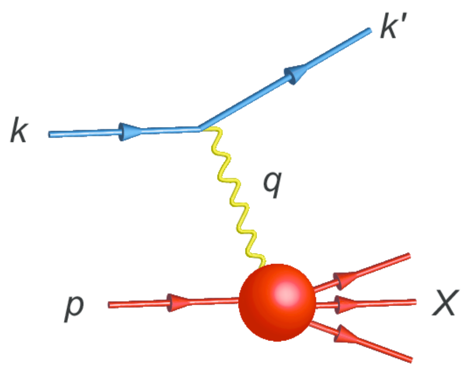
- **Running of the strong coupling**



strong force gets weaker at short distances

▶ High-energy regime

- quarks weakly coupled
- "seen" as individual entities by sufficiently energetic probes
- perturbation theory applicable !



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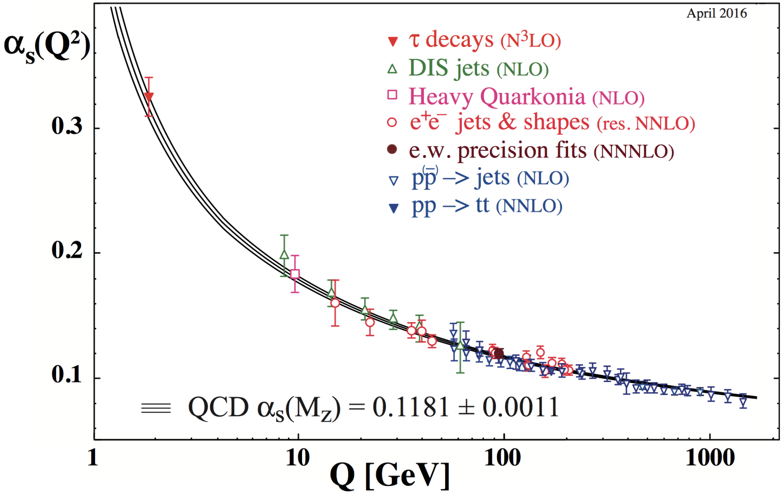
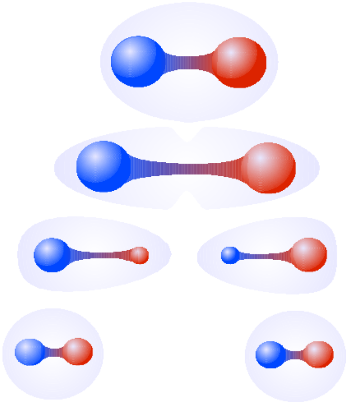


- Strong force get stronger with the distance
- Only bound states are observed (color singlets)
- As soon as there is enough energy, a new quark/anti-quark pair is created

▶ Low-energy regime

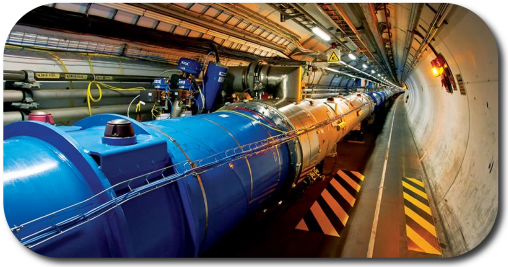
- quarks strongly coupled : mesons and hadrons
- relevant degrees of freedom are hadrons
- perturbation theory breaks down → need different techniques



Why lattice QCD ?

Strong interaction is omnipresent

- Hadrons structure and masses
- The « form factors » of hadrons
- The products of high energy collisions : $pp \to X$ (hadrons)
- No general exact solution to QCD



Goals of Lattice QCD

- validate QCD as fundamental theory of strong interaction
- understand confinement
- compute basic hadron properties
- compute electroweak amplitudes involving hadrons
- study exotic states of matter (quark-gluon plasma, ...)
- $\cdots$

Ken Wilson

Advantages

✓ Non-perturbative tool
✓ Rigorous calculation : it is not a model
✓ The precision can be systematically improved

Disadvantages

✗ Need supercomputers (expensive calculations)
✗ Gives a numbers (not an analytic expression)

Lattice QCD

QCD Lagrangian

QCD is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \sum_{f=1}^{N_f} \overline{\psi}_f(x)\left(i\not{D} - m_f\right)\psi_f(x) \quad , \quad \not{D} = \gamma^\mu\left[\partial_\mu - igA_\mu(x)\right]$$

- $\psi_f = (u, d, c, s, b, t)$: 6 quark flavors

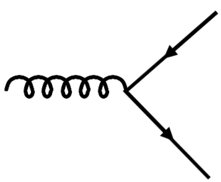
  $\rightarrow$ Spinor field : 1 color index ($a = 1, 2, 3$) + 1 Dirac index ($\alpha = 1, 2, 3, 4$) : $(\psi_f)^a_\alpha$

- $A^a_\mu(x)$: gluon field

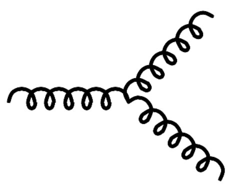
  $\rightarrow$ 1 color index + 1 Lorentz index

- $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$, $\quad f_{abc}$ are the structure constants of SU(3)

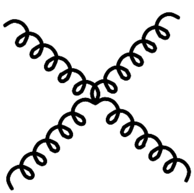
Looks similar to the QED Lagrangian, except for the additional color index : $f_{abc} \neq 0$



Analogous to photon exchange of QED



3-gluon vertex



4-gluon vertex

Gauge invariance

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \sum_{f=1}^{N_f} \overline{\psi}_f(x)\left(i\not{D} - m_f\right)\psi_f(x) \quad , \quad \not{D} = \gamma^\mu\left[\partial_\mu - igA_\mu(x)\right]$$

