Direct CP violation in $K^0 \to \pi\pi$
Standard Model Status

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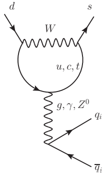
Outline

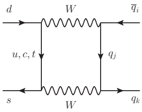
H. Gisbert, A. Pich, arXiv:1712.06147

1. **Theoretical Framework**
2. **Octet Enhancement. Phase Shifts**
3. **CP Violation: ε'/ε**

Theoretical Framework

- **Sensitivity to Short-Distance Scales:**





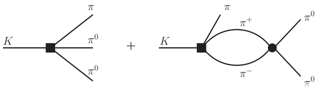
Charm mass prediction

Top quark

GIM cancellation

New Physics ?

- **Long-Distance Physics:**



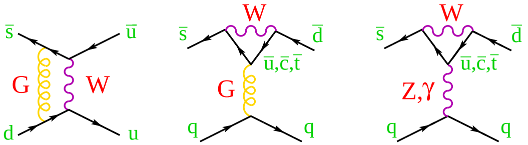
Chiral Dynamics

- **Multi-Scale Problem:**

$\log(M/\mu)$ **(OPE)** , $\log(\mu/m_\pi)$ **(χPT)**

Energy Scale	Fields	Effective Theory
M_W	W, Z, γ, g τ, μ, e, ν_i t, b, c, s, d, u	Standard Model
	$\Big\downarrow$ OPE	
$\lesssim m_c$	$\gamma, g\ ;\ \mu, e, \nu_i$ s, d, u	$\mathcal{L}_{\mathrm{QCD}}^{(n_f=3)}$, $\mathcal{L}_{\mathrm{eff}}^{\Delta S=1,2}$
	$\Big\downarrow$ $N_C \to \infty$	
M_K	$\gamma\ ;\ \mu, e, \nu_i$ π, K, η	χPT

$\Delta S = 1$ TRANSITIONS



$$\boxed{\mathcal{L}_{\rm eff}^{\Delta S=1} = -\frac{G_F}{\sqrt{2}}\, V_{ud}\, V_{us}^* \sum_i C_i(\mu)\; Q_i(\mu)}$$

$$\begin{aligned}
Q_1 &= (\bar{s}_\alpha u_\beta)_{\rm V-A}\, (\bar{u}_\beta d_\alpha)_{\rm V-A} & Q_2 &= (\bar{s}u)_{\rm V-A}\, (\bar{u}d)_{\rm V-A} \\
Q_{3,5} &= (\bar{s}d)_{\rm V-A} \textstyle\sum_q (\bar{q}q)_{\rm V\mp A} & Q_4 &= (\bar{s}_\alpha d_\beta)_{\rm V-A} \textstyle\sum_q (\bar{q}_\beta q_\alpha)_{\rm V-A} \\
Q_{7,9} &= \tfrac{3}{2}(\bar{s}d)_{\rm V-A} \textstyle\sum_q e_q (\bar{q}q)_{\rm V\pm A} & Q_{10} &= \tfrac{3}{2}(\bar{s}_\alpha d_\beta)_{\rm V-A} \textstyle\sum_q e_q\, (\bar{q}_\beta q_\alpha)_{\rm V-A} \\
Q_6 &= -8 \textstyle\sum_q (\bar{s}_L q_R)(\bar{q}_R d_L) & Q_8 &= -12 \textstyle\sum_q e_q\, (\bar{s}_L q_R)(\bar{q}_R d_L) \\
Q_{11,12} &= (\bar{s}d)_{\rm V-A} \textstyle\sum_\ell (\bar{\ell}\ell)_{\rm V,A} & Q_{13} &= (\bar{s}d)_{\rm V-A} \textstyle\sum_\nu (\bar{\nu}\nu)_{\rm V-A}
\end{aligned}$$

- $q > \mu$: $\quad C_i(\mu) = z_i(\mu) - y_i(\mu)\, (V_{td}\, V_{ts}^* / V_{ud}\, V_{us}^*)$

  $O(\alpha_s^n t^n)$, $O(\alpha_s^{n+1} t^n)$ $\qquad [t \equiv \log{(M/m)}]$ $\qquad$ Munich / Rome

- $q < \mu$: $\quad \langle \pi\pi | Q_i(\mu) | K \rangle$? $\qquad$ **Physics does not depend on μ**

CHIRAL PERTURBATION THEORY (χPT)

- **Expansion in powers of p^2/Λ_χ^2 :** $\mathcal{A} = \sum_n \mathcal{A}^{(n)}$ $(\Lambda_\chi \sim 4\pi F_\pi \sim 1.2\ \mathrm{GeV})$
- **Amplitude structure fixed by chiral symmetry**

$$\mathrm{SU(3)_L} \otimes \mathrm{SU(3)_R} \;\to\; \mathrm{SU(3)_V}$$

- **Short-distance dynamics encoded in Low-Energy Couplings**
- **O(p^2) χPT: Goldstone interactions (π, K, η)** $\Phi \equiv \frac{1}{\sqrt{2}}\,\vec{\lambda}\,\vec{\varphi}$

$$\boxed{\begin{gathered}\mathcal{L}_2^{\Delta S=1} = G_8\, F^4\, \mathrm{Tr}(\lambda L_\mu L^\mu) \,+\, G_{27}\, F^4 \left(L_{\mu 23} L^\mu_{11} + \frac{2}{3}\, L_{\mu 21} L^\mu_{13} \right) \\ G_R \equiv -\frac{G_F}{\sqrt{2}}\, V_{ud} V_{us}^*\, g_R \quad ; \quad L_\mu = -iU^\dagger D_\mu U \quad ; \quad \lambda \equiv \tfrac{1}{2}\,\lambda_{6-i7} \quad ; \quad U \equiv \exp\left\{ i\sqrt{2}\,\Phi/F \right\}\end{gathered}}$$

