An introduction to astroparticle theory: basic cosmology and the dark matter problem

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I. ON THE GENERAL NOTION OF ASTROPARTICLE PHYSICS

Broadly speaking, *astroparticle physics* encompasses:

- The application of methods and tools (including experimental ones!) from particle physics to astrophysical and cosmological systems. Sometimes the latter is more specifically called *particle cosmology*.
- The use of astrophysical and cosmological observations to learn about fundamental properties of matter constituents and interactions (according to Zeldovich, one of the founding fathers of particle cosmology, a theoretical "poor man's accelerator", which is actually often much more powerful than anything that can be built by human beings).

A number of classical problems of XX$^{\rm th}$ century physics falls under the umbrella of this relatively new discipline. Examples are the (nowadays solved) solar neutrino problem, the origin of cosmic rays, etc. In the following, I will concentrate on the paradigmatic case of "dark matter" (DM). This is a (still open) problem which clearly shows the above-mentioned interplay and involves a number of systems and research directions. But it is far from being unique: others major topics include the origin of matter-antimatter asymmetry ("baryogenesis"), the origin of structures in the universe and of its initial hot phase (inflationary physics, reheating, etc), the understanding of its current acceleration (dark energy and the like), etc.

Exercise: As in particle physics, astroparticle physics practitioners tend to use natural units (where $c = k_B = \hbar = 1$). On the other hand, differently from some convention frequent among people working on gravity, we retain $G_N \equiv M_P^{-2} = (1.22 \times 10^{19}\,{\rm GeV})^{-2}$. If you are unfamiliar with these units, practice a bit!

- Compute your typical body temperature (assuming you are still alive) in eV.
- Check the working frequency of your mobile phone. Rephrase it into eV.
- Compute your height in eV^{-1}.
- Compute your age in eV^{-1}.
- Compute your density (estimate it with $\mathcal{O}(10\%)$ error from what happens when you jump into the Annecy lake+ Archimedes' law) in eV^4.

II. INITIAL EVIDENCE AND BASIC PROPERTIES OF DARK MATTER

First (and historical) notion of "dark matter" (DM) to get accustomed to:

> *In a number of astrophysical bound systems, one finds a mismatch between the mass inferred by its gravitational effects and the mass inferred by other observables (electromagnetic ones), with the former much larger than the latter. The excess of the former with respect to the latter is dubbed DM.*

A couple of comments:

- For the moment there is no implication that DM is an exotic form of matter. It might still be ordinary matter which does not shine (e.g. dim stars, planets, cold and/or rarefied gas, etc.).
- The DM notion implicitly assumes that the theory of gravity used (Einstein GR, most often in its Newtonian limit, in fact!) is correct.
- The fact that it is denoted as "matter" (as opposed e.g. to radiation) has to do with the fact that its effects are inferred in bound systems, so that DM must "cluster" and form structures (this is very different, for instance, from "dark energy" that you will not probably touch upon in this school.)

Despite what you may have heard, you see that the DM hypothesis is born as a rather conservative explanation of observations. Let us sketch a few of the arguments leading to its (purely gravitational, then like now!) discovery, which goes back to the 30's of the XX$^{\rm th}$ century. For this, it is worth introducing some basic notions.

A. Vlasov and Jeans equations

You are probably familiar with Liouville's equation, expressing the conservation of phase space density of a system of N particles (function of 6N space and momentum variables, plus time) under Hamiltonian evolution.

For a system of particles only subject to an "average" gravitational force associated to the potential (per unit mass [1]) Φ, the phase space density factorizes into the product of N one particle phase space distributions function of one particle space and momentum variables, plus time. Each single-particle distribution function f obeys

$$\left[\frac{\partial}{\partial t}+\vec{v}\cdot\frac{\partial}{\partial\vec{x}}-\frac{\partial\Phi}{\partial\vec{x}}\cdot\frac{\partial}{\partial\vec{v}}\right]f=0\,. \tag{1}$$

If Φ, rather than being externally assigned, is generated by the particles themselves, the above equation takes the name of *Vlasov equation* (a similar formalism applies in plasma physics, by the way, with gravity replaced by electromagnetism). The above equation then constitutes a non-trivial problem, which must be solved in association to the *Poisson equation* (in the Newtonian limit):

$$\nabla^2\Phi=4\pi G_N n\equiv 4\pi G_N\int d^3\vec{v}\,f\,, \tag{2}$$

where we introduced the spatial density function $n(\vec{x})$ as the integral of the distribution function f over velocity. A more manageable set of equations only in spatial coordinates can be obtained with the method of moments, leading for the zeroth and first moment of velocity to the so-called *Jeans Equations* (summation over repeated index implicit)

$$\frac{\partial n}{\partial t}+\frac{\partial(n\bar{v}_i)}{\partial x_i}=0 \tag{3}$$

$$n\frac{\partial\bar{v}_j}{\partial t}+n\,\bar{v}_i\frac{\partial\bar{v}_j}{\partial x_i}=-n\,\frac{\partial\Phi}{\partial x_j}-\frac{\partial(n\,\sigma^2_{ij})}{\partial x_i}\,. \tag{4}$$

Above, we introduced the expectation value of the velocity dispersion squared $(v_i-\bar{v}_i)(v_j-\bar{v}_j)$

$$\sigma^2_{ij}\equiv\overline{v_iv_j}-\bar{v}_i\bar{v}_j\,. \tag{5}$$

Proof: The first equation follows trivially from integrating Eq. (1) over the velocity space and the divergence theorem on f, noting that this function goes to zero for large $\vec{v}$. The second equation follows from multiplying Eq. (1) times v_j and the divergence theorem on f, then integrating. The final form is obtained by subtracting from the equation thus obtained the first equation times $\bar{v}_j$. The explicit derivation is left as an exercise, or see [1], Sec. 4.8.

Some comments:

- Analogous equations are obtained in fluidodynamics, and are known as *(continuity and) Euler Equations:*

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0 \tag{6}$$

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{1}{\rho}\nabla P-\nabla\Phi\,. \tag{7}$$

  where we introduced the fluid density ρ, and the pressure P replaces the σ^2_{ij}, assuming only isotropic stress present. This is one reason why one sometimes speaks of DM "fluid", and uses other similar notions by analogy. Be aware of the limitations, though!

- Neither the Jeans equations nor the Euler ones are closed equations. They are just the first two moments of a hierarchy which is in principle infinite (i.e. one would need a third equation for the six independent quantities σ^2_{ij}, in turn dependent on the third moments of velocity, etc.) However, in some approximations of particular relevance (e.g. some symmetry assumptions, a specified equation of state. . .) a closure is possible.

- Eq. (1) is itself an approximation, notably:

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[1] We shall implicitly make throughout the hypothesis that DM constituents all share the same mass.

a) It only describes the one-point distribution function, not addressing higher-order correlations between particles, including so-called collisional/two-body/relaxation effects in gravitational interactions, i.e. the fact that the gravitational interactions among DM particles are "granular", not strictly-speaking mean-field ones (for those of you who are familiar with it, it is only the first equation of the *BBGKY hierarchy*).

   This is usually an excellent approximation, due to the very large number of DM "particles" involved, but one exception of some importance is the case of stellar mass or heavier Black Holes (for a recent popular science review of this newly popular idea, see [2].)

b) It is a classical (as opposed to quantum) equation.