- The Lagrangian is invariant under local rotations in color space : group SU(3)

$$\Omega(x)^\dagger\Omega(x) = 1$$

- $\Omega(x) = \exp\left(i\omega^a(x)T_a\right) \in$ SU(3) depends on the space-time position x

$$\begin{aligned}
\psi(x) &\longrightarrow \psi'(x) = \Omega(x)\psi(x) \\
\overline{\psi}(x) &\longrightarrow \overline{\psi}'(x) = \overline{\psi}(x)\Omega^\dagger(x) \\
A_\mu(x) &\longrightarrow A'_\mu(x) = \Omega(x)A_\mu(x)\Omega^\dagger(x) - \frac{i}{g}\partial_\mu\Omega(x)\,\Omega^\dagger(x)
\end{aligned}$$

- Covariant derivative obeys the simple transformation rule

$$D_\mu\psi(x) \to \Omega(x)D_\mu\psi(x)$$

- The Field strengh

$$F_{\mu\nu}(x) = \frac{i}{g}[D_\mu, D_\nu] \;\longrightarrow\; \Omega(x)F_{\mu\nu}(x)\Omega^\dagger(x)$$

- Parallel transporter :

$$U_P(z,y) = \mathcal{P}\exp\left(ig_0\int_P A_\mu\,\mathrm{d}x^\mu\right)$$

The path integral formalism

The vacuum expectation value of an observable $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U] \int \mathcal{D}[\overline{\psi}]\, \mathcal{D}[\psi]\ \mathcal{O}[\overline{\psi}, \psi, U]\ e^{iS[U,\overline{\psi},\psi]}$$

- In the previous formula we have to sum over all « paths » (field configurations)

  $\longrightarrow$ The weight is given by the action

  $\longrightarrow$ factor « i » : quantum interferences between paths

- In QED, on can use perturbation theory to compute $\langle \mathcal{O} \rangle$ order by order in the small coupling α_{QED}

$$S[U, \overline{\psi}, \psi] = S_0[U, \overline{\psi}, \psi] + \alpha_{\mathrm{QED}}\, S_{\mathrm{int}}[U, \overline{\psi}, \psi] + \cdots$$

  $\longrightarrow$ Leads to the diagrammatic expansion (Feynman diagrams)

- The strong coupling is not small (at small energies)

  $\longrightarrow$ perturbation theory does not work

Idea : evaluate the path integral numerically

The path integral formalism

The vacuum expectation value of an observable $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U] \int \mathcal{D}[\overline{\psi}]\, \mathcal{D}[\psi]\ \mathcal{O}[\overline{\psi}, \psi, U]\ e^{iS[U,\overline{\psi},\psi]}$$

Problems :

- ill-defined object
- factor « i » : large oscillations, difficult to integrate numerically

Solution :

- discretization of the theory on a hypercubic lattice

  $\longrightarrow$ this is just a choice of regularization of the theory (like dimensional regularization)

  $\longrightarrow$ finite number of degrees of freedom : path integral well defined

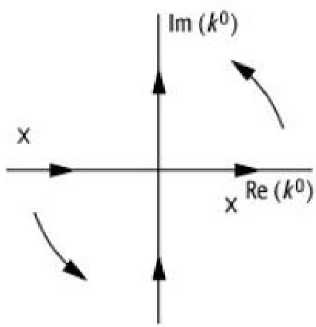
  $\longrightarrow$ regularization adapted to numerical calculations

- rotate to imaginary time (Wick rotation)

$$e^{iS[U,\overline{\psi},\psi]} \rightarrow e^{-S_E[U,\overline{\psi},\psi]}$$

  $\longrightarrow$ no oscillation anymore

  $\longrightarrow$ based on analytic properties of QFT



Lattice QCD

- QCD (euclidean) Lagrangien :

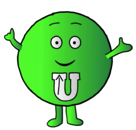
$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \sum_{i=1}^{N_f} \overline{\psi}_i(x)\left(\not{D} + m_i\right)\psi_i(x) \quad , \quad \not{D} = \gamma^\mu\left[\partial_\mu - igA_\mu(x)\right]$$

- Break-up spacetime into a 4D grid : lattice spacing a, spatial extent L, time extent T
- Lattice spacing : natural UV regulator for the theory

1) Rotational/translational Lorentz symmetries are broken

2) Gauge symmetry is preserved

Quark fields $\psi(x), \overline{\psi}(x)$ on each site



- $\psi_\alpha^a(x)$: $\alpha =$ Dirac index
  $a =$ color index

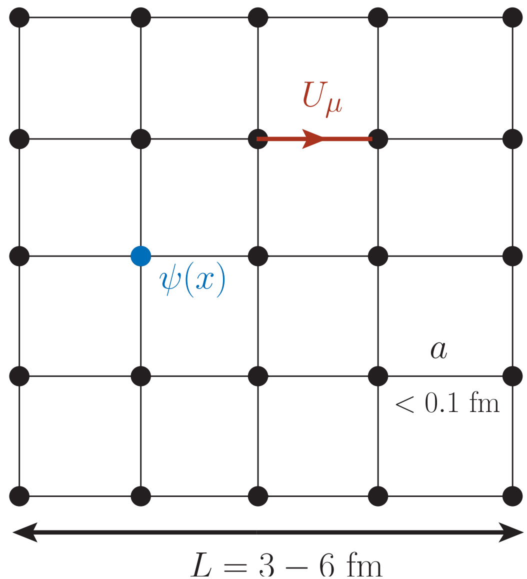
$\Rightarrow 3 \times 4 = 12$ complex numbers per site

