Duality along the RG flow

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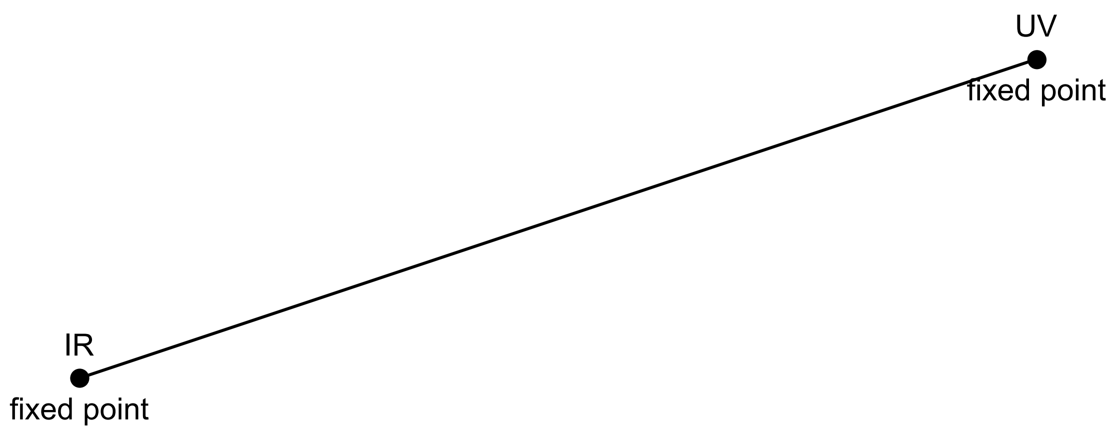
S. Abel, BB, F. Sannino, 1805.07611

Introduction

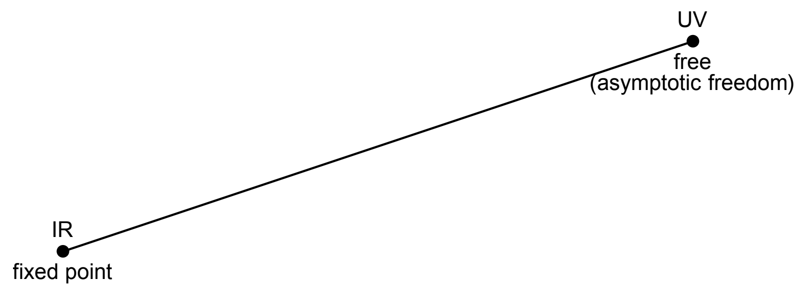
Field theories RG flow between fixed points in parameter space:



UV and IR fixed points could be weak (perturbative) or strong (non-perturbative)

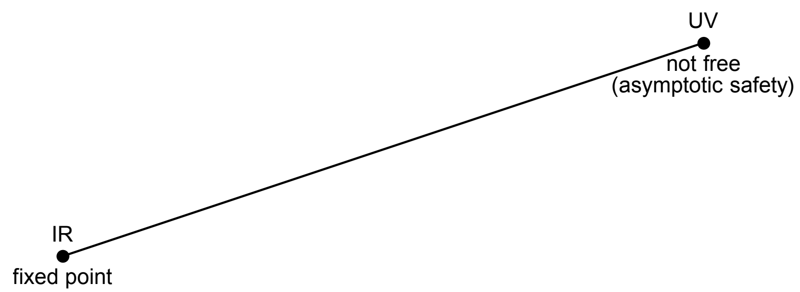
The usual paradigm is

UV fixed point free (asymptotic freedom)



However other possibilities (Litim, Sannino)

UV fixed point not free, either weak or strong (asymptotic safety)