   This is also ok for almost all DM candidates, one exception being e.g. "fuzzy" DM candidates whose de Broglie wavelength is of astrophysical scale, an idea recently becoming again popular following [3].

c) It also ignores short-range interactions of non-gravitational nature, which are in general responsible for source/sink terms at the RHS. This may be suitable to describe DM system, but is certainly inadequate to deal with DM production, or techniques for its non-gravitational detection (more on this later).

1. *Application I: Oort's method to infer local matter density*

Let us assume a steady state solution of Jeans equations, $\partial n/\partial t = 0$. Using the resulting Eq. (3) in Eq. (4) leads to

$$\frac{\partial (n\,\overline{v_i v_j})}{\partial x_i} = -n\,\frac{\partial \Phi}{\partial x_j}\,. \tag{8}$$

To a first approximation the Galaxy (as seen from the Sun) can be modeled as as disk, homogeneous in the $x-y$ direction and much more extended radially than vertically. In this limit, the dominant gradients in the gravitational potential are vertical (hence derivatives with respect to z), and the only component of the Eq. (8) which matters is the z component, leading to

$$\frac{1}{n}\frac{d(n\,\overline{v_z^2})}{d\,z} = -\frac{d\Phi}{d\,z}\,. \tag{9}$$

This equation implies that the gradient of the gravitational potential can be deduced provided that one has access to two ingredients: i) A *tracer* of the density as a function of height above the plane, like for example a class of bright stars. It is important to note that it only matters that their number density $n_{\rm tr} \propto n$, since Eq. (9) is left unchanged by a $z-$independent rescaling of n. ii) A measurement of the vertical velocity dispersion of this tracers, $\overline{v_z^2}$, always as a function of z. On the other hand, the variation in the gradient with z allows one to infer the density of gravitating mass via the Poisson Equation,

$$\frac{d^2\Phi}{d\,z^2} = 4\pi G_N\,n\,. \tag{10}$$

Back in the thirties, Oort used K giant stars as tracers to develop such a program [4]. Recently, such methods infer (modern notation/values) a total *gravitational* density around the solar system of the order of $10\times 10^{-21}\,{\rm kg\,m^{-3}}$, as opposed to the estimated value from "star counts" of the order of $4\times 10^{-21}\,{\rm kg\,m^{-3}}$. Actually the majority (or at least a fraction similar to the visible one) of the mass in our neighborhood might be in some *unseen* form.

2. *Application II: Spherical systems, applications to Galaxy Clusters*

What is usually reported as the first "awareness" of the DM problem came from clusters of galaxies. In particular, F. Zwicky applied the virial theorem (obeyed by particles bounded in a system by conservative forces), linking average kinetic and gravitational potential energy $2\langle T\rangle = -\langle U\rangle$, to the Coma cluster [5]. consider either one Galaxy of mass m orbiting a in a spherically symmetric system, at a distance r from a the center of the mass distribution, whose cumulative enclosed mass is $M(r)$. One has (using ergodic hypothesis)

$$2\frac{m\sum\overline{v_i^2}}{2} = \frac{G_N M(r) m}{r}\,, \tag{11}$$

hence from an estimate of the velocity dispersion (from Doppler shifts in galactic spectra) as a function of distance r (inferred e.g. with geometric methods plus astronomical distance ladder) one can estimate the total mass $M(r)$, to be compared e.g. with photometric estimates. The former was (and still is) systematically larger than the latter.

Let us approach the problem from the point of view of Vlasov-Jeans equation. Galaxy clusters are generically fairly round and slowly-rotating. It is useful to derive the analogous "Jeans equation" (do it as a exercise, following e.g. [1], 4.210 and following) assuming spherical symmetry and stationarity, one obtains

$$\frac{d(n\,\overline{v_r^2})}{d\,r}+2\frac{\beta}{r}n\overline{v_r^2}=-n\,\frac{d\Phi}{d\,r}\,. \tag{12}$$

where one also writes

$$\frac{d\Phi}{d\,r}=\frac{G_N M(r)}{r^2} \tag{13}$$

in terms of the $M(r)$, total mass enclosed within a distance r and we have introduced the velocity anisotropy parameter

$$\beta(r)=1-\frac{\overline{v_\theta^2}+\overline{v_\phi^2}}{2\overline{v_r^2}}\,. \tag{14}$$

One usually rewrites Eqs. (12,13) in the form

$$M(r)=-\frac{r\,\overline{v_r^2}}{G_N}\left[\frac{d\log\rho}{d\log r}+\frac{d\log\overline{v_r^2}}{d\log r}+2\beta(r)\right]. \tag{15}$$

We immediately see what is the problem to infer a mass (profile) from the above equation: while radial profiles of some tracers (galaxies) can be obtained relatively easily, one needs *two* functions of the velocity distribution, while only the velocity distribution along the line of sight is accessible (spectroscopic determination of motion via Doppler shift). In modern times, this problem is usually addressed in clusters of galaxies by using the baryonic gas "as tracer" of the underlying gravitating mass: *If we assume* hydrostatic equilibrium in the gas, it obeys

$$\frac{dP_{\rm gas}}{dr}=-\frac{G_N\,M(r)\,\rho_{\rm gas}}{r^2} \tag{16}$$

by further assuming the thermal hypothesis, using $P=k_B\,T\,N/V$ with N the number of particles in the gas, which is written for a ensemble of average mass μ in units of atomic mass units m_u, $N=m/(\mu\,m_u)$, one derives $P_{\rm gas}=(\mu\,m_u)^{-1}k_B\,T_{\rm gas}\,\rho_{\rm gas}$ and

$$M(r)=-\frac{k_B\,r\,T_{\rm gas}}{\mu\,m_u\,G_N}\left[\frac{d\log\rho_{\rm gas}}{d\log r}+\frac{d\log T_{\rm gas}}{d\log r}\right]. \tag{17}$$

From X-ray maps, one can deduce the enclosed mass via the gas density profile (correlated with intensity of the emission) and the temperature profile (related to the energy of measured X- rays), for an actual example see e.g. [6].

Compare Eq. (17) with Eq. (15): they would be formally identical in the case of velocity isotropy (i.e. $\beta=0$) and "thermal" assumption for the velocity distribution $\overline{v^2}\propto T$. If these hypotheses are often (not always!) justified for the baryonic gas which is subject to "collisions", generically they are not true for DM.

3. *Application III (sketch): Galactic rotation curves*

Already in the 30's (see for instance [7]) it was noted that a Galaxy like Andromeda presented an *unexpectedly large* circular velocity at large distances from its center. It was only by $\sim$ 1970, however—also thanks to a number of technical improvements, such as radioastronomy, 21 cm tracer, improved spectroscopic surveys—that some pioneers like V. Rubin and W. K. Ford Jr. embarked in a systematic campaign to obtain rotational curves of Spiral Galaxies up to their faint outer limits, eventually reporting a universal indication for a flattening of these curves, see e.g. [8]. We shall sketch very simply where the puzzle comes from: At large distances from the "center"(≡where virtually all of its luminous matter is concentrated) of the galaxy, one observes

$$v^2_{\rm rot,obs}\simeq const. \tag{18}$$

as opposed to the expectation

$$v^2_{\rm rot,exp}\simeq\frac{G_N\,M(r)}{r}\propto\frac{1}{r}\,, \tag{19}$$

where $M(r)$ is the mass enclosed within r, and the last equality is expected if all the mass is concentrated at small distance from the center of the distribution ($r \simeq 0$), like (after all!) for the Solar System. This problem could be solved by invoking a sizable amount of gravitating matter, roughly distributed in a spherical halo extending well beyond the luminous core of the galaxies, with profile $\rho(r) \propto r^{-2}$, so that $M(r) \simeq \int dr 4\pi\, r^2 \rho(r) \propto r$. By the way, the need for something of the sort was also obtained independently, via the first numerical simulations in the 70's, addressing the problem of the stability of disk galaxies: in articles such as [9] it was argued that rotationally supported systems (like the observable stars in many spiral galaxies, including the Milky Way) were unstable, and their disk appearance should be lost in a few dynamical scales in favour of elongated structures. A way out was identified in the possibility that large, roughly spherical halos mostly supported by random velocities, were extending way beyond the observed luminous region.