- **Loop corrections (χPT logarithms) unambiguously predicted**
- **LECs can be determined at $\mathrm{N_C} \to \infty$ (matching)**
- **O(p^2) LECs ($\mathrm{G_8}$, $\mathrm{G_{27}}$) can be phenomenologically determined**

$K \to 2\pi$ Isospin Amplitudes

$$A[K^0 \to \pi^+\pi^-] \equiv A_0\, e^{i\chi_0} + \frac{1}{\sqrt{2}} A_2\, e^{i\chi_2}$$

$$A[K^0 \to \pi^0\pi^0] \equiv A_0\, e^{i\chi_0} - \sqrt{2}\, A_2\, e^{i\chi_2}$$

$$A[K^+ \to \pi^+\pi^0] \equiv \frac{3}{2} A_2^+\, e^{i\chi_2^+}$$

1) $\Delta I = 1/2$ Rule: $\omega \equiv \frac{\mathrm{Re}\,(A_2)}{\mathrm{Re}\,(A_0)} \approx \frac{1}{22}$

2) Strong Final State Interactions: $\chi_0 - \chi_2 \approx \delta_0 - \delta_2 \approx 45^\circ$

$$\boxed{\varepsilon'_K = \frac{-i}{\sqrt{2}}\, e^{i(\chi_2-\chi_0)}\, \omega \left\{ \frac{\mathrm{Im}\,(A_0)}{\mathrm{Re}\,(A_0)} - \frac{\mathrm{Im}\,(A_2)}{\mathrm{Re}\,(A_2)} \right\}}$$

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$$\boxed{\begin{aligned} A_0\, e^{i\chi_0} &= \mathcal{A}_{1/2} \\ A_2\, e^{i\chi_2} &= \mathcal{A}_{3/2} + \mathcal{A}_{5/2} \\ A_2^+\, e^{i\chi_2^+} &= \mathcal{A}_{3/2} - \frac{2}{3}\mathcal{A}_{5/2} \end{aligned}}$$

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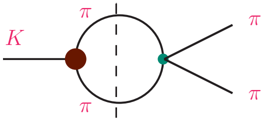
Implications of a Large Phase Shift

❶ **Unitarity:** $\delta_0(M_K) = (39.2 \pm 1.5)^\circ$ ➡ $A_0 \approx 1.3 \times \mathrm{Dis}\,(A_0)$



$$A_I \, \mathrm{e}^{i\delta_I} = \mathrm{Dis}\,(A_I) + i\,\mathrm{Abs}\,(A_I)$$

$$\tan\delta_I = \frac{\mathrm{Abs}\,(A_I)}{\mathrm{Dis}\,(A_I)}$$

$$A_I = \mathrm{Dis}\,(A_I)\,\sqrt{1+\tan^2\delta_I}$$

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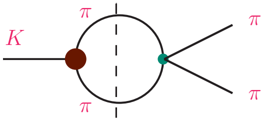


$$A_I \; \mathrm{e}^{i\delta_I} \;=\; \mathrm{Dis}\,(A_I) + i\,\mathrm{Abs}\,(A_I)$$

$$\tan\delta_I \;=\; \frac{\mathrm{Abs}\,(A_I)}{\mathrm{Dis}\,(A_I)}$$

$$A_I \;=\; \mathrm{Dis}\,(A_I)\;\sqrt{1+\tan^2\delta_I}$$

❷ **Analyticity:** $\Delta\,\mathrm{Dis}\,(A_I)[s] \;=\; \frac{1}{\pi}\int dt\;\frac{\mathrm{Abs}\,(A_I)[t]}{t-s-i\epsilon} \;+\; \mathrm{subtractions}$

$\boxed{\text{Large } \delta_0 \;\Rightarrow\; \text{Large } \mathrm{Abs}\,(A_0) \;\Rightarrow\; \text{Large correction to } \mathrm{Dis}\,(A_0)}$

Recent Lattice Results

Isospin limit:

RBC-UKQCD 1505.07863, 1502.00263

$$\sqrt{\frac{3}{2}}\,\mathrm{Re}\,A_2 = (1.50 \pm 0.04 \pm 0.14)\cdot 10^{-8}\ \mathrm{GeV} \qquad \mathrm{exp}: 1.482\,(2)\cdot 10^{-8}\ \mathrm{GeV} \quad 0.1\,\sigma$$

$$\sqrt{\frac{3}{2}}\,\mathrm{Im}\,A_2 = -(6.99 \pm 0.20 \pm 0.84)\cdot 10^{-13}\ \mathrm{GeV}$$

$$\sqrt{\frac{3}{2}}\,\mathrm{Re}\,A_0 = (4.66 \pm 1.00 \pm 1.26)\cdot 10^{-7}\ \mathrm{GeV} \qquad \mathrm{exp}: 3.112\,(1)\cdot 10^{-7}\ \mathrm{GeV} \quad 1.0\,\sigma$$

$$\sqrt{\frac{3}{2}}\,\mathrm{Im}\,A_0 = -(1.90 \pm 1.23 \pm 1.08)\cdot 10^{-11}\ \mathrm{GeV}$$

$$\mathrm{Re}\,(\varepsilon'/\varepsilon) = (1.38 \pm 5.15 \pm 4.59)\cdot 10^{-4} \qquad \mathrm{exp}: (16.8 \pm 1.4)\cdot 10^{-4} \quad 2.2\,\sigma$$

$$\delta_0 = (23.8 \pm 4.9 \pm 1.2)^\circ \qquad \mathrm{exp}: (39.2 \pm 1.5)^\circ \quad 2.9\,\sigma$$

$$\delta_2 = -(11.6 \pm 2.5 \pm 1.2)^\circ \qquad \mathrm{exp}: -(8.5 \pm 1.5)^\circ \quad 1.0\,\sigma$$