We will be interested in asymptotically safe (AS) theories with either

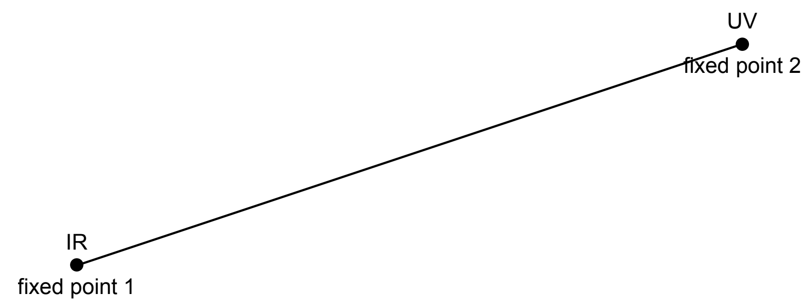
weak-weak fixed points (perturbative along the flow - Banks, Zaks)

or

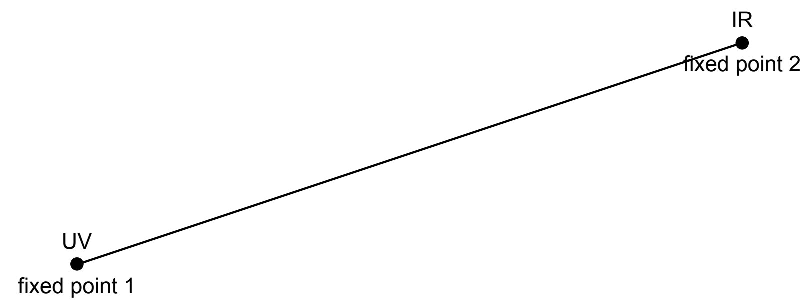
strong-strong fixed points (non-perturbative along the flow)

Constraints on RG flows

Suppose we have: fixed point 2 (UV) $\to$ fixed point 1 (IR)



Can we reverse the flow? fixed point 1 (UV) $\to$ fixed point 2 (IR)



The answer is NO and the reason is the a-theorem

The central charge a defined from the trace anomaly for stress-energy tensor $T^{\mu\nu}$ in curved background :

$$T^{\mu}{}_{\mu} = -a \times E_4 + \dots$$

Euler invariant

$$E_4 = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta} R_{\alpha\beta} + R^2$$

quadratic diffeomorphism invariant combination

The central charge can be connected (Komargodski-Schwimmer) to the dilaton-dilaton ($\phi\phi$) scattering:

$$a_{KS}(\mu) = a_{UV} - \int_{\mu}^{\infty} d\mu \frac{\sigma_{\phi\phi\to\phi\phi}(\mu)}{\mu^3}$$

Since cross-section $\sigma > 0 \to a_{KS}(\mu)$ decreasing from UV to IR (a-theorem)

$$\Delta a \equiv a_{UV} - a_{IR} > 0$$

Because of it

- RG flow is irreversible
- a_{KS} provides a measure for # of d.o.f.
- since cross-section σ is a physical quantity, so is $a_{KS}(\mu)$

a-central charge at the fixed point

In a generic field theory a can be calculated perturbatively.

In most cases this not useful because fixed point non-perturbative

Fortunately in supersymmetry central charges can be got exactly

$$
\begin{aligned}
(R_i, n_i) \quad & \ldots \quad (\mathrm{R-charge}, \#\ \mathrm{d.o.f.})\ \text{of chiral field } i \\
|G| \quad & \ldots \quad \text{dimension of gauge group } G = \#\ \text{of gauge fields} \\
a_1(R) \quad & \equiv \quad 3(R-1)^3 - (R-1)
\end{aligned}
$$

$$
a = \underbrace{2|G|}_{gaugino} + \underbrace{\sum_i n_i a_1(R_i)}_{chiral\ fields}
$$

Total a equal to sum of single a_1 (one for each chiral multiplet)