Evidence for DM at galactic scales is perhaps the most troublesome: on the plus sign, it is comforting that something similar to what happens at cluster scales (Mpc) and, as we will see, in cosmology (Gpc) is also found down to kpc scales (basically, the dwarf galaxy scale). Additionally, the details of the DM distribution in our Galaxy (see e.g. [10]) and nearby ones is important towards DM identification via direct or indirect techniques (see other lectures). Finally, at "sociological level", they came in at an epoch where people were more willing to take these evidences seriously, and the first particle physics models which could "naturally" accommodate for such phenomena were being elaborated. It is undeniable that since the end of the 70's or early 80's the DM problem has grown in prominence and consideration. However, these determinations are still relatively shaky with respect to the ones at larger scales, being affected by a number of assumptions (e.g. *assumed* symmetries of the system, steady state configurations) and by important, highly non-linear processes involving the baryonic material (such as feedbacks via supernovae explosions, star-burst episodes accompanied by winds, central black holes activities, etc.). The galactic scale is also an arena where modifications of gravity (MOND and the like), at least at phenomenological level, seem to account for a number of observations much more easily than naive expectations from vanilla DM models. It is also possible that the phenomena involving DM at these scales might depend to some extent by some "non-minimal" properties of the constituents of DM (like their non-gravitational interaction, some peculiar velocity distribution, etc.). This is an extremely active field of research although at the moment it remains unclear how to disentangle clearly such putative effects from more mundane (but difficult to model) non-linear baryonic effects. The latter, while difficult to model, have seen a major improvement these years thanks to the major advances in simulations allowed by currently available computer power.

III. NOTIONS OF COSMOLOGY

Consider the Newtonian toy model of a uniform sphere of "dust", i.e. a fluid for which $P \ll \rho$. The acceleration of a particle at distance a from the center of the sphere is

$$\ddot{a} = -\frac{G_N\, M}{a^2}\,,\ M = \frac{4\pi}{3}\rho a^3\,, \tag{20}$$

which, by integration, leads to

$$\frac{\dot{a}^2}{2} = \frac{G_N\, M}{a} - \frac{k}{2}\,. \tag{21}$$

Provided that we reinterpret it carefully, this is actually pretty close to the cosmological solution of interest obtained in general relativity (GR)! In GR, the dynamical quantity is not a point particle, but the spacetime itself; the key quantity is its metric $g_{\mu\nu}$, which is a *field* generalizing the Minkowski one $\eta_{\mu\nu}$, and responds to all types of energy (and pressure). You saw that in the Gravitational Wave lectures, to some extent. Solving Einstein equations is in general extremely complicated, notably due to their non-linear nature. Whenever a *symmetry principle* can be invoked, it restricts the form that the metric can have and greatly facilitate the task!

In cosmology, one postulates that on sufficiently large scales, spatial homogeneity and isotropy should hold (*Cosmological principle*). This is a "generalized Cpernican principle", whose accuracy must be checked a posteriori (and it has been. It works fairly well above about 70 Mpc). Under this principle, the metric $g_{\mu\nu}$ can be described in terms of: i) a single independent function of time, $a(t)$, a "scale factor" describing the stretching of space as function of time. ii) a single number $k = 1, 0, -1$, describing the curvature of the 3D space (current data are consistent with $k = 0$ at the 1% level, so we stick to this, so-called "flat" universe solution. But don't be confused, the 4D Universe is not flat! Only its spatial 3D hyper-slices are!) You see thus that we have reduced the metric dynamical variables from all the independent components of $g_{\mu\nu}$ to the single function $a(t)$! That's the power of symmetry!

The *cosmological principle* justifies seeking cosmologically relevant solutions to the Einstein eqs. in terms of a homogeneous and isotropic metric (at least statistically speaking and at large scales). The key equations for the time evolution of scale factor a and energy density ρ in the smooth (and *3D spatially flat*) universe are the so-called *Friedmann equations*

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N\,\rho}{3}\,, \tag{22}$$

$$\dot{\rho} + 3H(\rho + P) = 0\,, \tag{23}$$

which must be supplemented by an equation of state, typically of the form $P = P(\rho)$. *Note that in GR pressure is also a source of gravitation, contrarily to the Newtonian analogue eq. (21). Check by time-deriving eq. (22)*). The most relevant types of energy considered in cosmology are: matter ($P \simeq 0$), radiation ($P = \rho/3$), and cosmological constant ($P = -\rho$).

Exercise: Replace H in the second equation above in order to obtain an equation in terms of derivative of ρ wrt a (no time explicitly). Prove than that $\rho \propto a^{-3}$ for matter, that $\rho \propto a^{-4}$ for radiation, and that $\rho = const.$ for cosmological constant.

The expansion of the universe implies a decreasing density of "matter particles" according to $n \propto a^{-3}$. This is usually stated as a particle conservation within a volume of universe, if accounting for its stretching, or equivalently by saying that *the comoving density is conserved*, $n\,a^3 = const.$. Similarly, a physical distance d, divided by the factor a, is dubbed "comoving distance", d/a. (The usual convention is to set $a = 1$ at the present epoch.) The radiation undergoes instead a further dilution due to the stretching of its wavelength. This phenomenon is called cosmological redshift (z), and

$$1 + z = \frac{\lambda_{\rm obs}}{\lambda_{\rm emiss}} = \frac{a_0}{a_{\rm emiss}}\,. \tag{24}$$

The scale factor evolution writes as

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \Omega_\Lambda + \Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r,0}\left(\frac{a_0}{a}\right)^4\,, \tag{25}$$

where $H_0 \equiv h\,100$ km/s/Mpc is the current value of the Hubble expansion constant ($h \simeq 0.7$), and the $\Omega_{i,0}$ are the energy density contents in different species normalized to the critical one today, $\Omega_{i,0} \equiv \rho_{i,0}/\rho_c$, $\rho_c \equiv 3H_0^2/(8\pi G_N)$.

Exercise: Compute the value of ρ_c in SI as well as natural units, and of H_0 in natural units.

Exercise: Use Friedmann eqs. to show that in a universe whose energy content is dominated by matter, one has $a \propto t^{2/3}$, in one dominated by energy, one has $a \propto t^{1/2}$, whereas if a cosmological constant dominates, $a \propto \exp(H_0\,t)$.

Exercise: Based on the above, justify why one talks of a "Radiation era", followed by a "Matter era", and eventually by a "Cosmological constant" era.