Isospin Breaking in ε'/ε

$$\boxed{\begin{aligned}\varepsilon' &\sim \omega_+ \left\{ \frac{\mathrm{Im}\, A_0^{(0)}}{\mathrm{Re}\, A_0^{(0)}} \left(1 + \Delta_0 + f_{5/2}\right) - \frac{\mathrm{Im}\, A_2}{\mathrm{Re}\, A_2^{(0)}} \right\} \\ &\sim \omega_+ \left\{ \frac{\mathrm{Im}\, A_0^{(0)}}{\mathrm{Re}\, A_0^{(0)}} \left(1 - \Omega_{\mathrm{eff}}\right) - \frac{\mathrm{Im}\, A_2^{\mathrm{emp}}}{\mathrm{Re}\, A_2^{(0)}} \right\}\end{aligned}}$$

$$\omega \equiv \frac{\mathrm{Re}\, A_2}{\mathrm{Re}\, A_0} = \omega_+ \left(1 + f_{5/2}\right) \quad ; \quad \omega_+ \equiv \frac{\mathrm{Re}\, A_2^{+}}{\mathrm{Re}\, A_0} \quad , \quad \Omega_{IB} = \frac{\mathrm{Re}\, A_0^{(0)}}{\mathrm{Re}\, A_2^{(0)}} \cdot \frac{\mathrm{Im}\, A_2^{\mathrm{non-emp}}}{\mathrm{Im}\, A_0^{(0)}}$$

Cirigliano-Ecker-Neufeld-Pich 2003

$\times\ 10^{-2}$	$\alpha = 0$ LO	$\alpha = 0$ NLO	$\alpha \neq 0$ LO	$\alpha \neq 0$ NLO
Ω_{IB}	11.7	15.9 ± 4.5	18.0 ± 6.5	22.7 ± 7.6
Δ_0	-0.004	-0.41 ± 0.05	8.7 ± 3.0	8.4 ± 3.6
$f_{5/2}$	0	0	0	8.3 ± 2.4
Ω_{eff}	11.7	16.3 ± 4.5	9.3 ± 5.8	6.0 ± 7.7

$$\boxed{\begin{aligned}\Omega_{\mathrm{eff}} &= 0.06 \pm 0.08 \\ &\equiv \Omega_{IB} - \Delta_0 - f_{5/2}\end{aligned}}$$

$$\Omega_{\mathrm{IB}}^{\pi^0\eta} = 0.16 \pm 0.03$$

Simplified Estimate

1. **CP violation** $\longrightarrow$ **Penguin operators**

2. **Chirality** $\longrightarrow$ **Enhanced** $(V-A)\otimes(V+A)$ **operators**

$$Q_6 = -8\sum_q (\bar{s}_L q_R)(\bar{q}_R d_L) \qquad , \qquad Q_8 = -12\sum_q e_q (\bar{s}_L q_R)(\bar{q}_R d_L)$$

3. **Large-N_C:** $\langle J\cdot J\rangle = \langle J\rangle\langle J\rangle\,\{1+\mathcal{O}(1/N_C)\}$

$$\mathcal{M}_{LL} \equiv \langle\pi^+\pi^-|(\bar{s}_L\gamma^\mu u_L)(\bar{u}_L\gamma_\mu d_L)|K^0\rangle = \langle\pi^+|\bar{u}_L\gamma_\mu d_L|0\rangle\,\langle\pi^-|\bar{s}_L\gamma^\mu u_L|K^0\rangle = \frac{i\sqrt{2}}{4}\,F_\pi\,(M_K^2-M_\pi^2)$$

$$\mathcal{M}_{LR}(\mu) \equiv \langle\pi^+\pi^-|(\bar{s}_L u_R)(\bar{u}_R d_L)|K^0\rangle = \langle\pi^+|\bar{u}_R d_L|0\rangle\,\langle\pi^-|\bar{s}_L u_R|K^0\rangle = \frac{i\sqrt{2}}{4}\,F_\pi\left[\frac{M_K^2}{m_d(\mu)+m_s(\mu)}\right]^2$$

At $\mu = 1$ GeV, $\qquad \mathcal{M}_{LR}(\mu)/\mathcal{M}_{LL} \sim M_K^2/[m_s(\mu)+m_d(\mu)]^2 \sim 14$

$$\longrightarrow \quad \boxed{\mathrm{Re}(\varepsilon'/\varepsilon) \approx 2.2\cdot 10^{-3}\left\{B_6^{(1/2)}\,(1-\Omega_{\rm eff}) - 0.48\,B_8^{(3/2)}\right\}}$$

Strong Cancellation

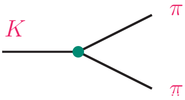
$$B_6^{(1/2)} = B_8^{(3/2)} = 1\,, \qquad \Omega_{\rm eff} = 0.06 \quad\longrightarrow\quad \mathrm{Re}(\varepsilon'/\varepsilon) \approx 1.0\cdot 10^{-3}$$

Buras et al: $\quad B_6^{(1/2)} = 0.57\,,\quad B_8^{(3/2)} = 0.76\,,\quad \Omega_{\rm eff} = 0.15 \quad\longrightarrow\quad \mathrm{Re}(\varepsilon'/\varepsilon) \approx 2.6\cdot 10^{-4}$

$\mathbf{O(p^2)}\quad \chi\mathbf{PT}$

$$\mathcal{L}_2^{\Delta S=1} = G_8\, F^4\, \langle \lambda L_\mu L^\mu \rangle \;+\; G_{27}\, F^4 \left(L_{\mu 23}\, L_{11}^\mu + \frac{2}{3}\, L_{\mu 21}\, L_{13}^\mu \right)$$

$$G_R \equiv -\frac{G_F}{\sqrt{2}}\, V_{ud} V_{us}^*\, g_R \quad ; \quad L_\mu = -iU^\dagger D_\mu U \quad ; \quad \lambda \equiv \frac{1}{2}\lambda_{6-i7} \quad ; \quad U \equiv \exp\left\{ i\sqrt{2}\,\Phi/F \right\}$$



$$\boxed{\begin{array}{c} \mathcal{A}_{1/2} = \sqrt{2}\, F_\pi \left(G_8 + \frac{1}{9}\, G_{27} \right) (M_K^2 - M_\pi^2) \\ \mathcal{A}_{3/2} = \frac{10}{9}\, F_\pi\; G_{27}\, (M_K^2 - M_\pi^2) \\ \mathcal{A}_{5/2} = 0 \qquad ; \qquad \delta_0 = \delta_2 = 0 \end{array}}$$

$$[\Gamma(K \to 2\pi) \;+\; \delta_I]_{\mathrm{Exp}} \quad \Rightarrow \quad |g_8| \approx 5.1 \quad ; \quad |g_{27}| \approx 0.29$$