This exact relation is due to the fact that

$T_{\mu\nu}$ and j_R^μ are different components of the same supermultiplet

$\rightarrow$ relations between $T^\mu{}_\mu$ and $\partial_\mu j_R^\mu$:

$$\begin{aligned} T^\mu{}_\mu &= -a\, E_4 + \dots \\ \partial_\mu j_R^\mu &= \underbrace{\left[Tr\, U(1)_R\right]}_{\propto \sum_i n_i (R_i - 1)} R_{\alpha\beta\gamma\delta} \tilde{R}^{\alpha\beta\gamma\delta} + \underbrace{\left[Tr\, U(1)_R^3\right]}_{\propto \sum_i n_i (R_i - 1)^3} F_{R\mu\nu} \tilde{F}_R^{\mu\nu} \end{aligned}$$

$U(1)_R$ symmetry unavoidable in supersymmetric fixed points (conformal theories): R charge part of the superconformal algebra

If we know the R-charges, we know the central charge a

How do we get the R-charges R_i?

In a general SCFT

$$R(\text{chiral superfield}) = \frac{2}{3} D(\text{chiral superfield})$$

$$\text{R charge} \;\leftrightarrow\; \text{anomalous dimension}$$

For a free theory $(D(\phi_{free}) = 1)$

$$R(\phi_{free}) = 2/3$$

For non-trivial SCFT the β functions must vanish:

- NSVZ β function is proportional to

$$T(G) + \sum_i T(r_i)(R_i - 1) = 0$$

$T \ldots$Dynkin index

- β function for superpotential Yukawa coupling y_a of

$$W = y_a \prod_i \phi_i^{q_{ia}}$$

is proportional to

$$\sum_i q_{ia} R_i - 2 = 0$$

Three possibilities:

1. # of constraints above bigger than number of chiral fields
   → no SCFT
2. # of constraints above equal to number of chiral fields
   → the solution to above equations unique and represents a possible candidate for CFT
3. # of constraints above smaller than number of chiral fields
   → one uses the above equations to express some R-charges with the others; then applies the a-maximization to calculate the remaining R-charges:

a-maximization:

$$\frac{\partial a}{\partial R_i} = 0$$

This gives same number of equations than unknowns R_i.

Equations are quadratic so there can be several real solutions. One should choose the one with

$$\frac{\partial^2 a}{\partial R_i \partial R_j} \qquad \text{all negative eigenvalues}$$

The constraints can be enforced by Lagrange multipliers:

$$\begin{aligned} a &= 2|G| + \sum_i a_1(R_i) \\ &+ \lambda_1 \left(T(G) + \sum_i T(r_i)(R_i - 1) \right) + \sum_{a>1} \lambda_a \left(\sum_i q_{ia} R_i - 2 \right) \end{aligned}$$

From

$$\frac{\partial a}{\partial \lambda_a} = 0 \quad , \quad a = 1, \ldots$$

we solve for R_i and plug in into a. This is the standard way for solving at the fixed point.

Another way (Kutasov) is:

First solve

$$\frac{\partial a}{\partial R_i} = 0 \to R_i = R_i(\lambda)$$

and then plug in into a:

$$a_K = a(R_i(\lambda), \lambda)$$

The interpretation now is different: at the fixed point again

$$\frac{\partial a_K}{\partial \lambda_a} = 0 \quad , \quad a = 1, \ldots$$

and the result is the same as before.

But one can interpret $a_K = a(R_i(\lambda), \lambda)$ as the a function along the flow with λ changing from λ_{IR} in IR to λ_{UV} in UV.