Glue field $U_\mu(x)$ on links : parallel transporter

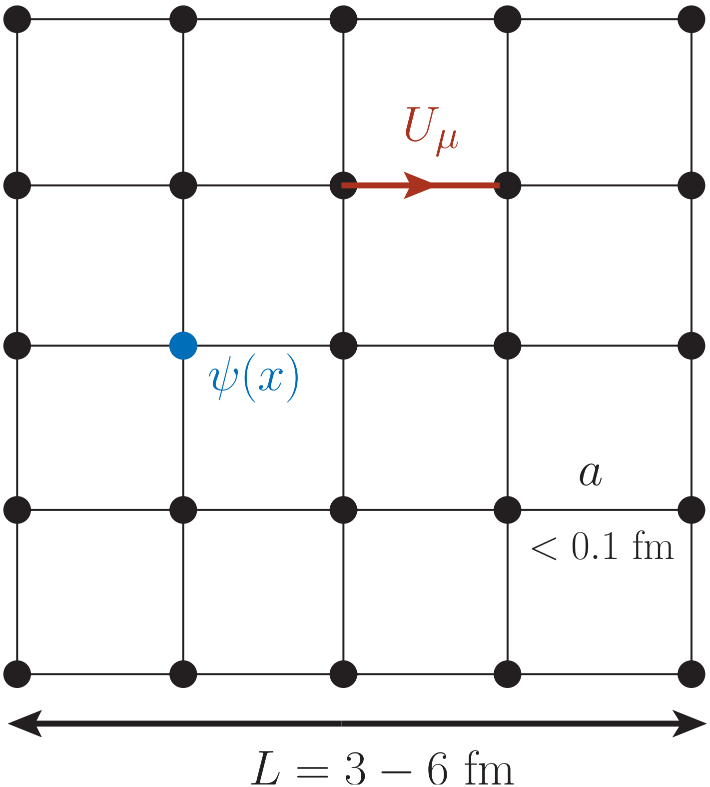
$$U_\mu(x) = \mathcal{P}e^{ig\int_x^{x+a\hat{\mu}} A_\nu(y)\mathrm{d}y^\nu} \in \mathrm{SU}(3)$$

- A field configuration $\{U_\mu\}$ is a set of SU(3) matrices

$\Rightarrow 9 \times 4 = 36$ complex numbers per site



Physical size a lattice



Typical lattice

- $L^3 \times T = 48^3 \times 96$
- $\approx 800 \times 10^6$ degrees of freedom
- $a \in [0.04 : 0.1]$ fm ($L \in 2 - 6$ fm)

Proton radius ≈ 0.9 fm

Wilson action for gluons

- In the continuum :

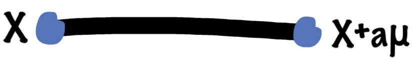
$$S_G^{\mathrm{cont}} = -\frac{1}{2}\int \mathrm{d}^4x \,\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right]$$

- We want to preserve gauge invariance symmetry

$$U_\mu(x) = \exp\left(iaA_\mu^a T^a\right) = 1 + iaA_\mu^a T^a + \ldots \in \mathrm{SU}(3)$$

Under a gauge transformation $\Omega(x) \in$ SU(3) : $\psi(x) \to \Omega(x)\psi(x)$

$$U_\mu(x) \to \Omega(x)U_\mu(x)\Omega^\dagger(x+a\hat{\mu})$$



$$U_\mu(x)U_\mu(x+a\hat{\mu}) \to \Omega(x)U_\mu(x)U_\mu(x+a\hat{\mu})\Omega^\dagger(x+2a\hat{\mu})$$

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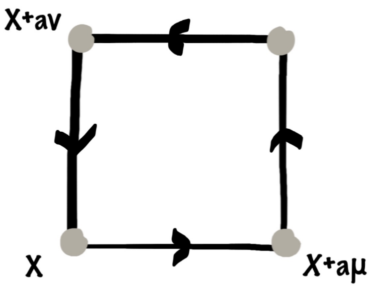

Plaquette :

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x)$$

$$P_{\mu\nu}(x) = 1 + ig_0a^2F_{\mu\nu} - \tfrac{1}{2}g_0^2a^4F_{\mu\nu}^2 + \mathcal{O}(a^6)$$

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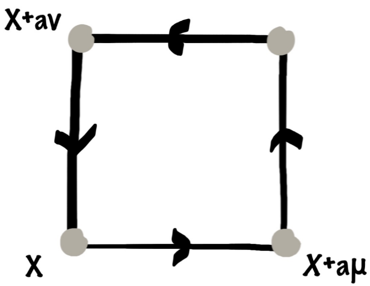

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$$P_{\mu\nu}(x) = 1 + ig_0a^2F_{\mu\nu} - \tfrac{1}{2}g_0^2a^4F_{\mu\nu}^2 + \mathcal{O}(a^6)$$

- We define the lattice action

$$S_G[U] = \frac{1}{g_0^2}\sum_{x\in\Lambda}\sum_{\mu,\nu}\mathrm{Re}\,\mathrm{Tr}\left[1 - P_{\mu\nu}(x)\right] = S_G^{\text{cont}}[U] + \mathcal{O}(a^2)$$

- Other choices are possible. They differ by an $\mathcal{O}(a^2)$ ambiguity.
  $\rightarrow$ Can be use to reduce discretization errors

Wilson fermions

- In the continuum (free case) :

$$S_F^{\text{cont}} = \int \mathrm{d}^4x \, \overline{\psi}(x) \left[\gamma_\mu \partial_\mu + m\right] \psi(x)$$

- $\psi(x)$ and $\psi(x)$ are defined on each site of the lattice
- Naive discretization :

$$S_F = a^4 \sum_{x \in \Lambda} \overline{\Psi}(x) \left[\gamma_\mu \partial^s_\mu + m\right] \Psi(x)$$

$$\partial^s_\mu \Psi(x) = \frac{\Psi(x + a\hat{\mu}) - \Psi(x - a\hat{\mu})}{2a}$$

- Interacting theory : $\partial_\mu \to \mathcal{D}_\mu = \partial_\mu - i g_0 A_\mu$

$$\nabla^s_\mu \Psi(x) = \frac{U_\mu(x)\Psi(x + a\hat{\mu}) - U^\dagger_\mu(x)\Psi(x - a\hat{\mu})}{2a}$$

- Gauge invariance is preserved : $\nabla_\mu \Psi(x) \to \Omega(x) \nabla_\mu \Psi(x)$
- But maybe too naive ...