$\mathbf{O(p^2, e^2p^0)}$ χPT

$\mathcal{Q} = \mathrm{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$

$$\mathcal{L}_2^{\Delta S=1} = G_8\, F^4\, \langle\lambda L_\mu L^\mu\rangle + G_{27}\, F^4 \left(L_{\mu 23}\, L^\mu_{11} + \frac{2}{3}\, L_{\mu 21}\, L^\mu_{13}\right)$$
$$+\; e^2\, F^6\, G_8\, g_{ew}\, \langle\lambda U^\dagger \mathcal{Q} U\rangle$$

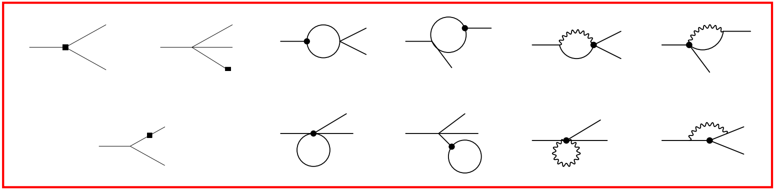
$$\mathcal{A}_{1/2} = \sqrt{2}\, F_\pi \left\{ G_8 \left[(M_K^2 - M_\pi^2)\left(1 - \frac{2}{3\sqrt{3}}\,\varepsilon^{(2)}\right) - \frac{2}{3}\, F_\pi^2\, e^2\, (g_{ew} + 2\, Z)\right] + \frac{1}{9}\, G_{27}\, (M_K^2 - M_\pi^2)\right\}$$

$$\mathcal{A}_{3/2} = \frac{2}{3}\, F_\pi \left\{ \left(\frac{5}{3}\, G_{27} + \frac{2}{\sqrt{3}}\,\varepsilon^{(2)}\, G_8\right)(M_K^2 - M_\pi^2) - F_\pi^2\, e^2\, G_8\, (g_{ew} + 2\, Z)\right\}$$

$$\mathcal{A}_{5/2} = 0 \qquad ; \qquad \delta_0 = \delta_2 = 0$$

$\varepsilon^{(2)} = (\sqrt{3}/4)\,(m_d - m_u)/(m_s - \hat{m}) \approx 0.011$; $Z \approx (M^2_{\pi^\pm} - M^2_{\pi^0})/(2\, e^2 F_\pi^2) \approx 0.8$

$\mathbf{O}\left[\mathbf{p}^4, (\mathbf{m_u} - \mathbf{m_d})\,\mathbf{p}^2, \mathbf{e}^2\mathbf{p}^0, \mathbf{e}^2\mathbf{p}^2\right]$ χPT



- **Nonleptonic weak Lagrangian:** $\mathcal{O}(G_F\, p^4)$

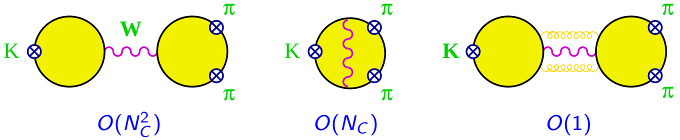
$$\mathcal{L}_{\rm weak}^{(4)} = \sum_i G_8\, N_i\, F^2\, O_i^8 + \sum_i G_{27}\, D_i\, F^2\, O_i^{27} + \text{h.c.}$$

- **Electroweak Lagrangian:** $\mathcal{O}(G_F\, e^2 p^{0,2})$

$$\mathcal{L}_{\rm EW} = e^2 F^6 G_8\, g_{ew}\, \mathrm{Tr}(\lambda U^\dagger \mathcal{Q} U) + e^2 \sum_i G_8\, Z_i\, F^4\, O_i^{EW} + \text{h.c.}$$

- $\mathcal{O}(e^2 p^{0,2})$ **Electromagnetic** $+$ $\mathcal{O}(p^4)$ **Strong:** $Z\,,\ K_i\,,\ L_i$

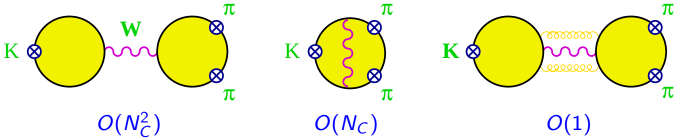
Weak Currents Factorize at Large N_C



$$A[K^0 \to \pi^0\pi^0] = 0 \quad \Longrightarrow \quad A_0 = \sqrt{2}\, A_2$$

No $\Delta I = \frac{1}{2}$ enhancement at leading order in $1/N_C$

Weak Currents Factorize at Large N_C



$$A[K^0 \to \pi^0\pi^0] = 0 \quad \Rightarrow \quad A_0 = \sqrt{2}\, A_2$$

No $\Delta I = \frac{1}{2}$ enhancement at leading order in $1/N_C$

- **Multiscale problem:** **OPE** $\quad \frac{1}{N_C} \log\left(\frac{M_W}{\mu}\right) \sim \frac{1}{3} \times 4$

Short-distance logarithms must be summed

- **Large χPT logarithms:** **FSI** $\quad \frac{1}{N_C} \log\left(\frac{\mu}{M_\pi}\right) \sim \frac{1}{3} \times 2$

Infrared logarithms must also be included $\quad [\delta_I \sim O(1/N_C)$, $\delta_0 - \delta_2 \approx 45^\circ]$