Example: perturbative SQCD with some matter ($W = 0$), only one Lagrange multiplier, λ_1:

$$a_K = 2(N_c^2 - 1) + \sum_i a_1(R_i) + \lambda_1 \left(N_c + \sum_i T_i (R_i - 1) \right)$$

$$\frac{\partial a_K}{\partial R_i} = 0 \quad \to \quad R_i(\lambda_1)$$

All $R_i(\lambda_1)$ (and so all anomalous dimensions) determined just by one function along the flow - $\lambda_1(\mu)$

- in the UV $\lambda_1 = 0$
- For small λ_1 the theory is perturbative and one finds the 1-loop relation

$$\lambda_1 = -\frac{g^2}{2\pi^2} + \mathcal{O}(g^4)$$

- one can repeat the calculation up to 3-loops getting agreement for the scheme independent part of the perturbative calculation of the anomalous dimensions
- the flow ends at IR CFT when (at some λ_1^*) NSVZ vanishes:

$$T(G) + \sum_i T(r_i)\,(R_i(\lambda_1^*) - 1) = 0$$

Duality

Seiberg type dualities connect theories with different gauge group and field content but same flavor structure.

The original (and simplest) example is SQCD:

ELECTRIC	MAGNETIC
$SU(N_c)\ :\ g$	$SU(N_f - N_c)\ :\ \tilde{g}$
$N_f * \left(Q + \tilde{Q}\right)$	$N_f * (q + \tilde{q}) + N_f^2 * M$
$W = 0$	$W = \tilde{y}\, qMq$

- valid only in the IR (in the fixed point)
- valid only in the conformal window $3N_c/2 \leq N_f \leq 3N_c$ (both electric and magnetic theories asymptotically free)
- quantum numbers of magnetic singlets $M \sim \tilde{Q}Q$

$$R(M) = 2R(Q)$$

This follows from magnetic superpotential $W \sim \tilde{q}qM$

$$2R(q)+R(M) = R(W) = 2$$

together with (at the fixed point)

$$2R(q)+2R(Q) = 2\left(1 - \frac{N_f - N_c}{N_f}\right) + 2\left(1 - \frac{N_c}{N_f}\right) = 2$$

- at least one theory must be strongly coupled (no duality possible between two different gauge theories at weak coupling)
- duality of type strong↔weak (one weak, the other strong)

If a theory has a nontrivial UV fixed point and a nontrivial IR fixed point, and we know the duals of both of them, then reasonable that they are *dual in the whole flow*

Consider the following simple example:

1. first at $\mu \to \infty$ have SQCD with SU(N_c) and $N_f + 1$ quarks in the conformal window $(3N_c/2 < N_f + 1 < 3N_c)$

   electric and magnetic theories different

2. run down to the IR, in the IR the usual duality between electric and magnetic theory

3. perturb the electric theory with a mass m for 1 quark pair; let this mass deep in the fixed point regime, duality still valid

$$
\begin{aligned}
W_E &= m\tilde{Q}_{N_f} Q^{N_f} \\
W_M &= m{M_{N_f}}^{N_f} + \tilde{q} q M \to \langle \tilde{q}_{N_f} q^{N_f} \rangle = -m
\end{aligned}
$$

4. right above the mass ($\mu \to m + 0$ - this is now the **new UV**) we thus have duality between:
   - electric theory is $SU(N_c)$ with $N_f + 1$ quark pairs
   - magnetic theory is $SU(N_f + 1 - N_c)$ with $N_f + 1$ quark pairs

   they are dual to each other (equivalent)
5. in the deep IR ($\mu \to 0$) (this is now the **new IR**) again (a new) duality
   - electric theory is $SU(N_c)$ with N_f quark pairs
   - magnetic theory is $SU(N_f - N_c)$ with N_f quark pairs

   they are dual to each other (equivalent)

$$t \equiv \log(\mu/m)$$

Duality valid at UV ($t = 0$ or $\mu = m$) and IR ($t = -\infty$ or $\mu = 0$).