Doublers

- In the free theory, the Dirac operator is given by

$$D = \gamma_\mu \partial^s_\mu + m \quad \longrightarrow \quad \widetilde{D}(p) = \frac{i}{a}\sum_\mu \gamma_\mu \sin(a p_\mu) + m \quad , \quad p_\mu = \frac{\pi}{L} n_\mu$$

- The propagator (inverse of the Dirac operator) is given by

$$\widetilde{D}^{-1}(p) = \frac{-i a^{-1} \sum_\mu \gamma_\mu \sin(a p_\mu) + m}{a^{-2} \sum_\mu \sin^2(a p_\mu) + m^2}$$

  - pole at $p^2 = -m^2$ when $a \to 0$
  - 15 other poles $p_\mu \in \left[0, \frac{\pi}{a}\right]$, $\mu \in [0, 3]$

- Interacting theory : doublers can interact with each other via loop corrections
- It is important to remove them properly

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- Interacting theory : doublers can interact with each other via loop corrections
- It is important to remove them properly

Solution

$$D_W = \gamma_\mu \nabla_\mu^s + m - \frac{a}{2}\nabla_\mu^* \nabla_\mu$$

$$S_F = a^4 \sum_{x\in\Lambda} \overline{\psi}(x) D_W \psi(x)$$

breaks chiral symmetry

→ Wilson fermions : $\langle \mathcal{A}^{\text{lat}}(x)\rangle = \langle \mathcal{A}^{\text{cont}}(x)\rangle + \mathcal{O}(a)$

→ We can add a new term to remove leading discretization errors (Wilson-Clover fermions)

$$\langle \mathcal{A}^{\text{lat}}(x)\rangle = \langle \mathcal{A}^{\text{cont}}(x)\rangle + \mathcal{O}(a^2)$$

Discretizations of the fermionic action

- There are many different actions
- They are all equivalent in the continuum limit ($\rightarrow$ QCD !)
- But they have different features at finite value of the lattice spacing

Action	Advantages	Disadvantages
Staggered	✓ computationally very fast	✗ fourth root problem ✗ complicated Wick contractions ✗ taste mixing
Wilson-Clover	✓ computationally fast $(\times 10)$	✗ breaks chiral symmetry ✗ needs operator improvement
Twisted mass fermions	✓ computationally fast $(\times 10)$ ✓ automatic O(a)-improvement	✗ breaks chiral symmetry ✗ violation of isospin
Domain wall	✓ improved chiral symmetry	✗ computationally expensive $(\times 100)$
Overlap fermions	✓ exact chiral symmetry	✗ computationally expensive $(\times 100)$

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Overlap fermions	✓ exact chiral symmetry	✗ computationally expensive $(\times 100)$

- We now have a discrete formulation of the action ✓

- How to perform the path integral ?

Gauge invariance

- In the continuum, the fundamental fields are
  - $\psi(x)$ (fermions)
  - $A_\mu(x)$ (gluons)

$$\mathcal{L}_F = \overline{\psi}(x)\left(\not{D} + m\right)\psi(x) \quad , \quad \not{D} = \gamma^\mu\left[\partial_\mu - igA_\mu(x)\right]$$

- Naive discretization :

$$S_F = \frac{a^3}{2}\sum_{x\in\Lambda}\left(\overline{\psi}_i(x)\gamma_\mu\psi(x+a\hat{\mu}) - \overline{\psi}(x)\gamma_\mu\psi(x-a\hat{\mu})\right) - a^4 ig\sum_{x\in\Lambda}\overline{\psi}(x)A_\mu(x)\psi(x) + ma^4\sum_{x\in\Lambda}\overline{\psi}(x)\psi(x)$$

- This discretization couples the quark field at neighboring sites
- Breaks local gauge invariance
- Gauge invariance is easy to implement using the link variables $U_\mu(x)$

Wick contractions

- In LQCD, expectation values are given by the (finite) path integral (S_E is the euclidean action) :

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U_\mu] \mathcal{D}[\psi, \overline{\psi}] \; \mathcal{O}[U_\mu, \psi, \overline{\psi}] \; e^{-S_E[U_\mu, \psi, \overline{\psi}]}$$

- $S_E = S_G + S_F$
- Lattice spacing : natural UV regulator for the theory [rigorous definition of the path integral]
- Integration over fermionic variables

$$S_F = a^4 \sum_{x \in \Lambda} \overline{\psi}(x) D_W \psi(x)$$

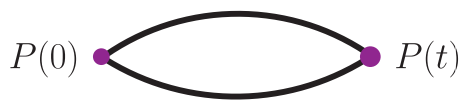
$\rightarrow$ Action quadratic (like in the free theory)

$\rightarrow$ can be computed using Wick contractions

$$\int D[\psi] D[\overline{\psi}] \; e^{-S_F} = \det D_W$$

$$\int D[\psi] D[\overline{\psi}] \; \psi_i(y) \; \overline{\psi}_j(x) \; e^{-S_F} = -\left(D_W^{-1} \right)_{ij} \det D_W$$

$$C_2(x) = \sum_{\vec{x}} \langle \left(\overline{\psi} \gamma_5 \psi \right)^\dagger (t, \vec{x}) \; \left(\overline{\psi} \gamma_5 \psi \right)(0) \rangle$$