Multi-Scale Problem: Summation of logarithms needed

A large $\log(M_1/M_2)$ compensates a $1/N_C$ suppression

❶ **Short-distance:** $\frac{1}{N_C}\log\left(M_W/\mu\right)$ Bardeen-Buras-Gerard

$$\Longrightarrow \begin{cases} g_8^\infty = 1.13 \pm 0.05_\mu \pm 0.08_{L_5} \pm 0.05_{m_s} \\ g_{27}^\infty = 0.46 \pm 0.01_\mu \end{cases}$$

Cirigliano et al, Pallante et al

❷ **Long-distance (χPT):** $\frac{1}{N_C}\log\left(\mu/m_\pi\right)$ Kambor et al, Pallante et al

$$g_8^{\mathrm{LO}} = 5.0 \Longrightarrow g_8^{\mathrm{NLO}} = 3.6$$

$$g_{27}^{\mathrm{LO}} = 0.285 \Longrightarrow g_{27}^{\mathrm{NLO}} = 0.286$$

Cirigliano et al

❸ **Isospin Violation:** $g_{27}^{\mathrm{NLO}} = 0.297$ Cirigliano et al

$\mathbf{N_C \to \infty}$

$$g_8 = \left(\frac{3}{5}\, C_2 - \frac{2}{5}\, C_1 + C_4\right) - 16\;\, L_5\, \left(\frac{\langle \bar q\, q\rangle(\mu)}{F_\pi^3}\right)^2 C_6(\mu)$$

$$g_{27} = \frac{3}{5}\, (C_2 + C_1)$$

$$e^2\, g_8\, g_{ew} = -3\, \left(\frac{\langle \bar q\, q\rangle(\mu)}{F_\pi^3}\right)^2 \left[C_8(\mu) \; + \; \frac{16}{9}\; C_6(\mu)\; e^2\, (K_9 - 2\, K_{10})\right]$$

$$\frac{\langle \bar q\, q\rangle(\mu)}{F_\pi^3} = \frac{M_{K^0}^2}{(m_s + m_d)(\mu)\, F_\pi} \left\{1 - \frac{8 M_{K^0}^2}{F_\pi^2}\, (2 L_8 - L_5) + \frac{4 M_{\pi^0}^2}{F_\pi^2}\, L_5\right\}$$

- **Equivalent to standard calculations of $\mathbf{B_i}$**
- **μ dependence only captured for $\mathbf{Q_{6,8}}$**

Anomalous Dimension Matrix

$$\gamma_s^{(0)} = \begin{pmatrix}
-\frac{3}{N_c^2} & \frac{3}{N_c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{N_c} & -\frac{3}{N_c^2} & -\frac{1}{3N_c^2} & \frac{1}{3N_c} & -\frac{1}{3N_c^2} & \frac{1}{3N_c} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{11}{3N_c^2} & \frac{11}{3N_c} & -\frac{2}{3N_c^2} & \frac{2}{3N_c} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{N_c} - \frac{n_f}{3N_c^2} & \frac{n_f}{3N_c} - \frac{3}{N_c^2} & -\frac{n_f}{3N_c^2} & \frac{n_f}{3N_c} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{N_c^2} & -\frac{3}{N_c} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{n_f}{3N_c^2} & \frac{n_f}{3N_c} & -\frac{n_f}{3N_c^2} & {\color{red}-3} + \frac{n_f}{3N_c} + \frac{3}{N_c^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{N_c^2} & -\frac{3}{N_c} & 0 & 0 \\
0 & 0 & \frac{-n_u + \frac{n_d}{2}}{3N_c^2} & \frac{n_u - \frac{n_d}{2}}{3N_c} & \frac{-n_u + \frac{n_d}{2}}{3N_c^2} & \frac{n_u - \frac{n_d}{2}}{3N_c} & 0 & {\color{red}-3} + \frac{3}{N_c^2} & 0 & 0 \\
0 & 0 & \frac{1}{3N_c^2} & -\frac{1}{3N_c} & \frac{1}{3N_c^2} & -\frac{1}{3N_c} & 0 & 0 & -\frac{3}{N_c^2} & 0 \\
0 & 0 & \frac{-n_u + \frac{n_d}{2}}{3N_c^2} & \frac{n_u - \frac{n_d}{2}}{3N_c} & \frac{-n_u + \frac{n_d}{2}}{3N_c^2} & \frac{n_u - \frac{n_d}{2}}{3N_c} & 0 & 0 & 0 & -\frac{3}{N_c^2}
\end{pmatrix}$$

Only γ_{66} and γ_{88} survive the large–N_C limit

Anatomy of ε'/ε calculation

$$\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right) = -\frac{\omega_+}{\sqrt{2}|\varepsilon|}\left\{\frac{\mathrm{Im}\, A_0^{(0)}}{\mathrm{Re}\, A_0^{(0)}}\,(1-\Omega_{\mathrm{eff}}) - \frac{\mathrm{Im}\, A_2^{\mathrm{emp}}}{\mathrm{Re}\, A_2^{(0)}}\right\}$$

1. **$O(p^4)$ χPT Loops: Large correction** (NLO in $1/N_C$) FSI

$$\mathcal{A}_n^{(X)} = a_n^{(X)}\left[1 + \Delta_L \mathcal{A}_n^{(X)} + \Delta_C \mathcal{A}_n^{(X)}\right]$$

Pallante-Pich-Scimemi
Gisbert-Pich

$$\Delta_L \mathcal{A}_{1/2}^{(8)} = 0.27 \pm 0.05 + 0.47\, i \quad ;$$

$$\Delta_L \mathcal{A}_{1/2}^{(27)} = 1.03 \pm 0.63 + 0.47\, i \quad ; \quad \Delta_L \mathcal{A}_{3/2}^{(27)} = -0.04 \pm 0.05 - 0.21\, i$$

$$\Delta_L \mathcal{A}_{1/2}^{(g)} = 0.27 \pm 0.01 + 0.47\, i \quad ; \quad \Delta_L \mathcal{A}_{3/2}^{(g)} = -0.50 \pm 0.19 - 0.21\, i$$

