Physique de la turbulence à l’échelle de Kolmogorov: 
(Enquête autour de 5 énigmes de la turbulence)

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Définition: Turbulence describes the state of a fluid (liquid or gas) in which velocity is in a swirling state.

\[ \nabla \cdot \mathbf{u} = 0 \]
\[ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} \]

\[ \text{Re} = \frac{(u \nabla u)}{\nu \Delta u} = \frac{LU}{\nu} \]

Turbulence: \( \text{Re} \gg 1 \)

Open basic question:

Navier-Stokes equations are they well-posed?
Are they any singularities?

Puzzle # 0 of turbulence
Figure 1.1: Vortices affect fluid behavior on all scales. (a) quantum vortices in a superfluid [130] (b) bathtub vortex [152] (c) tornado [109] (d) hurricane [106] (e) sun spot vortices [110] (f) spiral galaxy [105] (numbers approximate)

R. Fuchs, PhD thesis
Vortices hierarchy on the Earth
Vorticity hierarchy in the lab

Turbulent flow in SPIV

4 fast cameras

Laser

Turbulent flow in 4D-PTV

ANR EXPLOIT: F. Daviaud, P. Debue, B. Dubrulle, J-M. Foucault, J-P. Laval, Y. Ostovan, V. Padilla, V. Valori, C. Wiertel
Self-similarity of vortices in turbulence

Seymour Narrows, Between Vancouver and Quadra Islands

Fig. 6.2. The turbulence spectra, measured by Grant, Stewart and Moilliet (1962) and scaled according to the Kolmogorov parameters. The viscous dissipation rate $\dot{\varepsilon}$ varied over a range of values of the order 100. The straight line represents variation as $k^{-5/3}$. The top few points are believed to be rather high on account of the low frequency heaving motions of the ship.
3 puzzles set by the vortices hierarchy

Number of degrees of freedom

\[ N = \left( \frac{L_{\text{max}}}{L_{\text{min}}} \right)^3 \]

3 more puzzles of turbulence
Practical importance for DNS and theories
Solving # 1 and # 2 puzzles:

How are Lmax and Lmin selected?
The forcing matters

\[ \nabla \cdot u = 0 \]
\[ \partial_t u + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \Delta u \]

Energy dissipation!

Without forcing, unique equilibrium state \( u=0 \)

With forcing (possibly multiple) out-of-equilibrium stationary state
The dissipation matters

\[ \partial_t \int \frac{u^2}{2} \, dx^3 = \int F \cdot u \, dx^3 - \nu \int (\nabla u)^2 \, dx^3 \]

Energy balance in a turbulent flow

\[ dE = dW + dQ \]

Cf Joule’s experiment:
Work measured by Torques applied at Shafts
=Heat flux measured By keeping T constant

\[ P_{\text{inj}} = P_{\text{diss}} \equiv \rho L^3 \varepsilon \]

Rousset et al, RSI 85, 103908 (2014);
Energy balance in a turbulent flow

\[ \partial_t \int \frac{u^2}{2} \, dx^3 = \int F \cdot u \, dx^3 - \nu \int (\nabla u)^2 \, dx^3 \]

1st characteristic scale: \( L \): size of the stirrer

Injection scale

\[ P_{inj} = P_{diss} \equiv \rho V \varepsilon \]
Solution of puzzle # 2

\[ \partial_t \int \frac{u^2}{2} \, dx^3 = \int F \cdot u \, dx^3 - \nu \int (\nabla u)^2 \, dx^3 \]

Energy balance in a turbulent flow

Dimensional analysis

\[ \varepsilon = [m^2 s^{-3}] \]
\[ \nu = [m^2 s^{-1}] \]

\[ P_{\text{inj}} = P_{\text{diss}} \equiv \rho V \varepsilon \]

2nd characteristic scale

\[ \eta = \left( \frac{\nu^3}{\varepsilon} \right)^{1/4} \]

Kolmogorov scale
Preliminary phenomenological picture

Number of degrees of freedom \( N = \left( \frac{L}{\eta} \right)^3 \)

\[ E(k) \eta^{-5/3} \]

\[ \eta = \left( \frac{v^{3/4}}{\varepsilon^{1/4}} \right) \]

Radius propeller

Blade height

Lmax

Lmin

Debue et al., PRF (2018)
Solving # 3 puzzle:

How is spectral exponent selected?
The spectral exponent is very robust but not given by dimensional analysis.