A. The three pillars of Hot Big Bang Cosmology

If one only observes the "local universe", so that the effects of time evolution of the density are negligible, from eq. (22) one derives that $\dot{a} = H\,a$. Hence, as long as local motions can be neglected with respect to the cosmological "flow", we have a proportionality of recession velocity $\sim \dot{a}$ wrt distance $\sim a$. Albeit in the local universe proper motions introduce a scatter, Hubble managed to clearly observe this correlation, nowadays known as Hubble law, which is one of the three pillars of the so-called *Hot Big Bang Cosmology*.

You are certainly familiar with the notion that the total electromagnetic energy density associated to a black-body is proportional to T^4 (for the radiated energy, this is known as Stephan-Boltzmann law). Together with the result that $\rho \propto a^{-4}$ for radiation, we get the prediction that $T\,a = const.$. Since a has been growing with time (Hubble law), this implies that the universe was hotter in the past. If we can extrapolate, we infer that it must have been so hot that e.g. it was once dissociated in a plasma state, and attaining (local) thermal equilibrium.

This notion leads to the expectation of a residual blackbody radiation, emitted when the universe was that young and hot (notably, predicted by George Gamow). This has been measured by Penzias and Wilson in 1964 (Nobel prize 1978), and is now known as Cosmic Microwave Background (CMB), since the once extremely hot thermal photons associated to $e - p$ combination into hydrogen have now cooled down to microwave wavelengths, due to a stretching factor exceeding 1000.

Exercise: Convert the CMB temperature of ~ 2.7K in eV. Compare with the ionization energy of the hydrogen. Estimate the redshift at which the (bulk of) CMB photons had comparable energies.

Finally, extrapolating even further back, we expect the content of the universe can reach even higher temperatures, comparable to the ones found in stellar cores or even higher. At very early times, the universe has experienced thermonuclear reactions in equilibrium! The model is able to predict the relative abundance of nuclear ashes (notably light elements, such as deuterium, helium-3, helium-4) coming out of such a phase, so called Big Bang nucleosynthesis (BBN). The remarkable agreement with abundances inferred in pristine environments (helium in very old stars, deuterium in very old gas clouds...) is the third pillar.

IV. MORE MODERN EVIDENCE FOR DARK MATTER

None of the evidences recalled in Sec. I convincingly argues in favour of some exotic species constituting the DM. After all, it is not an unusual situation in astrophysics not to have a complete "account" of what is known to exist: For instance, we do know that a large number of small bodies populate the outer solar system beyond Neptune (possibly including also planetary-sized objects) but we are very far from a complete census of them. Most of the stars in the universe are very dim and escape detection (actually, even in our own Galaxy!). A large chunk of the baryonic material in the universe (whose existence we infer from cosmological arguments) is still unidentified ("missing baryons"), possibly residing as hot and very rarefied gas around galactic halos, or in between them [11]. And so on.

The situation has however changed dramatically with some modern evidence for DM:

> *Cosmological evidence strongly suggests that the DM phenomenon: a) should be attributed to a (yet) unidentified gravitating species, rather than to a new, unaccounted force among known constituents. b) cannot be accounted for via the known particles and forces of the standard model of particle physics, requiring some new ingredient.*

These conclusions are heavily based on cosmological perturbation theory, applied to the CMB anisotropies and clustering properties of the large scale structures (LSS). The quantitative tools to study those are more advanced, and cannot be introduced but with dedicated lectures. Here I will just summarize a few facts and the overall logic for these conclusions.

- Of course, the current universe at small scales is far from homogeneous and isotropic. Where does Clusters, Galaxies or stars come from? We know that that they can form by gravitational instability: if instead of being initially perfectly homogeneous and isotropic, the universe had minor departures from that state, under the pull of gravity and the right timescale and conditions they may grow into the structures we see today.

- We observe $\sim 10^{-5}$ anisotropies in the CMB, as well as a peculiar angular power spectrum, with peaks and throughs. The CMB was "freed" at an epoch corresponding to $z \simeq 1100$, before which photons and baryons (mostly protons, via their respective couplings with electrons) were strongly coupled. These anisotropies reflect (roughly with decreasing importance): i) the temperature fluctuations at those recombination times ii) the gravitational redshift (blueshift) experienced in climbing up (down) the potential wells at the recombination time (*early Sachs Wolfe effect*, ESW) iii) a Doppler effect due to the relative velocity. iv) the same effect as ii), but integrated along the line-of-sight during the whole history of the universe/photon propagation, influenced by the following growth of structures (*integrated Sachs Wolfe effect*, ISW).

- Due to the e.m. coupling, the perturbations in the baryon density are linked to the ones in photons and of the same order at $z \simeq 1100$. If we were to run the Universe forward with those initial conditions (and no other component), the time between there and now would simply be insufficient for structures to grow to the observed level, under purely gravitational instability.

- It is conceivable that the level of structures quantified e.g. by the power-spectrum of density contrast of LSS, might still be attained by assuming some sort of Modified Gravity (MG), with a stronger than standard growth of perturbations (this is at least qualitatively achieved in some so-called TeVeS theories [12] [2]). However, the baryonic fluid keeps memory of its former coupling to the photons, via peculiar oscillations in Fourier space ("acoustic" waves in the plasma). These are not due to gravity, but to the e.m. coupling, and one can hardly see how to "undo" them in MG. An account of this issue is given in [13]. Even the possibility that these features are missed due to observational problems and "smoothing" must be discarded, since we *do see* these features, at a "suppressed" level (since baryons only make a small fraction of the total matter), in the LSS (this are the so-called baryon acoustic oscillations or BAO).

- These problems disappear if we add a single ingredient: a DM "fluid" which gravitates but is not electromagnetically coupled to photons before recombination. Even if the initial perturbations created by some early universe mechanism were shared by photons, baryons and DM (as we believe it is the case to a high degree of approximation, in the

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[2] Note that this *does require* extra fields. DM would thus be interpreted in MG as a manifestation of one or more "classical" fields, as opposed to "particles", i.e. excitations of a quantum field. But the need for extra ingredients in addition to the Standard Model of Physics would be the same, with the difference that most of the extended gravity sectors that might still make sense classically are known to be flawed at the quantum level. This problem is not present in any presently viable DM theory. Also, some authors intend DM exclusively as particles having other interactions in addition to gravity, but this is an unnecessary requirement. Be aware, thus, that the difference between modifying gravity with extra fields or adding extra matter fields with purely gravitational couplings is less sharp-cut than usually sold, see e.g. [14].

so-called inflationary model with adiabatic initial fluctuations), by the time photons and baryons recombine, the DM density contrast has already grown somewhat: first logarithmically with the scale factor, in the radiation-dominated phase, then linearly in the matter dominated phase. The baryons can thus fall within the already significant DM potential wells soon after the photons recombine, and structures can form sufficiently early to agree with observations.

- A quantitative determination of how much DM is needed with respect to baryons is remarkably consistent across observables (CMB, LSS, Clusters). For instance, at least at the 10% level, the $\Omega_b/\Omega_{\rm DM}$ ratio inferred from CMB agrees with the one inferred from clusters of galaxies, suggesting that we are observing the same phenomenon at different scales. Additionally, since the CMB parameters are inferred in a *linear perturbation* phase, when departures from homogeneity and isotropy are only tiny, this indicates that the DM cannot be attributed to unidentified, but otherwise "ordinary" baryonic matter. In fact, Ω_b inferred from CMB is even consistent with the Ω_b deduced from primordial nucleosynthesis considerations, going back to a much earlier period.