$\rightarrow$ The results depends on $D^{-1}[U_\mu]$

$\rightarrow$ One needs to compute the inverse of a huge matrix !

$$\boxed{\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U_\mu] \; \langle \mathcal{O} \rangle_F \, [U_\mu] \; \det D \; e^{-S_G}}$$

Monte Carlo sampling

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U_\mu] \; \langle \mathcal{O} \rangle_F [U_\mu] \; \det D \; e^{-S_G}$$

- Assumes $\det D \; e^{-S_G} > 0$: it acts as a weight function
- Stochastic evaluation using Monte-Carlo methods :

1) generate n gauge configurations $\{U_\mu^{(i)}\}$ with a probability distribution $\mathcal{P} \propto e^{-S_G + \ln \det D}$

2) on each gauge configuration, compute $\langle \mathcal{O} \rangle_F [U_\mu^{(i)}]$ (Wick contractions, require $D^{-1}[U_\mu^{(i)}]$)

3) compute expectation values :

$$\overline{\mathcal{O}} = \sum_{i=1}^{n} \langle \mathcal{O} \rangle_F [U_\mu^{(i)}] = \langle \mathcal{O} \rangle + \delta\mathcal{O} \;\; \rightarrow \;\; \text{statistical error}$$

The calculation is done in two parts :

Generation of gauge configurations

- independent of the observable
- very expensive ($n = \mathcal{O}(10^3)$)
- need to be done only once for all
- Lattice collaborations : ETMC, CLS, UKQCD, ...

Computation of the observable

- write the different Wick contractions
- compute $D^{-1}[U_\mu]$ on each gauge configuration
- inverse of a huge matrix
- compute the correlation function

Supercomputers

CINES
Centre Informatique National
de l'Enseignement Supérieur

Setting the scale

Consider QCD with two degenerate quarks

- The lattice action depends on two free parameters :
  - $\rightarrow$ The quark mass : $m = m_u = m_d$
  - $\rightarrow$ The value of the coupling constant : g_0
  - $\rightarrow$ Those are bare parameters : $g_0(a), m(a)$ (lattice spacing = regulator)
- Numerically : only dimensionless quantities can be computed : am_π, af_π, ...

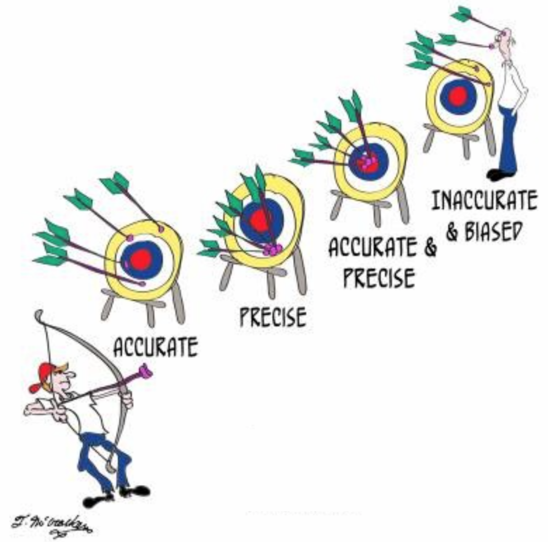
How to tune the bare parameters on a lattice simulation ?

- Tune the quark mass such that $\left(\frac{m_\pi}{m_p}\right)_{\text{lat}} = \left(\frac{m_\pi}{m_p}\right)_{\text{exp}}$
- Determine a in physical units from $(am_\pi)_{\text{lat}}$ and m_π^{phys}
  - $\rightarrow$ The continuum limit must be taken using a constant line of physics
  - $\rightarrow$ $m_\pi, m_p \ll a^{-1}$ while keeping $\left(\frac{m_\pi}{m_p}\right)_{\text{lat}}$ constant
- In principle the result depends on the observables chosen to set the scale (not unique)
  - $\rightarrow$ At low energies, small effect : decoupling
  - $\rightarrow$ Some observables are better than others (statistical precision, experimental accuracy, ...)