By dimensional analysis:
\[ E(k) = [m^3 s^{-2}] \]

Experiments for a wide class of forcing conditions:
\[ E(k) = \varepsilon^{2/3} \eta^{5/3} (k\eta)^{-1-2h} \]

How is the exponent \( h=1/3 \) selected?

\[ v(k) \approx k^{-h} \]
The spectral exponent is very robust but not given by dimensional analysis.

**Numerical simulations NSE**

- **By dimensional analysis**
  \[ E(k) = \frac{[m^3 s^{-2}]}{\varepsilon^{2/3} \eta^{5/3}} \]
  
- **How is the exponent \( h = 1/3 \) selected?**
  
**Experiments for a wide class of forcing conditions**

- **#1 Suspect: the energy dissipation**
  \[ E(k) = \varepsilon^{2/3} \eta^{5/3} (k\eta)^{-1-2h} \]
  
- **\( \nu(k) \approx k^{-h} \)**
A closer look at energy dissipation

Re = 3500: Turbulence

Saint-Michel et al, POF 26, 125109 (2014)

VKE + VKS collaboration + SHREK collaboration
Non-dimensional energy dissipation per unit mass is constant at large Reynolds
Independent of viscosity?

\[ \epsilon = \nu < (\nabla u)^2 > = \nu < \omega^2 > \approx \frac{\epsilon}{\nu} \]

\[ \lim_{\nu \to 0} < \omega^2 > = \infty \]

Building of very large gradients at small scale... Singularity?
How to measure them/quantify them/understand this?

P. K. Yeung et al. PNAS 2015;112:41:12633-12638
Local energy budget for irregular fields

\[ D(u) = \lim_{\ell \to 0} \frac{1}{4} \int_{r \leq \ell} d^3 r \ \nabla \phi_{\ell}(r) \cdot \delta u_r |\delta u_r|^2 \]

Inertial dissipation = singularity/large gradient detector!

\[ \frac{1}{2} \partial_t \mathbf{u}^2 + \text{div} \left( \mathbf{u} \left( \frac{1}{2} \mathbf{u}^2 + p \right) - \nu \nabla \mathbf{u} \right) = D(u) - \nu \nabla u^2 \]


\[ \delta u = u(x+r) - u(x) \quad \text{Velocity increment} \]
Applying the DR detector in von Karman flow

\[ D_\ell(u) = \frac{1}{4} \int_V d^3r \left( \nabla G_\ell(r) \right) \cdot \delta u(r) \left| \delta u(r) \right|^2, \]

\[ G_\ell(r) = \frac{1}{N} \exp(-1/(1 - (r/2\ell)^2)), \]

-Kuzzay D. et al. (2017), Nonlinearity
LECTURES ON THE ONSAGER CONJECTURE

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(Communicated by [the associate editor name])

ABSTRACT. These lectures give an account of recent results pertinent to celebrate Onsager conjecture. The conjecture states that the minimal space regularity needed for a weak solution of the Euler equation to conserve energy is 1/3. Our presentation is based on the Littlewood-Paley method. We start with quasi-local estimates on the energy flux, introduce Onsager criticality, find a positive solution to the conjecture in Besov spaces of smoothness 1/3. We highlight important connections with the scaling laws of turbulence.

Results for dyadic models and a complete resolution of the Onsager conjecture for these is discussed, as well as recent attempts to construct dissipative solutions for the actual equation.

The article is based on a series of four lectures given at the 11th school "Mathematical Theory in Fluid Mechanics" in Károly, Czech Republic, May 2009.

"...in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available."

