Numerical relativity

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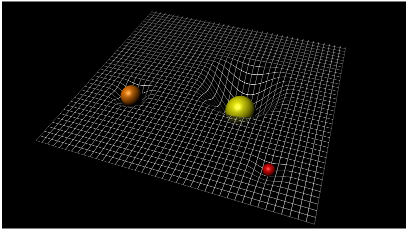
Numerical relativity ?

The object of numerical relativity is to solve Einstein equations with computers.

- Complementary to semi-analytic approaches (post-Newtonian, perturbative techniques).
- Largely motivated by the gravitational wave detectors.
- Main application : binary black hole computations.
- Many other applications (supernovae, neutron stars, boson stars, ADS/CFT conjecture, alternative theories...)

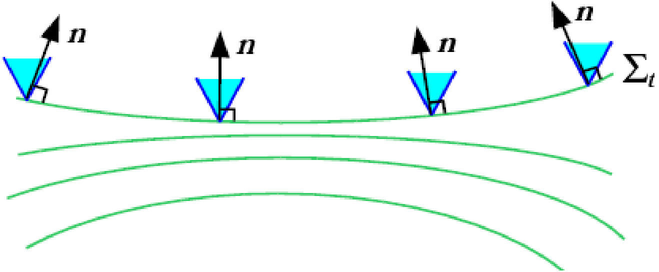
Einstein's equations

- Geometry is described by a 4D, Lorentzian metric $g_{\mu\nu}$.
- The coupling between geometry and energy is described by Einstein's equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$
- $G_{\mu\nu}$ is Einstein's tensor, non-linear, and second order in $g_{\mu\nu}$.
- In their standard form, Einstein's equations are not well suited for numerical resolution.



Foliation of spacetime

The 3+1 formalism is the most widely used way to write Einstein equations for NR. It makes explicit the split between space and time. Spacetime is foliated by a family of spatial hypersurfaces

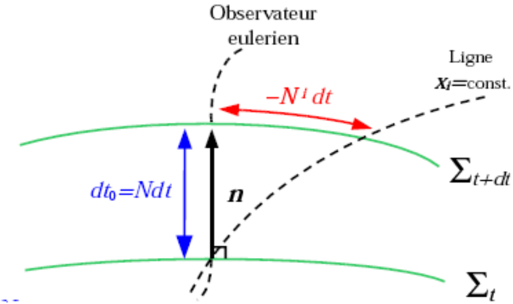


- Coordinate system of Σ_t : (x_1, x_2, x_3).
- Coordinate system of spacetime : (t, x_1, x_2, x_3).

Greek indices 4D $(0, 1, 2, 3)$ and Latin 3D $(1, 2, 3)$.

Unit normal

The unit normal can be written as $n^\mu = \left(\frac{1}{N}, -\frac{B^i}{N}\right)$.



- N is the lapse, a choice of time coordinate.
- B^i is the shift, a choice of spatial coordinates.

Projections

- Projection on the normal of a vector $\vec{V}$ is given by $n_\mu V^\mu$.
- Projection operator on the hypersurfaces $\gamma_\mu^\nu = g_\mu^\nu + n_\mu n^\nu$.
- Projection of a vector $\vec{V}$ is given by $\gamma_\mu^\nu V^\mu$.
- Projection the 4D metric is

$$\gamma_\alpha^\nu \gamma_\beta^\mu g_{\mu\nu} = \gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$$

The 4D line-element reads

$$\mathrm{d}s^2 = -\left(N^2 - B^i B_i\right)\mathrm{d}t^2 + 2B_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

Extrinsic curvature K_{ij}

- Describes the part of the geometry not accounted for by the induced metric.
- It describes the variation of normal projected on the hypersurface.

$$K_{ij} = -\gamma_i^\mu \gamma_j^\nu \nabla_\mu n_\nu$$

- In the 3+1 framework, it is given by

$$\left(\partial_t - \mathcal{L}_{\vec{B}}\right) \gamma_{ij} = -2NK_{ij}$$

- It is known as the second fundamental form.
- The first fundamental form is the induced metric γ_{ij}
- Both forms live on the hypersurface (i.e. contractions with $\vec{n}$ are null).

Link between 4D and 3D quantities

In order to derive the 3+1 version of Einstein's equations, one needs to relate the 4D quantities to the 3D ones.

- 4D quantities : $g_{\mu\nu}$, n^α, ${}^4R^\alpha_{\beta\mu\nu}$, ∇_α...
- 3D quantities : γ_{ij}, K_{ij}, D_i, N, B^i ...