Another important piece of evidence has been obtained in the last decades via gravitational lensing: on the one hand, detailed analyses of lensing due to clusters on background galaxy images requires "gravitating mass" in-between the galaxies of the clusters (and more smoothly distributed than them), dominating the total potential. On the other hand, there are several cases where colliding clusters of galaxies (bullet clusters, train-wreck, etc.) show segregation effects, with the cluster gas (which dominates the total baryons in clusters, with stars being one order of magnitude less important) forming shocked fronts detected via X-rays, while lensing shows the gravitating mass passing through the shock front unscathed, just like galaxies (and collisionless species) do (for a review, see e.g. [15]). Actually, recently even secondary effects on the CMB due to lensing of the gravitational structures crossed have provided another independent cross-check of the DM scenario.

V. DM PROPERTIES AND BASICS ON MODELS AND PRODUCTION MECHANISMS

A. Cosmologically inferred properties

Cosmological and astrophysical observations do not only tell us about the existence of the DM phenomenon, also tell us something about DM properties. Here is a(n incomplete) list:

1. DM abundance

Data tell us *how much* DM is out there, as summarized via the parameter $\Omega_{\rm DM}$. Later on, we shall dwell on how it can be computed, in some given models.

2. DM velocity distribution

DM must have a *non-relativistic* velocity distribution, so it cannot be "hot" as the SM neutrinos are (which is one of the reasons SM neutrinos cannot make the DM, besides not having a sufficiently large mass).This has essentially to do with the properties of large-scale structures that we observe: a too fast DM candidate would not settle into potential wells below a characteristic scale, known as *free-streaming scale*. For a given DM candidate, one can estimate this scale (no proof given here, but should sound reasonable to you!)

$$\lambda_{fs}(t) = a(t)\lambda_{fs}^{\rm com} \simeq a(t)\int_{t_{\rm kd}}^{t} dt' \frac{\overline{v}_{\rm kd}}{a(t')^2}\,. \qquad (26)$$

where $\overline{v}_{\rm kd}$ is a typical velocity the particle has at the time it decouples from the plasma, $t_{\rm kd}$. One clearly has to ask it to be at most as large as kpc, where DM structures have been detected.

3. DM lifetime

DM must be remarkably long-lived for particle physics standards, since it has to survive at least for the age of the universe (actually, at least one order of magnitude longer, see e.g. [16, 17] and possibly much more, if its decay products contain "visible" SM byproducts). Apart from some comments below, we will not indulge too much on this, but be aware that it is an extremely important constraint for model-building, see e.g. [18].

4. DM mass

DM is detected in virialized structures as small as dwarf galaxies, characterized by kpc-size and velocity dispersions of O(10) km/s or less. Its behaviour appears correctly described by classical mechanics at and above those scales. This suggests that associated quantum phenomena must be shorter scales, e.g. that its De Broglie wavelength satisfies

$$\lambda_{DB} = \frac{h}{m\,v} \lesssim 1\,\mathrm{kpc} \Rightarrow m \gtrsim 10^{-21}\,\mathrm{eV}\left(\frac{\lambda_{DB}}{1\,\mathrm{kpc}}\right)\left(\frac{10\,\mathrm{km/s}}{v}\right). \tag{27}$$

If DM is fermionic, a much stronger bound applies: i) DM obeys Fermi-Dirac statistics ("Pauli principle"), so that its initial phase-space density has the upper limit (g being its multiplicity)

$$f \leq \frac{g}{h^3}\,. \tag{28}$$

Additonally, under purely gravitational interaction the phase space density is preserved (Liouville theorem) and it can also be shown that for any observable the coarse grained phase space density must be lower than the real one, so that the above relation allows to derive a bound from current observations. This argument, initially developped in [19], currently allows to put a lower limit to fermionic DM of about 0.4 keV, see [20].

Exercise: Consider a spherical, uniform system, gravitationally bound, of mass M and radius R, made of cold fermionic particles of mass m, with spin multiplicity g. Compute the lower limit on m. Some hints/guidelines: *gravitationally bound* implies that all particles should have a velocity lower than the gravitational escape velocity (compute the latter for such a system!) *cold fermions*: the system contains $N = M/m$ particles. But the "coldest" fermion system still settles in "Fermi levels": compute the Fermi velocity as a function of M,R, m, g. *estimate:* Given the dependence on M, R, which systems are optimal to set bounds on m? Search the literature for typical values of M, R of astrophysical systems (clusters of Galaxies, Milky Way-like Galaxy, dwarf spheroidals) and to derive a(n order of magnitude) bound. *Bonus:* Direct (terrestrial) experiments tell us that neutrinos have a maximum allowed mass of about 2.2 eV. What is the maximum overall mass they could contribute to a Milky Way-sized halo (assuming that they can be "maximally packed" into it)? What if the forthcoming Katrin experiment pushes the upper limit to 0.2 eV, as expected?

5. DM self-interactions: elastic collisions

DM must be collisionless, at least if compared with typical collisional rates of baryonic gases. Too important collisions would spoil e.g. the segregated DM distribution inferred from colliding clusters like the bullet cluster, or modify a number of other observables, as reviewed in [21]. Typical bounds thus obtained are $\sigma/m \lesssim 0.1 - 1\,\mathrm{cm}^2/\mathrm{g}$, with some dependence on the velocity (in the simplest cases, some sort of "hard-sphere" scattering limit is assumed, although interactions mediated by massive particles are also considered in some cases). For comparison 1cm^2/g=1.78 b/GeV, so that these bounds are quite loose from the particle physics (as opposed to atomic physics) point of view. Note that a self-interaction near these bounds may also be beneficial for some phenomenological consequences, so that models predicting such relatively large cross-sections are currently pursued, as well as possible specific signatures, see e.g. [22].

Exercise: i) Why are the above bounds on σ/m, rather than on σ? (*hint*: The physical quantity constrained is the interaction rate. How does it write in terms of fundamental quantities?) ii) Estimate the *geometric* cross section for hydrogen atomic scattering, and compare it with the above bounds.

6. DM self-interactions: inelastic collisions

DM should be dissipationless, i.e. to a large extent it cannot dissipate its energy by emitting some sort of radiation (including "dark" one). Otherwise, DM would behave similarly to baryons, for instance forming flat disks rather than spheroidal halos. This is relevant if DM interactions are mediated by relatively light particles, which are kinematically accessible in present-day halos, despite the relatively low DM velocities. Please do not get confused by articles like [23], which may superficially make you think that DM can be dissipative: they in fact only refer to *at most* a small fraction of DM, quantified e.g. in less than 5% in [24], see also [25].

Exercise: Assume the DM particle has a mass of 1 TeV, and that the DM velocity distribution in a halo is Maxwellian. Estimate below which mass of the "dark radiation/mediator" particle its on-shell production is possible in half of the DM-DM collisions in: i) the Milky Way Galaxy ii) The Coma cluster of galaxies.

7. DM interactions with the SM

DM cannot obviously interact too much with SM particles... otherwise it would not be "dark"! For most popular DM models direct, indirect or collider searches provide the most stringent bounds on these couplings (see below). But a number of bounds have been derived from astro-cosmo observables also on interaction with SM. Definitively, cosmological bounds on the DM interaction rate with photons [26] and neutrinos [27] are the most stringent ones: If too large, they would damp power-spectrum features at small scales. Considerations from astrophysics and cosmology (examples are related to the stability of structures such as galactic disks) also exist on interactions with the nucleons, see e.g. [28] or [29] for macroscopic DM candidates.