2. **$O(p^4)$ LECs fixed at $N_C \to \infty$: Small correction** $\Delta_C \mathcal{A}_n^{(X)}$

3. **Isospin Breaking** $O\left[(m_u - m_d)\, p^2, e^2 p^2\right]$: **Sizeable correction**

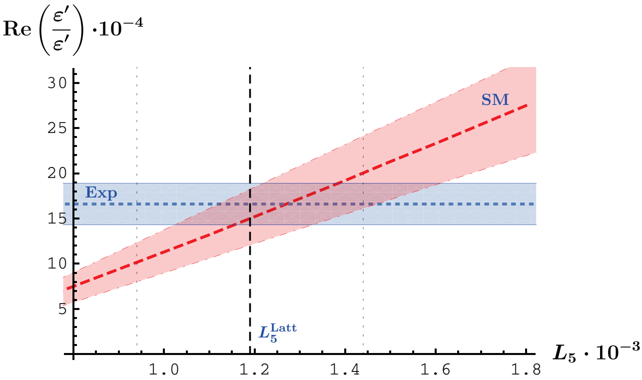
$$\Omega_{\mathrm{eff}} = 0.06 \pm 0.08$$

Cirigliano-Ecker-Neufeld-Pich

4. **$\mathrm{Re}(g_8)$, $\mathrm{Re}(g_{27})$, $\chi_0 - \chi_2$ fitted to data**

SM Prediction of ε'/ε

H. Gisbert, A. Pich, arXiv:1712.06147



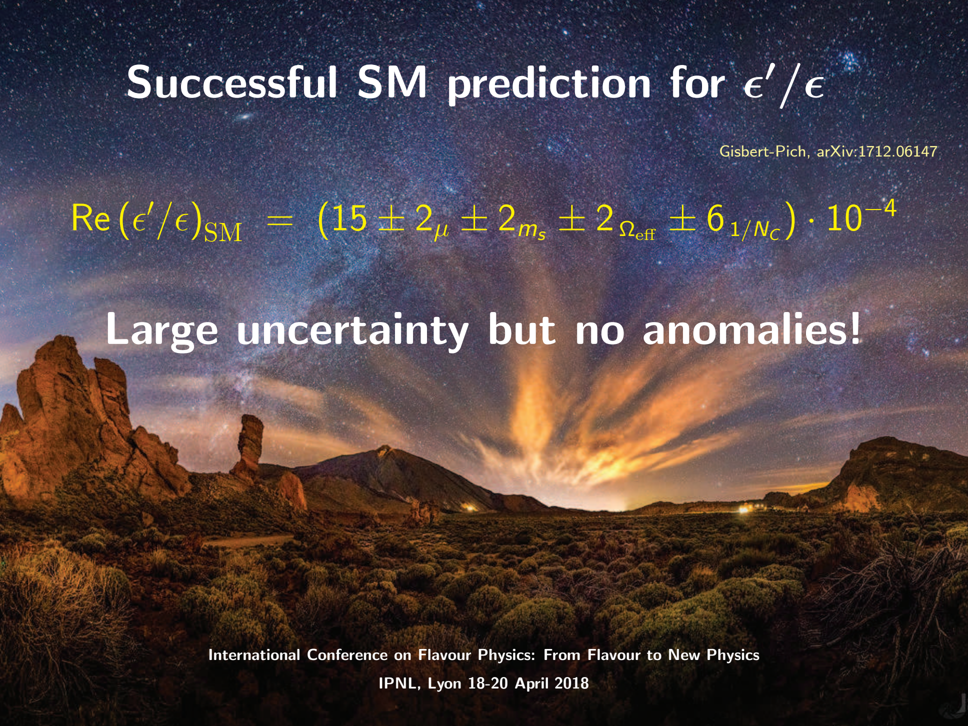
$$\mathrm{Re}\,(\epsilon'/\epsilon)_{\mathrm{SM}} \;=\; \left(15 \pm 2_{\mu} \pm 2_{m_s} \pm 2_{\Omega_{\mathrm{eff}}} \pm 6_{1/N_C}\right) \cdot 10^{-4}$$

Outlook: Needed Improvements

- **Wilson coefficients at NNLO** — Cerda et al
- **Updated value of Ω_{eff}** — Cirigliano et al
- $g_8\ g_{\text{ew}}$ **at NLO in $1/N_C$** — Rodriguez-Sanchez, A.P.
- g_8 **and higher-order LECs at NLO** — New ideas needed
- χ**PT logarithms at NNLO** — Feasible
- **Improved lattice input** — Eagerly expected

Difficult, but worth while enterprise

Best strategy: χ**PT (amplitudes) + Lattice (LECs)**



Successful SM prediction for ϵ'/ϵ

Gisbert-Pich, arXiv:1712.06147

$$\mathrm{Re}\,(\epsilon'/\epsilon)_{\mathrm{SM}} \;=\; \left(15 \pm 2_{\mu} \pm 2_{m_s} \pm 2_{\Omega_{\mathrm{eff}}} \pm 6_{1/N_C}\right) \cdot 10^{-4}$$

Large uncertainty but no anomalies!