Some examples

- Gauss relation :

$$\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \gamma^\sigma_\delta \ {}^4R^\rho_{\sigma\mu\nu} = R^\gamma_{\delta\alpha\beta} + K^\gamma_\alpha K_{\delta\beta} - K^\gamma_\beta K_{\alpha\delta}.$$

- Codazzi relation :

$$\gamma^\gamma_\rho n^\sigma \gamma^\mu_\alpha \gamma^\nu_\beta \ {}^4R^\rho_{\sigma\mu\nu} = D_\beta K^\gamma_\alpha - D_\alpha K^\gamma_\beta.$$

Projection of Einstein's equations

The 3+1 equations are obtained by projecting $G_{\mu\nu} = 8\pi T_{\mu\nu}$ on n^α and on the hypersurfaces.

- projections onto $n^\mu n^\nu$, the Hamiltonian constraint:
$$R + K^2 - K_{ij}K^{ij} = 16\pi E.$$
- projection onto $n^\mu \gamma_i^\nu$, the momentum constraint:
$$D_j K_i^j - D_i K = 8\pi P_i.$$
- projection onto $\gamma_i^\nu \gamma_j^\mu$, the evolution equation:
$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathcal{L}_{\vec{B}}\right) \quad & K_{ij} = \\ -D_i D_j N \quad & +N\left(R_{ij} + KK_{ij} - 2K_{ik}K_j^k + 4\pi\left[(S-E)\gamma_{ij} - 2S_{ij}\right]\right). \end{aligned}$$

- E, P_i and S_{ij} are the various projections of $T_{\mu\nu}$.

Evolution problem

- Give yourself a set of initial data $\gamma_{ij}\,(t=0)$ and $K_{ij}\,(t=0)$.
- The fields at latter time are obtained by two first order equations :
  - Definition ofK_{ij} : $\left(\partial_t - \mathcal{L}_{\vec{B}}\right)\gamma_{ij} = ...$
  - Evolution equation : $\left(\partial_t - \mathcal{L}_{\vec{B}}\right)K_{ij} = ...$
- Similar to writing the Newton's law in terms of $\vec{x}$ and $\vec{v}$:
  - $\gamma_{ij} \leftrightarrow \vec{x}$.
  - $K_{ij} \leftrightarrow \vec{v}$.
  - Definition of $K_{ij} \leftrightarrow \partial_t\vec{x} = \vec{v}$.
  - Evolution equation $\leftrightarrow$ Newton's second law.
- Need to prescribe lapse N and shift B^i : *choice of coordinates*.

Form well adapted for numerical simulations (Runge-Kutta type algorithms).

Constraint equations

- The evolution equations are not all of Einstein's equations.
- The constraints are 4 equations, that do not contain ∂_t.
- Such equations are absent in Newtonian dynamics but not for Maxwell.

Type	Einstein	Maxwell
Constraints	Hamiltonian $R + K^2 - K_{ij}K^{ij} = 0$ Momentum : $D_j K^{ij} - D^i K = 0$	$\nabla \cdot \vec{E} = 0$ $\nabla \cdot \vec{B} = 0$
Evolution	$\frac{\partial \gamma_{ij}}{\partial t} - \mathcal{L}_{\vec{B}} \gamma_{ij} = -2NK_{ij}$ $\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\vec{B}} K_{ij} = -D_i D_j N + N\left(R_{ij} - 2K_{ik}K^k_j + KK_{ij}\right)$	$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \mu_0}\left(\vec{\nabla} \times \vec{B}\right)$ $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

A two-step process

Initial value problem

- Find a set of fields $\gamma_{ij}\left(t=0\right)$ and $K_{ij}\left(t=0\right)$ fulfilling the constraints.
- It means solving coupled non-linear elliptic equations.
- Difficulty to make the link with physical situations.

The evolution

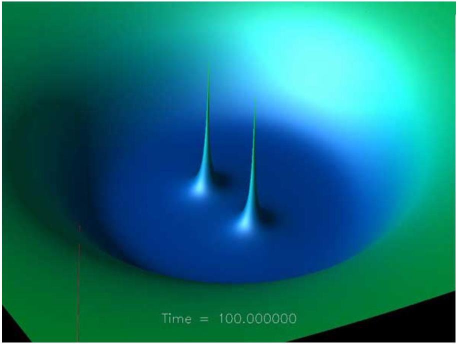
- Solve hyperbolic evolution equations.
- Difficulties :
  - maintaining the accuracy (i.e. the constraints).
  - Dealing with formation of shocks, black holes.
  - Treating outgoing wave boundary conditions.