VI. NOTIONS OF THERMODYNAMICS IN THE EARLY UNIVERSE

The cosmological principle also implies that, at least in the "cosmic" frame, the phase space distribution of particles only depends on $f = f(p,t)$, with $p = |\mathbf{p}|$, or equivalently $f = f(p,a)$. Actually, for species in thermal equilibrium, the previous arguments suggest that we can replace with $f = f(p,T)$, since we argued that $T\,a \simeq const.$, i.e. one can use the temperature of a thermal relativistic species (typically photons) as a "clock", replacing the scale factor or time as independent variable.

Provided that the interaction rates of the processes keeping a given species in thermal contact with the bath are fast enough (compared to the Universe expansion), at least locally the Universe attains thermodynamical equilibrium (why local thermodynamical equilibrium quantities seem to be shared at a "global" level is one of the puzzles suggesting the inflationary theory!). The phase-space distribution then is well known from basic statistical theory,

$$f(p) = \frac{1}{\exp\left(\frac{E(p)-\mu}{T}\right) \pm 1}\,, \tag{29}$$

where the upper $(+)$ sign refers to fermions, the lower $(-)$ to bosons.Temperature T is the intensive quantity (or a "multiplier") associated to the exchange of energy with the bath, μ the analogous one related to exchange of particles. If processes leading to *energy* exchange are "fast", one says that *kinetic* equilibrium is attained. If processes leading to *particle* exchange are "fast", one says that *chemical* equilibrium is attained. *Thermal* equilibrium usually is meant to imply *both*. At equilibrium, the above-introduced quantities attain a simple expression in terms of T (we set $\mu = 0$ in the following, and g denotes the spin multiplicity, e.g. 2 for photons, 4 for the electron-positron component, etc.):

- Number density, relativistic limit

$$n = g\int f\frac{d\vec{p}}{(2\pi)^3} = \frac{g}{2\pi^2}T^3\mathcal{J}_{\pm}(2) \tag{30}$$

- Energy density, relativistic limit

$$\rho = g\int E f\frac{d\vec{p}}{(2\pi)^3} = \frac{g}{2\pi^2}T^4\mathcal{J}_{\pm}(3) \tag{31}$$

- Pressure, relativistic limit

$$P = \frac{1}{3}\int \frac{p^2}{E} f\frac{d\vec{p}}{(2\pi)^3} = \frac{\rho}{3}\,. \tag{32}$$

In the above equations we defined (remember that for integer ℓ, $\Gamma(1+\ell) = \ell!$)

$$\mathcal{J}_{\pm}(\ell) \equiv \int_0^\infty dy\frac{y^\ell}{e^y \pm 1} = \begin{cases}\Gamma(1+\ell)\zeta(1+\ell) & -\,,\ \text{bosons}\,,\\ (1-2^{-\ell})\Gamma(1+\ell)\zeta(1+\ell) & +\,,\ \text{fermions}\,.\end{cases} \tag{33}$$

In the non-relativistic limit ($T \ll m$, $E \simeq m + p^2/(2\,m)$), the thermodynamical quantities are the same for bosons and fermions and one has the Maxwell-Boltzmann distribution, $f_{\rm MB}(p) = \exp(-p^2/(2\,m\,T))\exp(-m/T)$. From that follows

- Number density, non-relativistic limit

$$n = g\int f\frac{d\vec{p}}{(2\pi)^3} = g\left(\frac{m\,T}{2\pi}\right)^{3/2}\exp\left(-\frac{m}{T}\right) \tag{34}$$

- Energy density, non-relativistic limit (remember $E=m+p^2/(2\,m)$)

$$\rho = g\int E f\frac{d\vec{p}}{(2\pi)^3} = n\left(m+\frac{3}{2}T\right)\,. \tag{35}$$

- Pressure, non-relativistic limit

$$P=\frac{1}{3}\int\left[\frac{p^2}{m}+\mathcal{O}\left(\frac{p^2}{m^2}\right)\right]f\frac{d\vec{p}}{(2\pi)^3}=n\,T\,. \tag{36}$$

The entropy density, which for species at equilibrium writes

$$s=\frac{\rho+P-\mu\,n}{T}\,. \tag{37}$$

is also frequently used in the literature.

It is sometimes useful to rewrite the total density in the relativistic period as

$$\rho_{\rm tot}=\frac{\pi^2}{30}g_{\rm eff}(T)T^4 \tag{38}$$

or equivalently (via first Friedmann Eq.)

$$H^2=\frac{4\pi^3}{45\,M_P^2}g_{\rm eff}(T)T^4\,, \tag{39}$$

where

$$g_{\rm eff}(T)\simeq\sum_{\rm i=relat.bos.}g_i\left(\frac{T_i}{T}\right)^4+\frac{7}{8}\sum_{\rm j=relat.ferm.}g_j\left(\frac{T_j}{T}\right)^4 \tag{40}$$

counts the degrees of freedom (only the relativistic ones, the non-relativistic ones contributing negligibly in the relativistic era) and accounting for the different statistics as well as for the possibility that different species attain different temperatures. This is actually the case, to a good approximation, for neutrinos and photons at temperatures well below the electron mass, with $(T_\nu/T_\gamma)^3\simeq 4/11$, while being the same for $T\gg m_e$.

Also note that the entropy density is always dominated by the relativistic species, $s\simeq(4/3)\rho/T$ and in absence of particle creations/annihilations, it is conserved! It is useful to write it as

$$s=\frac{2\pi^2}{45}h_{\rm eff}(T)T^3\,, \tag{41}$$

where

$$h_{\rm eff}(T)\simeq\sum_{\rm i=relat.bos.}g_i\left(\frac{T_i}{T}\right)^3+\frac{7}{8}\sum_{\rm j=relat.ferm.}g_j\left(\frac{T_j}{T}\right)^3\,. \tag{42}$$

The quantity s is used (interchangeably with n_γ or T^3, more or less) to normalize densities to a comoving quantity.

Exercise: Check explicitly the above integrals.

VII. PREDICTING THE DM ABUNDANCE - WIMPS

How to produce DM? To compute that, we know for sure that we need to go beyond Eq. (1), since the initial condition $f=0$ (no DM to start with) is a solution of that equation. We shall not develop an advanced theory for that, but will just proceed heuristically, writing down the simplest equation for the number density of particles (i.e. integral of f over velocities) which provides a working DM production mechanism. This formalism basically goes back to [30]. We need to extend the SM with a (meta)stable particle X, directly (or indirectly, but let's neglect that here) coupled to the plasma of SM particles populating the early universe. In symbols, for a self-conjugated particle we have $XX\leftrightarrow({\rm SM\,thermal\,bath})$. Let us imagine that the particle is in thermal (kinetic and chemical) equilibrium. We also know it should be non-relativistic, at the epochs we care about. If m_X is its mass, we only need to know its number density at the epochs of interest, since $\Omega_X \propto m_X n_X$. In particular,

$$\Omega_X\, h^2 = \frac{\rho_X\, h^2}{\rho_c} = \frac{m_X\, n_X\, h^2}{\frac{3H^2}{8\pi G_N}} = \frac{m_X\, s_0\, Y_0}{1.054\times 10^4\,\mathrm{eV\,cm^{-3}}} = 0.274\frac{m_X}{\mathrm{eV}}Y_0\,, \tag{43}$$

where we used ($h_{\rm eff} \simeq 2 + 3\times 2(4/11)7/8 \simeq 3.91$ comes from accounting for γ's and ν's)

$$s_0 = 2889\left(\frac{T_{\gamma,0}}{2.725}\right)^3 \mathrm{cm}^{-3}\,. \tag{44}$$