International Conference on Flavour Physics: From Flavour to New Physics

IPNL, Lyon 18-20 April 2018

Modelling (some) non-factorizable $1/N_C$ corrections

Buras-Gérard, 1507.06326

$$B_6^{(1/2)} = 1 - \frac{3}{2}\left[\frac{F_\pi}{F_K - F_\pi}\right]\frac{(m_K^2 - m_\pi^2)}{(4\pi F_\pi)^2}\ln(1 + \frac{\Lambda^2}{\tilde{m}_6^2}) = 1 - 0.66\ \ln(1 + \frac{\Lambda^2}{\tilde{m}_6^2})$$

$$B_8^{(1/2)} = 1 + \frac{(m_K^2 - m_\pi^2)}{(4\pi F_\pi)^2}\ln(1 + \frac{\Lambda^2}{\tilde{m}_8^2}) = 1 + 0.08\ \ln(1 + \frac{\Lambda^2}{\tilde{m}_8^2})$$

$$B_8^{(3/2)} = 1 - 2\frac{(m_K^2 - m_\pi^2)}{(4\pi F_\pi)^2}\ln(1 + \frac{\Lambda^2}{\tilde{m}_8^2}) = 1 - 0.17\ \ln(1 + \frac{\Lambda^2}{\tilde{m}_8^2})$$

$$\Rightarrow \qquad B_6^{(1/2)} \le B_8^{(3/2)} < 1$$

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$$\longrightarrow \qquad B_6^{(1/2)} \le B_8^{(3/2)} < 1$$

- FSI ($1/N_C$) not included $\longrightarrow$ $\delta_I = 0$
- Part of 1-loop χPT corrections (?)
- Difficult to account in a matching calculation

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$\rightarrow$ $B_6^{(1/2)} \leq B_8^{(3/2)} < 1$

Not true in QCD

- FSI $(1/N_C)$ not included $\rightarrow$ $\delta_I = 0$
- Part of 1-loop χPT corrections (?)
- Difficult to account in a matching calculation

BBG Model

Bardeen-Buras-Gerard

$$\boxed{\mathcal{L}_{\rm eff} = \frac{f_\pi^2}{4}\left\{\langle D_\mu U^\dagger D^\mu U\rangle + r\,\langle m\,(U+U^\dagger)\rangle - \frac{r}{\Lambda_\chi^2}\,\langle m\,(D^2U + D^2U^\dagger)\rangle\right\}}$$

1. Equivalent to $\mathcal{O}(p^2)$ χPT + L_5 term $\qquad (L_i = 0\,,\; i \neq 5)$
   Most L_i are leading in N_C ➔ $\mathcal{L}_{\rm eff}$ **does not represent large-N$_C$ QCD**

2. **Cut-off loop regularization: M $\sim (0.8 - 0.9)$ GeV**
   $f_\pi^2(M^2) = F_\pi^2 + 2\, I_2(m_\pi^2) + I_2(m_K^2)$, $\qquad I_2(m_i^2) = \frac{1}{16\pi^2}\left[M^2 - m_i^2\,\log\left(1+\frac{M^2}{m_i^2}\right)\right]$

3. **Large-N$_C$ factorization assumed to hold in the IR ($\mu = 0$):** $\langle J\cdot J\rangle = \langle J\rangle\langle J\rangle$

4. **M identified with SD renormalization scale μ: $C_i(\mu)$ running**
   Meson evolution ⟷ **Quark evolution**

5. **Vector meson loops included through Hidden U(3) Gauge Symmetry**
   Could partially account for $L_{1,2,3,9,10}$
   L_8 still missing ➔ $\langle\bar{q}q\rangle$, $Q_{6,8}$ **not quite correct even at large-N$_C$**

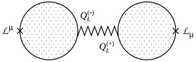
6. **$\pi\pi$ re-scattering completely missing** ➔ $\delta_{0,2} = 0$, **FSI absent**

Dynamical understanding of the $\Delta I = 1/2$ rule

AP – E. de Rafael, PL B374 (1996) 186

$$\mathcal{L}_{\rm eff} = -\frac{G_F}{\sqrt{2}} F^4 \left[a \ \mathrm{Tr}(Q_L^{(-)} L_\mu)\, \mathrm{Tr}(Q_L^{(+)} L^\mu) + b \ \mathrm{Tr}(Q_L^{(-)} L_\mu Q_L^{(+)} L^\mu) + c \ \mathrm{Tr}(Q_L^{(-)} Q_L^{(+)} L_\mu L^\mu) \right]$$

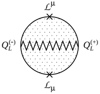
$\mathcal{O}(N_C^2)$



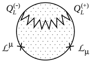
a

$$Q_L^{(+)} = \begin{pmatrix} 0 & V_{\rm ud} & V_{\rm us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad Q_L^{(-)} = Q_L^{(+)\dagger}$$

$\mathcal{O}(N_C)$



b



c

$$g_8 = \frac{3}{5}(a+b) - b + c$$

$$g_{27} = \frac{3}{5}(a+b)$$

$$a = 1 + \mathcal{O}\left(\frac{1}{N_c^2}\right) \quad ; \quad c = \mathrm{Re}C_4 - 16\, L_5\, \mathrm{Re}C_6(\mu^2) \left[\frac{<\bar\psi\psi>}{f_\pi^3}\right]^2 + \mathcal{O}\left(\frac{1}{N_c^2}\right) \simeq 0.3 \pm 0.2$$

$$|g_{27}| \simeq 0.29 \quad \longrightarrow \quad b \simeq -0.52 + \mathcal{O}\left(\frac{1}{N_c^2}\right) \quad \longrightarrow \quad g_8 \simeq 1.1 + \mathcal{O}\left(\frac{1}{N_c^2}\right)$$

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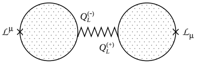

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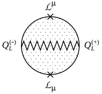
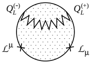


b



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$\mathbf{b < 0}$ predicted through explicit calculations

AP–E. de Rafael, NP B358 (1991) 311
Bardeen-Buras-Gerard, Bijnens-Prades, Bertolini et al

Confirmed through inclusive QCD analysis

M. Jamin–AP, NP B425 (1994) 15

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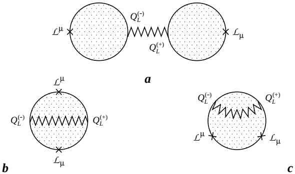

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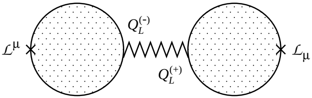
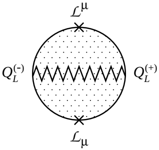
Confirmed recently by lattice calculations — RBC-UKQCD, PRL 110 (2013) 15, 152001; PRD 91 (2015) 7, 074502

“A qualitative picture towards the understanding of the underlying physics begins to emerge”