We thus assume that duality between the electric and magnetic theory is valid in the whole interval $-\infty < t < 0$ $(0 < \mu < m)$

	IR $(t < 0)$	UV $(t > 0)$
magnetic theory	N_f flavours $\tilde{N}_c$ colours	$N_f + 1$ flavours $\tilde{N}_c + 1$ colours
electric theory	N_f flavours N_c colours	$N_f + 1$ flavours N_c colours

$$\tilde{N}_c \equiv N_f - N_c$$

Magnetic theory

If we choose

$$N_f = 3\tilde{N}_c - 1$$

then the magnetic theory is weakly coupled

$$\beta_1 = 3\tilde{N}_c - N_f = 1$$

and we can calculate the flow perturbatively

$$\tilde{\alpha}_g \equiv \frac{\tilde{N}_c \tilde{g}^2}{(4\pi)^2} \quad , \quad \tilde{\alpha}_y \equiv \frac{\tilde{N}_c \tilde{y}^2}{(4\pi)^2}$$

Up to 2 loops

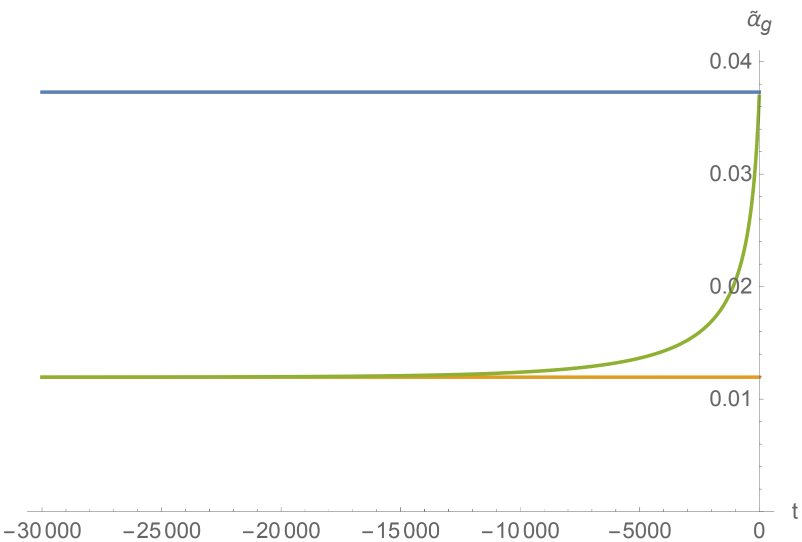
$$\frac{d}{dt}\begin{pmatrix} \tilde{\alpha}_g(t) \\ \tilde{\alpha}_y(t) \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}_g(t) & \tilde{\alpha}_y(t) \end{pmatrix} M \begin{pmatrix} \tilde{\alpha}_g(t) - \tilde{\alpha}_g(-\infty) \\ \tilde{\alpha}_y(t) - \tilde{\alpha}_y(-\infty) \end{pmatrix}$$

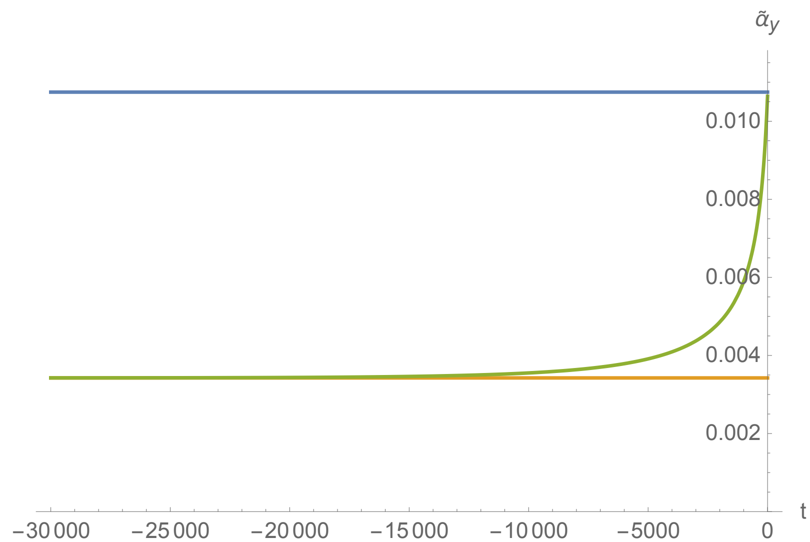
$M \ldots$ perturbatively calculable 2×2 matrix