L. Onsager, 1949

1. Lecture 1: motivation, Onsager criticality.

1.1. Onsager's original conjecture.

The motion of an ideal homogeneous (with constant density 1) incompressible fluid is described by the system of Euler equations given by

\[ \frac{1}{2} \frac{\partial u}{\partial t} + \text{div} \left( u \left( \frac{1}{2} u^2 + p \right) - \nu \nabla u \right) = D(u) - \nu \nabla u \]

\[ \nabla \cdot u = 0, \]

where \( u \) is a divergence-free velocity field, and \( p \) is the internal pressure. We assume that the fluid domain \( \Omega \) here is either periodic or the entire space. It is an easy consequence of the antisymmetry of the nonlinear term in (1) and incompressibility of the fluid that the law of energy conservation holds for smooth solutions:

\[ \int_\Omega |u(t)|^2 \, dx = \int_\Omega |u_0|^2 \, dx, \quad \text{for all} \ t \geq 0. \]

2000 Mathematics Subject Classification. Primary: 76F02, 76B03; Secondary: 42B37.

Key words and phrases. Euler equation, Navier-Stokes equation, weak solutions, turbulence, Onsager conjecture, Besov spaces, dyadic models.

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If \( h \leq 1/3 \) \( \Rightarrow \) Dissipation through irregularities (singularities) Without viscosity!

If \( h > 1/3 \) \( \Rightarrow \) Euler equation conserves energy, Dissipation in Navier-Stokes by viscosity.

Kolmogorov spectrum exponent is the « critical value » !!!!
Is this the only one? -> look statistics of velocity increments

Kolmogorov: \( h=1/3; D=\varepsilon=\text{cte} \)
**Puzzle # 4: $h = \frac{1}{3}$ is not unique!**

\[ \delta u \sim \ell^{1/3} \Rightarrow S_n(\ell) = \langle (\delta u)^n \rangle \geq \ell^{n/3} \]

Not observed!

Kolmogorov spectrum exponent $h=1/3$ is the « critical value » !!!!
It is not unique...

What is the physics involved?

Saw et al et al. JFM (2018)
Local energy balance in turbulence

\[ \partial_t E^\ell + \partial_j J_j^\ell = -\frac{1}{4} \int \nabla \phi^\ell(\xi) \cdot \delta u(\delta u)^2 d\xi + \nu \partial^2 E^\ell \]

\[ \equiv -D^I_\ell - D^\nu_\ell, \]

\[ \delta u \approx \ell^h \Rightarrow D^I_\ell \approx \ell^{3h-1} \text{ and } D^\nu_\ell \approx \nu \ell^{2h-2} \]

\[ \frac{D^I_\ell}{D^\nu_\ell} \approx \frac{\ell^{3h-1}}{\nu \ell^{2h-2}} \approx \frac{\ell^{h+1}}{\nu} \Rightarrow \eta_h \approx \text{Re}^{-1/(1+h)} \]

\[ \eta_{-1} = 0 \]

Kuzzay et al. et al. (2017), Nonlinearity
Solving puzzle # 4:
Is $h=1/3$ the only possible exponent?

Look at the physics of turbulence at Kolmogorov scale
Large events of $D(u)$
2D Structure of large events of inertial dissipation

\[ D_{\text{DR}}^\ell / \epsilon, \text{Fr} = 0.0207 \]

\[ D_{\text{DR}}^\ell / \epsilon, \text{Fr} = 0.0754 \]

Color map: M. Farge, 1990 L'Aéronautique et l'Astronautique, 140, 24-33
Saw, Kuzzay et al. (2016), Nature-Comm. 7
Structure of large events of inertial dissipation

\[ h = 1/3 \]

\[ h = -1 \]

Hierachy of \( h \)!

\( D_{DR}^{\ell}/\epsilon, \ Fr = 0.0418 \)
3D Structure of large events of inertial dissipation

Swirling filament

High enstrophy in the Vicinity of the event

P. Debue PhD Thesis; ANR EXPLOIT
Solving # 0 puzzle:

Are these footprints of singularities?