Once a matter species is decoupled, we know that its evolution obeys comoving particle conservation condition,

$$\frac{dn_X}{dt} + 3H\, n_X = 0 \Leftrightarrow n_X \propto a^{-3}\,. \tag{45}$$

In the opposite limite, where the particles are fully coupled to the rest of the plasma, the abundance of the species (assumed non-relativistic) at thermal equilibrium is given by the Boltzmann distribution,

$$n_{X,\rm eq} = g\left(\frac{m_X\, T}{2\pi}\right)^{3/2}\exp\left(-\frac{m_X}{T}\right)\,, \tag{46}$$

where we implicitly assume that the rate Γ of interactions keeping the equilibrium are fast with respect to the Hubble expansion H, and that on much longer timescales than the interaction ones, the temperature adiabatically responds to the universe expansion, $T = T(a)$. In the opposite limit $\Gamma \ll H$, the number density of particles is frozen-out and particles are simply comovingly conserved (see sec. III), with $n_X \propto a^{-3}$. If the reactions keeping X in thermal contact are $2 \leftrightarrow 2$, an equation with the correct (limiting) behaviour is

$$\frac{dn_X}{dt} + 3H\, n_X = -\langle\sigma v\rangle[n_X^2 - n_{X,\rm eq}^2] \tag{47}$$

with $\Gamma \propto \langle\sigma v\rangle$ (to be specified below). Let us simplify a bit this equation by rewriting it in terms of:

i) the comoving quantity $Y \equiv n_X/s$ under the assumption that the entropy is conserved, so that

$$\frac{dY}{dt} = -\langle\sigma v\rangle\, s\,[Y^2 - Y_{\rm eq}^2]\,. \tag{48}$$

Exercise: Prove the above, by computing explicitly $\frac{dY}{dt} = \frac{d}{dt}\left(\frac{n_X}{s}\right) = \frac{d}{dt}\left(\frac{n_X\, a^3}{s\, a^3}\right) = \frac{1}{s\, a^3}\frac{d}{dt}(n_X\, a^3) = \frac{1}{s}(\dot n_X + 3H\, n_X)$.

ii) by introducing the dimensionless independent variable $x \equiv m/T$ (any mass scale is fine, although often one takes $m = m_X$)

$$\frac{dY}{dx} = -\frac{\langle\sigma v\rangle(x)\, s(x)}{x\, H(x)}\,[Y^2 - Y_{\rm eq}^2]\,. \tag{49}$$

Exercise: Prove the above, by computing explicitly $\frac{d}{dt}(a^3 s) = 0 \Rightarrow \frac{d}{dt}(aT) = 0 \Rightarrow 0 = \frac{d}{dt}(a/x) \Rightarrow \frac{dx}{dt} = Hx$.
Exercise: Formulae which can be applied also in more general cases, e.g. with entropy non-conservation:

$$\frac{dY}{dx} = -\sqrt{45\pi}M_P\, m\frac{h_{\rm eff}(x)\langle\sigma v\rangle}{\sqrt{g_{\rm eff}(x)}x}\left(1 - \frac{1}{3}\frac{d\log h_{\rm eff}}{d\log x}\right)[Y^2 - Y_{\rm eq}^2]\,, \tag{50}$$

can be found in [31, 32]. Check them out.

We can further re-write

$$\frac{x}{Y_{\rm eq}}\frac{dY}{dx} = -\frac{\Gamma_{\rm eq}}{H}\left[\left(\frac{Y}{Y_{\rm eq}}\right)^2 - 1\right]\,. \tag{51}$$

with $\Gamma_{\rm eq} \equiv n_{\rm eq}\langle\sigma v\rangle$. In this form, the equation clearly shows the importance of the ratio $\Gamma_{\rm eq}/H$ in determining the outcome. Qualitatively, we expect that when (*if!*) $\Gamma_{\rm eq}/H \gg 1$ the abundance tracks equilibrium, $Y = Y_{\rm eq}$. When $\Gamma_{\rm eq}/H \ll 1$ (which at some point will happen, given the difference temperature dependence of the two) the RHS is vanishingly small, and Y is frozen (constant). An analytical estimate of this constant value can be obtained by searching for x_F such that

$$\Gamma_{\rm eq}(x_F) = H(x_F) \quad \text{freeze-out condition}\,. \tag{52}$$

If it turns out that freeze-out happens when the particle X is relativistic, the computation of Y is relatively simple, since Y is constant in this regime, and given by

$$Y = \frac{n_X}{s} = \frac{45\zeta(3)\,g}{2\pi^4}\frac{\{1(B),3/4(F)\}}{h_{\rm eff}(x_F)}\,, \Rightarrow \Omega_X\,h^2 = 0.0762\frac{m}{\rm eV}\frac{\{1(B),3/4(F)\}}{h_{\rm eff}(x_F)} \tag{53}$$

where x_F is given by Eq. (52), i.e.

$$x_F\sqrt{g_{\rm eff}(x_F)} = \langle\sigma v\rangle(x_F) m\, M_P\{1(B),3/4(F)\}\frac{g\zeta(3)\sqrt{45}}{2\pi^{7/2}}\,. \tag{54}$$

For the case of neutrinos ($h_{\rm eff} = 10.75$, $g = 2,\ldots$), this gives $\Omega_\nu h^2 = \sum m_\nu/(94\,{\rm eV})$, which makes them unsuitable DM candidates (even forgetting about their "hotness"!) given the current upper limits on their mass.

If—as in practice required for a good DM candidate—the particle decouples when non-relativistic, Y varies in time. Although Eq. (51) cannot be integrated in closed form (it is a *Riccati* equation) we estimate the relic abundance as $Y_0 = Y_{\rm eq}(x_F)$, with

$$Y_{\rm eq}(x_F) = \frac{g}{h_{\rm eff}}\frac{45}{2\pi^2(2\pi)^{3/2}}x_F^{3/2}e^{-x_F}\,, \tag{55}$$

with the freeze-out condition Eq. (52) giving

$$x_F^{1/2}e^{-x_F} = \sqrt{\frac{4\pi^3\,g_{\rm eff}(x_F)}{45}}\frac{(2\pi)^{3/2}}{g\,M_P\,m\langle\sigma v\rangle(x_F)}\,, \tag{56}$$

implying

$$Y_{\rm eq}(x_F) = \sqrt{\frac{45\,g_{\rm eff}(x_F)}{\pi}}\frac{x_F}{h_{\rm eff}(x_F)M_P\,m\langle\sigma v\rangle(x_F)} \sim \mathcal{O}(1)\frac{x_F}{M_P\,m\langle\sigma v\rangle(x_F)}\,, \tag{57}$$

where x_F can be obtained from the Eq. (56), i.e. re-writing it as $x_F^{(i+1)} = G(\ldots,\ln x_F^{(i)})$ and starting from a first guess $x_F \sim 10$ (a more typical value for popular candidates is $x_F \sim 30$). Note