AP – E. de Rafael, PL B374 (1996) 186





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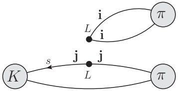
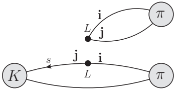
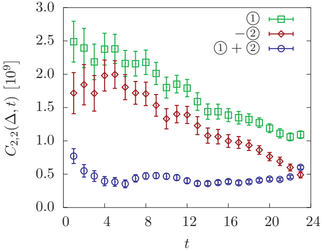
“Emerging understanding of the $\Delta I = 1/2$ Rule from Lattice QCD”

RBC-UKQCD, PRL 110 (2013) 15, 152001



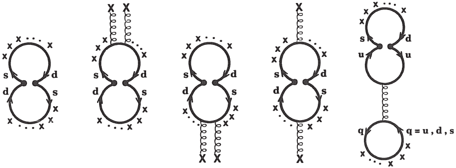


$$b \approx -0.7\, a$$



Effective Action Model: Bosonization in Gluonic Background

AP–E. de Rafael, NP B358 (1991) 311



$$\Delta \;=\; \frac{1}{N_C}\left[1-\frac{N_C}{2}\,\frac{\langle\frac{\alpha_s}{\pi}\,G^2\rangle}{16\pi^2 f_\pi^4}+\mathcal{O}(\alpha_s^2 N_C^2)\right] \;<\; 0$$

$$g_{27} \;\approx\; \frac{3}{5}\, C_+(\mu^2)\;\left\{1+\Delta+\mathcal{O}(1/N_C^2)\right\}$$

$$g_8 \;\approx\; \frac{1}{2}\, C_-(\mu^2)\;\left\{1-\Delta+\mathcal{O}(1/N_C^2)\right\}+\frac{1}{10}\, C_+(\mu^2)\;\left\{1+\Delta+\mathcal{O}(1/N_C^2)\right\}+c$$

$$c \;=\; C_4(\mu^2)-16\, C_6(\mu^2)\, L_5\left[\frac{\langle\bar\psi\psi\rangle}{f_\pi^3}\right]^2+\mathcal{O}(1/N_C^2)$$

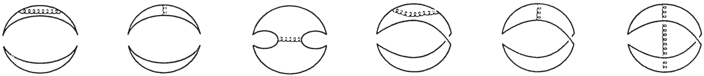
$$\boxed{b \;=\; \frac{1}{2}\, C_+(\mu^2)\;\left\{1+\Delta+\mathcal{O}(1/N_C^2)\right\}-\frac{1}{2}\, C_-(\mu^2)\;\left\{1-\Delta+\mathcal{O}(1/N_C^2)\right\} \;<\; 0}$$

$\mu\sim m_c\,,\quad \langle\frac{\alpha_s}{\pi}\,G^2\rangle\sim 330\ \mathrm{MeV}^4 \quad\Longrightarrow\quad b\sim -0.6+\mathcal{O}(1/N_C^2)$

Two-point Functions

AP–E. de Rafael, NP B358 (1991) 311, PL B374 (1996) 186
M. Jamin–AP, NP B425 (1994) 15

$$\Psi^{\Delta S=1,2}(q^2) \equiv i \int d^4x\, e^{iq\cdot x} \langle 0| T \left(\mathcal{H}_{\mathrm{eff}}^{\Delta S=1,2}(x),\, \mathcal{H}_{\mathrm{eff}}^{\Delta S=1,2}(0)^\dagger\right) |0\rangle = \sum_{ij} C_i\, C_j^*\, \Psi_{ij}(q^2)$$



$$\frac{1}{\pi}\,\mathrm{Im}\Psi_{\pm\pm}(t) = \theta(t)\,\frac{2}{45}\,N_c^2 \left(1 \pm \frac{1}{N_c}\right) \frac{t^4}{(4\pi)^6}\,\alpha_s(t)^{-2a_\pm}\, C_\pm^2(M_W^2) \left[1 + \frac{3}{4}\,\frac{\alpha_s(t) N_c}{\pi}\,\mathcal{K}_\pm\right]$$

$$a_\pm = \pm \frac{9}{11N_c}\,\frac{1 \mp 1/N_c}{1-6/11N_c}$$

$$\mathcal{K}_+ = 1 - \frac{30587}{3630}\,\frac{1}{N_c} + \frac{164936}{19965}\,\frac{1}{N_c^2} - \frac{51591}{14641}\,\frac{1}{N_c^3} + \frac{440193}{322102}\,\frac{1}{N_c^4} + \cdots = -\frac{3649}{3645}$$

$$\mathcal{K}_- = 1 + \frac{30587}{3630}\,\frac{1}{N_c} + \frac{169706}{19965}\,\frac{1}{N_c^2} + \frac{70335}{14641}\,\frac{1}{N_c^3} + \frac{1810209}{322102}\,\frac{1}{N_c^4} + \cdots = +\frac{18278}{3645}$$

Phenomenological $K \to \pi\pi$ Fit

Cirigliano-Ecker-Neufeld-Pich

	LO-IC	LO-IB	NLO-IC	NLO-IB
Re g_8	4.96	4.99	3.62 ± 0.28	3.61 ± 0.28
Re g_{27}	0.285	0.253	0.286 ± 0.029	0.297 ± 0.029
$\chi_0 - \chi_2$	47.5°	47.8°	$(47.5 \pm 0.9)^\circ$	$(51.3 \pm 0.8)^\circ$

IC $\equiv [m_u - m_d = \alpha = 0]$; IB $\equiv [m_u - m_d \neq 0\,,\ \alpha \neq 0]$

Isospin Limit: $[\delta_0 - \delta_2]_{K\to\pi\pi} = (52.5 \pm 0.8_{\mathrm{exp}} \pm 2.8_{\mathrm{th}})^\circ$

$\pi\pi \to \pi\pi$: $\delta_0 - \delta_2 = (47.7 \pm 1.5)^\circ$ Colangelo–Gasser–Leutwyler '01