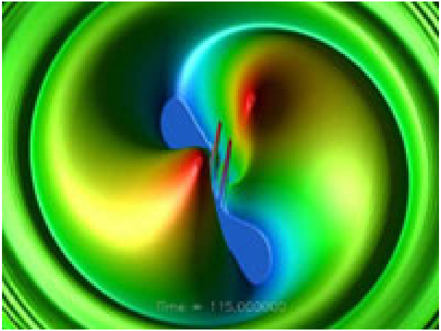
The *holy grail* of numerical relativity : the binary black hole case

A few important milestones

- 1975-1977 : First head-on collision (Smarr & Eppley).
- 1994-1998 : *The Grand Challenge* : improved head-on collision.
- 1999-2000 : Grazing collision (AEI, PSU).
- 2005-2006 : First orbits (Caltech, UTB, NASA)
- 2007-2014 : many orbits, study of various physical effects.
- 2015 : the detection

After decades of struggling ... a breakthrough





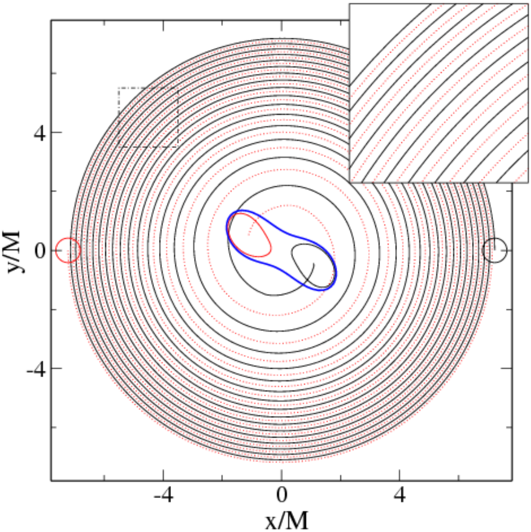
F. Pretorius 2005

Reason for the breakthrough ?

Several factors

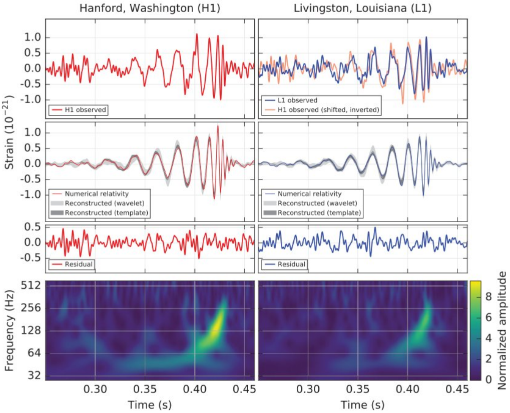
- Better formulation : BSSN formalism.
- Improved gauge choice.
- High order numerical algorithms.
- Better treatment of the holes.
- Bigger computers.

Simplest cases are well controlled



simulation from Cornell group

Very good match with observations



Not only time-evolution

Some situations do no require time evolution

- Stationary configurations (boson stars, neutron stars...)
- Configurations with some time symmetry: helical Killing vector (quasi-circular orbit, geons).
- (Quasi-)periodic solutions (oscillons, oscillatons).

- Mathematically : prescribe ∂_t operator (0, $\Omega\partial_\varphi$...)
- Produces elliptic system.
- Additional difficulties : regularity, boundary conditions...

Quasi-circular binaries

Properties

- Assume the orbits are closed and circular.
- Not exact due to gravitational waves emission.
- Enables to remove time by $\partial_t \longrightarrow \Omega\partial_\varphi$.
- Good approximation for widely separated objects.
- GW can be killed by the so-called conformal flatness approximation $\gamma_{ij} = \Psi^4 f_{ij}$.

Mathematical problem

- 5 unknown fields.
- 5 coupled, non-linear, elliptic equations.
- non-trivial boundary conditions on the horizons.

Laws of thermodynamics for corotating BBH

Zeroth law

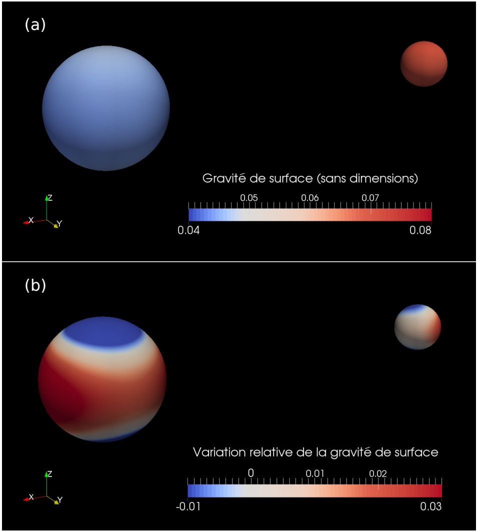
The surface gravity is constant (but different) on the horizons.

$$\kappa^2 = -\frac{1}{2} \nabla^\alpha k^\beta \nabla_\alpha k_\beta \Big|_{\mathcal{H}}$$

where k^α is the null generator of the horizon.

There exist also 2 other laws...

Surface gravity



Conclusions

Numerical relativity has overcome many hurdles and work has just begun to extract meaningful astrophysical informations from the simulations. (Baumgarte and Shapiro).

- Gravitational supernovae
  - Many other ingredients (EOS, neutrinos...)
  - Not many relevant results in full GR.
  - Some works with approximations of GR (conformal flatness).
- Links with cosmology
- Critical phenomenons
- Alternative theories
- Different geometries (fields in AdS for instance)