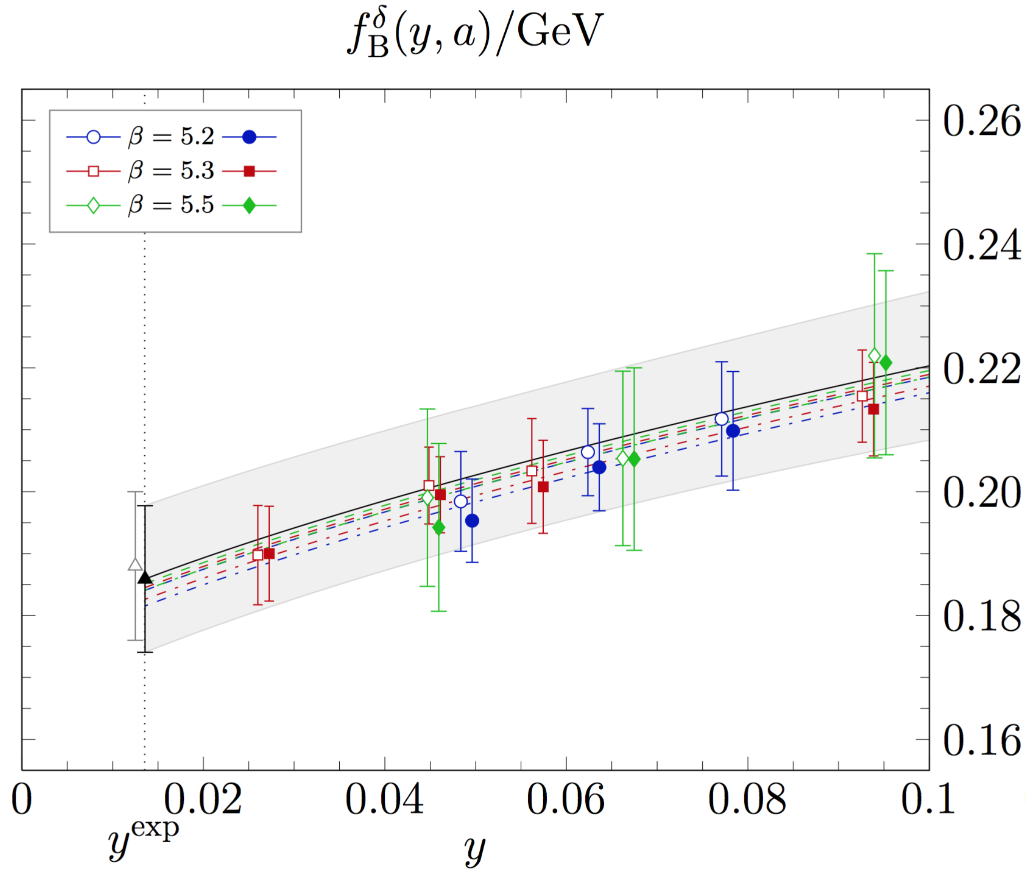
Statistical error

- Monte-Carlo algorithm : statistical error $\rightarrow \sim 1/\sqrt{N_{\text{meas}}}$

Systematic errors



- Finite lattice spacing : $a \neq 0$
  $\rightarrow$ Symanzik's improvement programme
- Finite volume V
  $\rightarrow$ one should take the infinite volume limit
  $\rightarrow$ χPT can help in some cases (pion dominates FSE)
- Unphysical quark masses
  $\rightarrow$ It is difficult to simulate light quarks
  (algorithmic performances, need large volume)
  $\rightarrow$ Use different values of the quark masses
  $\rightarrow$ Again, χPT can help
  $\rightarrow$ Today : many simulations at physical point (But volume effects ...)
- Number of dynamical quarks : « N_f »
  $\rightarrow$ $N_f = 0$: quenched approximation. Neglect all fermion loops. Cheap but lost unitarity.
  $\rightarrow$ $N_f = 2$: only two light quarks u and d in the sea with $m_u = m_d$.
  $\rightarrow$ Nowadays most simulations used $N_f = 2 + 1$, $N_f = 2 + 1 + 1$.


$f_B^\delta(y,a)/\mathrm{GeV}$
$\beta = 5.2$
$\beta = 5.3$
$\beta = 5.5$
0.26
0.24
0.22
0.20
0.18
0.16
0
0.02
0.04
0.06
0.08
0.1
y^{exp}
y

Conclusion

- Lattice QCD is a specific regularization of QCD
  $\rightarrow$ hypercubic lattice : finite number of degrees of freedom
  $\rightarrow$ adapted to numerical simulations : quarks live on sites, gluons on links
  $\rightarrow$ Explicitly gauge invariant
  $\rightarrow$ Not unique : different formulations exists (important cross-check)
  $\rightarrow$ it is not a model ! But :

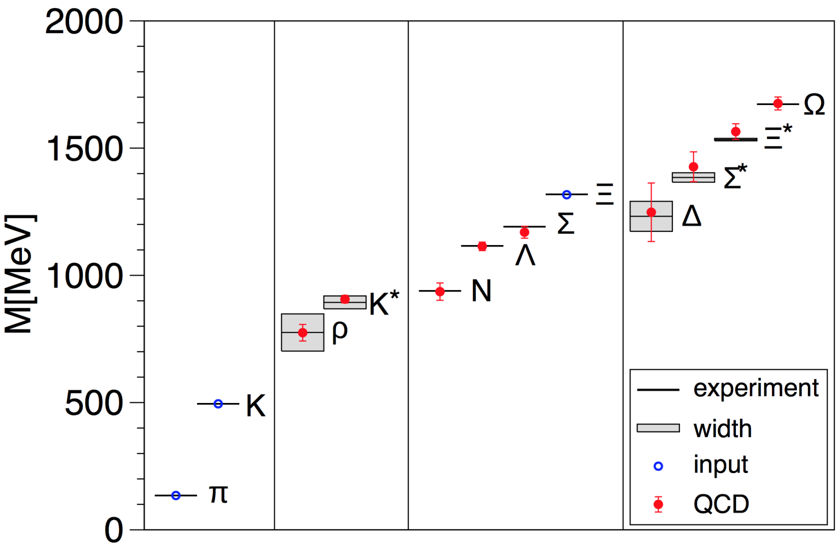
- The path integral is estimated stochastically using Monte Carlo sampling
  $\rightarrow$ Statistical errors

- Finite lattice spacing, finite volume, $N_f = 2, 2+1, ...$
  $\rightarrow$ Systematic errors

In principle errors are under control and can be systematically reduced

How to compute an observable in LQCD : meson masses

- This is one of the simplest quantity to extract on the lattice
- Check if QCD can indeed reproduced the experimental patern (with correct quantum numbers)



- Predicts new particle
- Information about the internal structure of resonances (??)