$(\tilde{\alpha}_g, \tilde{\alpha}_y)(-\infty) \ldots$ perturbatively calculable fixed points values

This can be easily numerically solved

$\tilde{N}_c = 100 \quad (\to N_f = 299,\ N_c = 199)$





Magnetic theory solved

Electric theory

What about the electric theory?

$$\alpha_g \equiv \frac{N_c g^2}{(4\pi)^2} \quad , \quad f(x) \equiv \frac{1}{1-2x}$$

Formally we can write the RGE:

$$\frac{d\alpha_g(t)}{dt} = -\frac{6}{N_c}\alpha_g^2(t) f(\alpha_g(t)) \underbrace{\left(N_c + N_f\left(R_Q(t)-1\right)\right)}_{\text{NSVZ } \beta \text{ function}}$$

Problem: theory non-perturbative so we do not know $R_Q(t)$ except in the fixed points (R-charges of Q in the conformal field theories)

$$\begin{aligned} R_Q(0) &= 1 - \frac{N_c}{N_f+1} && \text{(UV)} \\ R_Q(-\infty) &= 1 - \frac{N_c}{N_f} && \text{(IR)} \end{aligned}$$

How can duality help?

We need a physical quantity which have to be the same in the electric and magnetic theory.

What about the a central charge?

Remember, we had two ways to define it outside the fixed points:

1) Komargodski, Schwimmer: through dilaton-dilaton ($\phi\phi$) scattering

$$a(\mu) = a_{UV} - \int_\mu^\infty d\mu \frac{\sigma_{\phi\phi\to\phi\phi}(\mu)}{\mu^3}$$

- since cross-section $\sigma > 0 \to a(\mu)$ decreasing from UV to IR (a-theorem)
- since cross-section σ is a physical quantity, so is $a(\mu)$

Non-perturbative electric $a_{el}(\mu)$ and perturbative magnetic $a_{mag}(\mu)$ should match along the whole flow

$$a_{el}(\mu) = a_{mag}(\mu)$$

Good to prove a-theorem but not easy to calculate

2) Kutasov: with Lagrange multipliers

$$a_{el} = 2(N_c^2 - 1) + 2N_f N_c a_1(R_Q) - \lambda_g (N_c + N_f(R_Q - 1))$$

Maximisation:

$$\frac{da_{el}}{dR_Q} = 2N_f N_c a_1'(R_Q) - \lambda_g N_f = 0$$

$$\rightarrow \lambda_g = 2N_c a_1'(R_Q)$$

$\rightarrow$

$$\begin{aligned} a_{el}(R_Q) &= 2(N_c^2 - 1) \\ &+ 2N_f N_c \left[a_1(R_Q) - a_1'(R_Q)\left(R_Q - R_Q(-\infty)\right)\right] \end{aligned}$$

Still we do not know what $R_Q(t)$ along the flow is

No proof that Komargodski-Schwimmer and Kutasov definitions coincide.