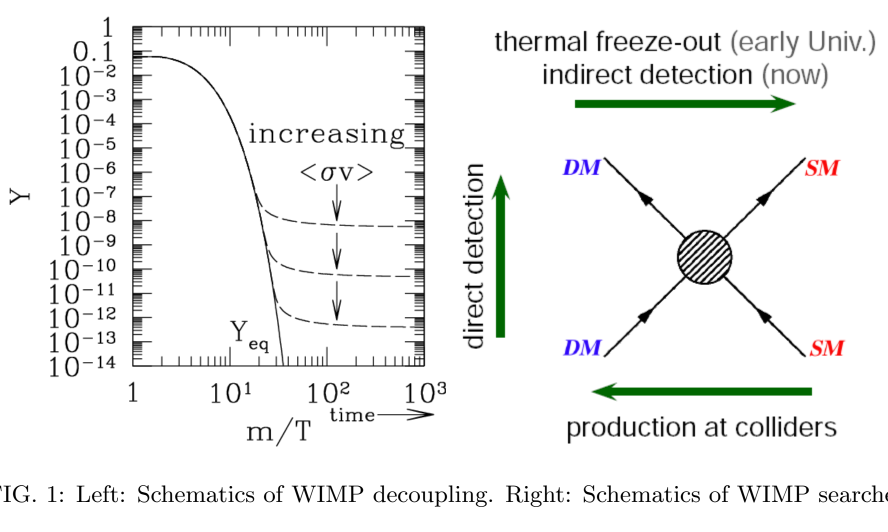

FIG. 1: Left: Schematics of WIMP decoupling. Right: Schematics of WIMP searches

- The more a particle interacts, the less of it there is: it makes sense, since the particle stays longer in thermal contact.

- a numerical estimates, assuming $\langle\sigma v\rangle \simeq const.$, leads to $\Omega_X\, h^2 = 0.1\,\mathrm{pb}/\langle\sigma v\rangle$. For electroweak scale masses and couplings (hence the name of *WIMPs, weakly interacting massive particles*) one gets the right value,

$$\Omega_X\, h^2 \simeq \frac{\alpha^2}{m^2} = 1\,\mathrm{pb}\left(\frac{200\,\mathrm{GeV}}{m}\right)^2 . \tag{58}$$

  Since however the pre-factor depends from widely different cosmological parameters (Hubble parameter, CMB temperature) and the Planck scale, one might wonder if this is a mere coincidence or it hints to a more physical link. *This is usually dubbed WIMP miracle*.

- One may reverse the argument above, which is in fact what happened with WIMP models and largely explains the popularity of the DM problem among particle physicists. If one has a strong prior for new TeV scale physics (with ew. strength coupling) due to the *hierarchy* problem (why is the Higgs light, compared e.g. to the Planck scale, given that its mass should receive quantum corrections proportional to new mass scales?). Precision ew data (e.g. from LEP) suggest that tree-level couplings SM-SM-BSM should be avoided. A straightforward solution (not unique!) is to impose a discrete "parity" symmetry e.g.: R-parity in SUSY, K-parity in extra-dimensional models, T-parity in Little Higgs. New particles only appear in pairs! On the one hand, this has some additional phenomenological benefits (e.g. respecting proton stability bounds. . .), on the other hand, it automatically makes the lightest particle of this new sector stable! In a sense, some WIMP DM (albeit we do not know if too few, too many . . . or just the right amount) is "naturally" expected for consistency of the currently favored framework for BSM physics at EW scale. Beware of the reverse induction, though: LHC is current our best tool to test this paradigm, but if no new physics is found at EW scale it is at best the WIMP scenario to be disfavored, not the "existence of DM".

- The simple treatment presented above fails in some specific cases, such as cohannihilations with other particle(s) close in mass, or when the cross-sections strongly depend on energy, for instance in presence of resonances, thresholds, etc. For more details see for instance [33, 34]. Nowadays, relic density calculations have reached a certain degree of sophistication and are automatized with publicly available software, such as MicrOMEGAs (http://lapth.cnrs.fr/micromegas/) or DarkSUSY (http://www.physto.se/˜edsjo/darksusy/)

Exercise: By using any software of your choice (including symbolic ones like Mathematica, Maple, etc.), write a simple code solving the relic abundance equation. Compare with the analytical approximations discussed during the lecture. Feel free to explore what happens under different conditions (e.g. different dependences for the cross section; epochs of entropy variations. . .).Have a look e.g. at [35] for comparison and for some "tricks" on how to make the computation more efficient (notably if you find, as you probably should, problems of numerical stiffness).

A word of caution: While WIMPs do work as DM candidates, and are still the leading option in actual searches, they are far from being unique. A whole model-builder zoology has developed over the years, with a success in the literature proportional to the number of our failed attempts at identifying DM: Feebly Interacting DM (FIMP), Strongly Interacting DM (SIMP), sterile neutrinos (produced via oscillations from the active ones), light scalar fields (e.g. axions) produced by the so-called misalignment mechanism, etc. In a limiting case, one should remember in time-dependent gravitational backgrounds, spontaneous particle creation takes place ("an initial vacuum state is no longer a vacuum state at later times"). If very massive ($m \sim 10^{13}$ GeV) stable particles exist, depending on the very early history of the universe this mechanism may produce them to a level accounting for DM. Such a candidate is virtually undetectable other than gravitationally, i.e., the way we already did detect DM. Unfortunately, this is not the unique class of "invisible" candidates. The logical possibility that we *might never experimentally know* the fundamental properties of DM is a perhaps disturbing, yet true lesson to be kept in mind. That does not mean that we should stop searching, hopefully we are not so unlucky!

One of the unique features of the WIMP scenario is that: i) they can be searched with many strategies; ii) the parameters probed can be compared across different types of searches. Although this has to be taken with a grain of salt in actual models, it has to do with the fact that the basic "toy diagram" (see right panel in Fig. 1) enters WIMP annihilations, scattering with the SM, or pair production in SM particle collisions. The fact that in many cases the detectors needed for these searches were already being developed for other purposes has also been a major plus in searching for this type of DM, on which you have heard (and will hear) more in this school by other lecturers.

1. More on the annihilation cross-section

In a collision between particle 1 and 2, one can define the average

$$\langle\sigma v\rangle \equiv \int \mathrm{d}^3\mathbf{v}_1 \int \mathrm{d}^3\mathbf{v}_2 h(\mathbf{v}_1)h(\mathbf{v}_2)\,\sigma(v_{\rm rel})v_{\rm rel}\,. \tag{59}$$

where the cross section only depends on the absolute value of the relative velocity. Remember that for any velocity distribution h one has

$$\int d^3\mathbf{v}\, h(\mathbf{v}) = 1\,. \tag{60}$$

By exploiting this fact, the independence of the cross-section with respect to translations and changing variables, we can factorize our the movement of the CM with respect to the relative velocity, hence simplifying into

$$\langle \sigma v \rangle \equiv \int d^3\mathbf{v}\, h(\mathbf{v})\, \sigma(v) v\,. \tag{61}$$

In a thermal-environment, $h(\mathbf{v}) \propto \exp[-v^2/(2\,T)]$. Away from particle-physics peculiarities (such as resonances, thresholds, etc.), keeping in mind that DM has to be non-relativistic, it makes sense to expand $\sigma(v)v$ into a power-series of velocities. In fact, partial wave expansion (see any standard textbook, including e.g. [36]) leads to a series in v^{2L}, L being the orbital angular momentum of the pair: in spectroscopic notation, $L = 0$ is the "S-wave" contribution, $L = 1$ the "P-wave", etc. In this approximation, the average over the termal distribution leads to $\langle \sigma v \rangle = a + bT + \ldots$

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