Open basic question:

*Navier-Stokes equations are they well-posed?*
*Are they any singularities?*
A possible suspect of NS singularity

Symmetry \((t, x, u) \rightarrow (\gamma^2 t, \gamma x, \gamma^{-1} u) (\nu \neq 0)\)

\[ u(\gamma^2 t, \gamma x) = \gamma^{-1} u(t, x) \quad \text{homogeneous solutions of NS of degree -1} \]

**Axisymmetric case:** classification started by Landau in 1944

\[
\begin{align*}
U_r &= -\frac{1}{r} \left\{1 + \frac{2(\gamma + 1)}{(\gamma + 1) \ln \left(\frac{1+\cos \theta}{2}\right)} - 2 ight. \\
&\quad + \left. \frac{1 - \cos \theta}{1 + \cos \theta} \frac{3(\gamma + 1)}{(\gamma + 1) \ln \left(\frac{1+\cos \theta}{2}\right) - 2} \right\} , \quad (1) \\
U_\theta &= \frac{1}{r} \left(1 - \frac{\cos \theta}{\sin \theta}\right) \left(1 + \frac{2(\gamma + 1)}{(\gamma + 1) \ln \left(\frac{1+\cos \theta}{2}\right) - 2} \right) \\
U_\phi &= \begin{cases} \\
\frac{1}{r \sin \theta} (b \ln (1 + \cos \theta) + a) & \text{for } \gamma = -1 \\
\frac{1}{r \sin \theta} \left(\frac{b}{\ln (1 - \cos \theta)} + a\right) & \text{for } \gamma > -1 \\
\end{cases}
\end{align*}
\]

where \(\gamma \geq 1, a, \) et \(b\) are free parameters. Note singular behaviour for \(\theta = \pi\).

Li, Yan Yan, (2016)
H. Faller M2 Thesis
3D footprints of the -1 degre homogeneous solution
2D footprints of the -1 degre homogeneous solution
Conclusions and perspectives

• Turbulence is plagued with puzzles of fundamental importance. One of them is connected with possible quasi-singularities or singularities of the NSE.
• We have found possible footpints of such quansi singularities. They are linked with coherent structures, living below the Kolmogorov scale,
• -> Impact on DNS of NSE!!!!
Impact on DNS of turbulence

Resolution of events with exponent $h$

$$\eta_h = L \, Re^{-1/(1+h)}$$

$$N = \left( \frac{L}{\eta_h} \right)^3 \sim Re^{3/(1+h)}$$

Kolmogorov

$$\eta_{1/3} = L \, Re^{-3/4}$$

$$N = \left( \frac{L}{\eta_{1/3}} \right)^3 \sim Re^{9/4}$$

$\eta = 0.4 \, mm$

$N \sim 10^7$

MFR with $h_{\text{min}} = -0.2$

$$\eta_{-1/5} = Re^{-5/4} = \eta Re^{-1/2} = 5 \times 10^{-3} \, mm$$

$$N = \left( \frac{L}{\eta_{-1/5}} \right)^3 \sim 8 \times 10^{12}$$

Re$=6 \times 10^3$

L$=10 \, cm$
Conclusions and perspectives

• Turbulence is plagued with puzzles of fundamental importance. One of them is connected with possible quasi-singularities or singularities of the NSE.
• We have found possible footpints of such quansi singularities. They are linked with coherent structures, living below the Kolmogorov scale,

• -> Impact on DNS of NSE!!!!

• **Perspective:**
  The study of the dynamics and properties of these structures is underway, using 4D-PTV.
  We are currently building a larger experiment to explore sub-Kolmogorov regimes, to look for more singular structures).
Present experiment

- $Re = 10^6$
- $\eta = 0.01 \text{ mm}$
- $\Delta x \sim \eta$

GVK experiment

- $Re = 10^6$
- $\eta = 0.05 \text{ mm}$
- $\Delta x \sim \eta/5$

- $Re = 6 \times 10^3$
- $\eta = 0.4 \text{ mm}$

Possibility to explore sub-Kolmogorov scales
Detection of stronger velocity gradients and quasi-singularities
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VKS collaboration

SHREK collaboration