However possible to prove they do coincide for simple cases

We assume that they coincide in general

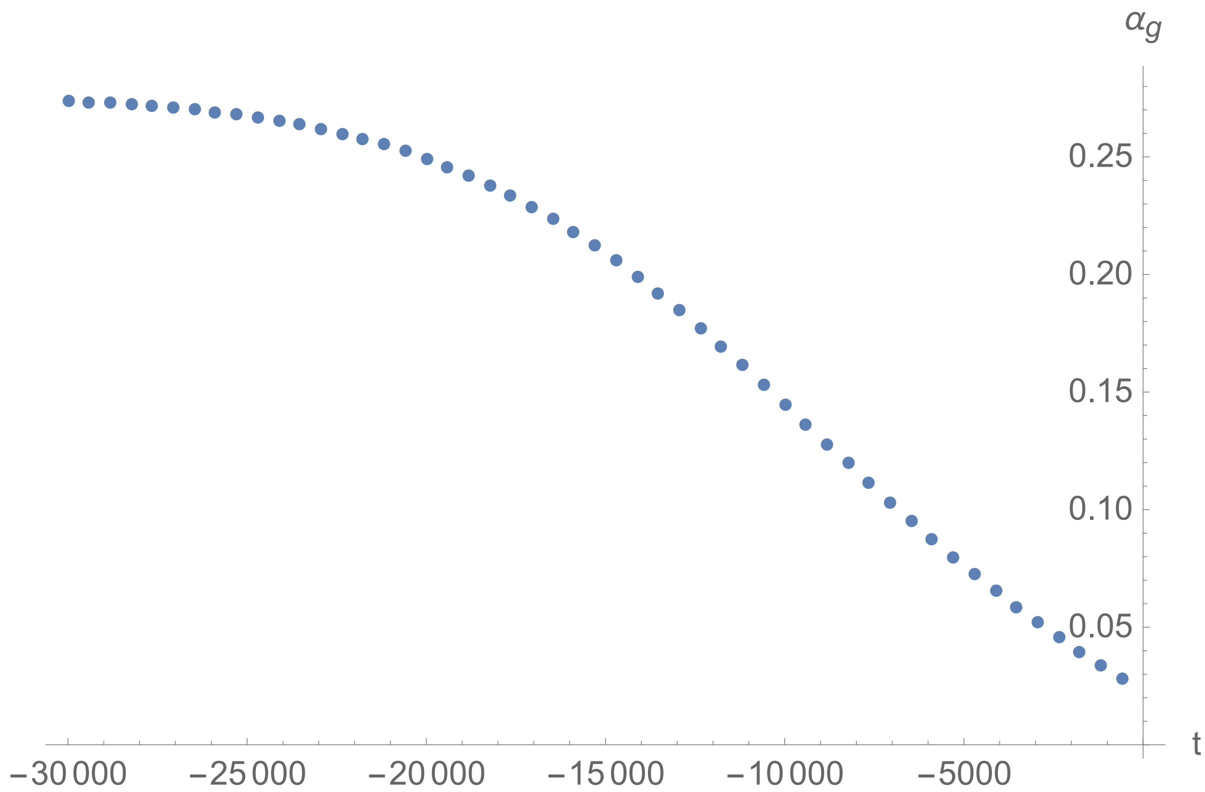
$$\to a_{el}(R_Q(t)) = a_{mag}(\tilde{\alpha}_g(t), \tilde{\alpha}_y(t))$$

r.h.s. perturbatively calculable $\to R_Q(t)$.

Then from formal RGE

$$\frac{d\alpha_g(t)}{dt} = -\frac{6}{N_c}\alpha_g^2(t) f(\alpha_g(t)) \left(N_c + N_f \left(R_Q(t) - 1\right)\right)$$

$\to \alpha_g(t)$ once we choose $\alpha_g(0) < 0.0216$



α_g
0.25
0.20
0.15
0.10
0.05
t
−30 000
−25 000
−20 000
−15 000
−10 000
−5000

Notice that

- magnetic $\tilde{\alpha}_g$ decreasing towards IR (more perturbative)

  but

  electric α_g increasing towards IR (more non-perturbative)

  $\rightarrow$ strong-weak duality

- There is only one $\alpha_g(0)$ which is correct, duality is exact only there.

  Unfortunately we do not know which one.

Conclusion

- a_{KS} physical because uniquely connected to dilaton-dilaton cross-section
- a_K defined along the flow as function of the R-charges (anomalous dimensions)
- some hint that $a_{KS}(\mu) = a_K(\mu)$
- assuming it correct $\rightarrow$ explicit example of Seiberg duality in the whole flow (modulo one number)