Meson or baryon masses

- Construct an operator $\mathcal{O}$ with quantum numbers of a given particle (spin, parity, momentum, ...)
- $\mathcal{O}(x) = \overline{\psi}(x)\gamma_5\psi(x)$: pseudoscalar (pion quantum numbers)

$$\langle 0|\mathcal{O}|\pi\rangle \neq 0$$

- Projection at given momentum : $\mathcal{O}(t,\vec{p}) = \sum_{\vec{x}} \mathcal{O}(\vec{x},t)\ e^{i\vec{p}\vec{x}}$
- Two-point function $\langle \mathcal{O}^\dagger(t)\mathcal{O}(0)\rangle$

  - Lorentz-invariance : $\mathcal{O}(t) = e^{Ht}\mathcal{O}(0)e^{-Ht}$
  - Completeness relation : $1 = \sum_n \frac{1}{2E_n}|n\rangle\langle n|$

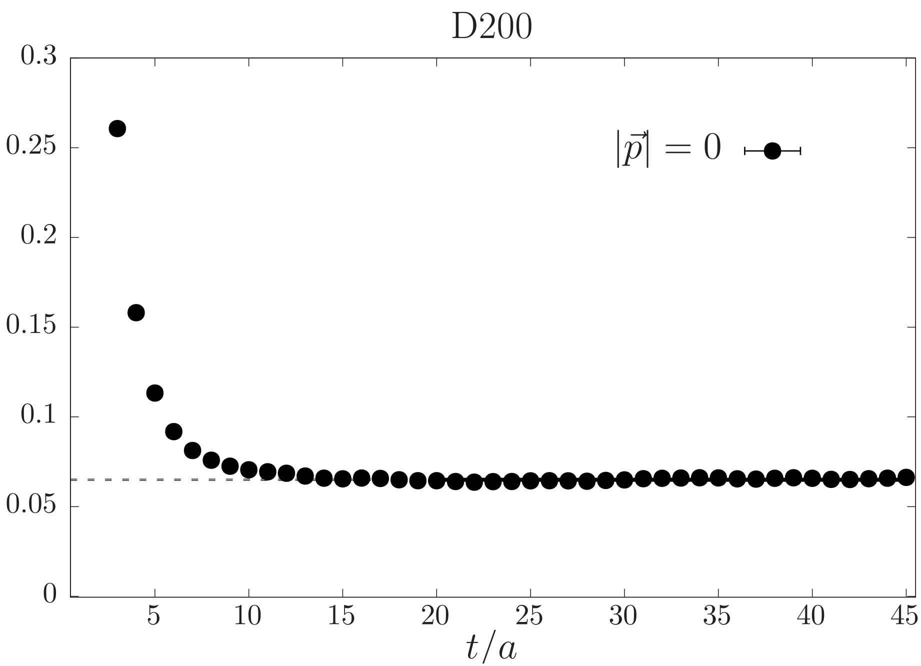
$$\begin{aligned}
\langle \mathcal{O}^\dagger(t)\mathcal{O}(0)\rangle &= \sum_n \langle 0|\mathcal{O}^\dagger(t)|n\rangle \frac{1}{2E_n}\langle n|\mathcal{O}(0)|0\rangle \\
&= \sum_n \langle 0|e^{Ht}\mathcal{O}^\dagger(0)e^{-Ht}|n\rangle \frac{1}{2E_n}\langle n|\mathcal{O}(0)|0\rangle \\
&= \sum_n \langle 0|\hat{\mathcal{O}}^\dagger|n\rangle \frac{1}{2E_n}\langle n|\hat{\mathcal{O}}|0\rangle\ e^{-E_n t}
\end{aligned}$$

$$\langle \mathcal{O}^\dagger(t)\mathcal{O}(0)\rangle = \frac{Z_\pi^2}{2E_\pi}\ e^{-E_\pi t} \times (1 + \text{exponentially suppressed terms})$$

Pion mass

- Choose $\mathcal{O}(x) = \overline{\psi}(x)\gamma_5\psi(x)$
- Compute $C^{(2)}(t) = \langle \mathcal{O}(t)\mathcal{O}(0)\rangle \to \frac{|Z_\pi|^2}{2E_\pi}\, e^{-E_\pi t}$

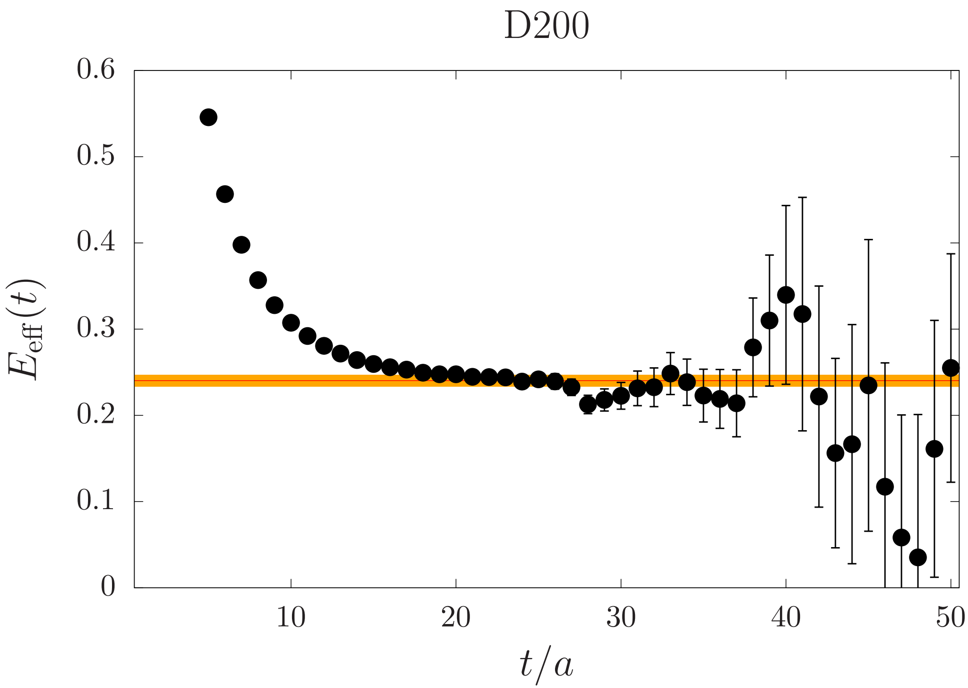
$$am_{\text{eff}} = \log\left(\frac{C^{(2)}(t)}{C^{(2)}(t+a)}\right)$$



- For $t/a > 15$ we observe a plateau : extraction of the pion mass am_π in lattice units
- At short time separation : contribution from excited states.
  $\to$ can be further reduced using $\langle 0|\mathcal{O}|\pi\rangle \neq 0$ and $\langle 0|\mathcal{O}|3\pi\rangle \approx 0$

Vector meson mass (« ρ »meson)

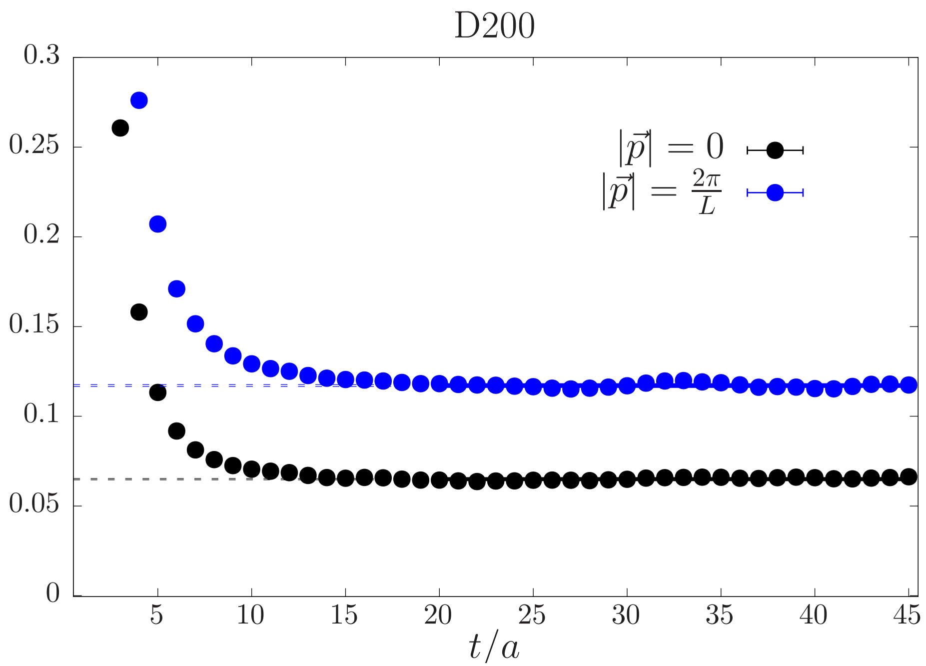
- Choose $\mathcal{O}(x) = \overline{\psi}(x)\gamma_k\psi(x)$ and project on vanishing momentum



- Similar behavior (exponential decay)
- Signal deteriorates at large time separation t/a : noise problem
- In this case noise/signal $\propto \exp\left[(m_V - m_\pi)t\right]$
- It is therefore important to use optimal interpolating operators $\mathcal{O}$ with large overlap

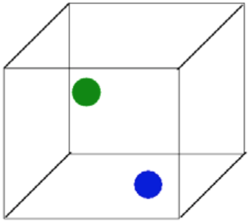
Pion mass : non-vanishing momentum

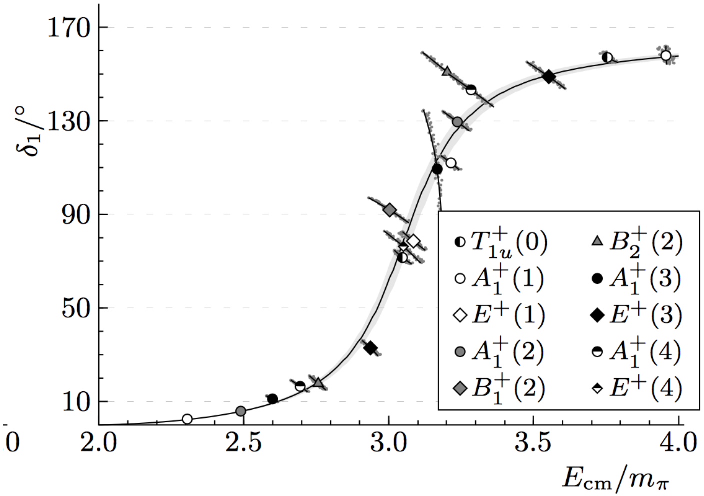
- Finite volume : $L^3 \times T$
- momentum takes only discrete values : $\vec{p} = \frac{2\pi}{L}\vec{n}$
- $L/a = 64$ with $L = 4.2$ fm $\Rightarrow \Delta p \approx 300$ MeV : no continuous spectrum !



- Statistical error increase with $|\vec{p}|$
- the range of $|\vec{p}|$ is also bounded (lattice spacing plays the role of the UV regulator)
- This can be a limitation in form factor calculations

- This method works for stable particle (via QCD)
- E_n are eigenvalues of the Hamiltonian in finite volume
- The ρ meson, and most mesons, are resonances
- Requires more sophisticated methods





- In a finite box the two pion interacts with each other
- Discrete spectrum (and discrete momenta)
- There is a strong resonance $\rho(780)$
- The interaction between the pions depends on the volume

[Bulava et al. '18]

Formalism developed by [Luescher '91]

Finite volume spectrum $\leftrightarrow$ mass and width of the resonance