



journée de la Société Française de Physique
division Champs et Particules

METHODES THEORIQUES POUR LES ONDES GRAVITATIONNELLES

Luc Blanchet

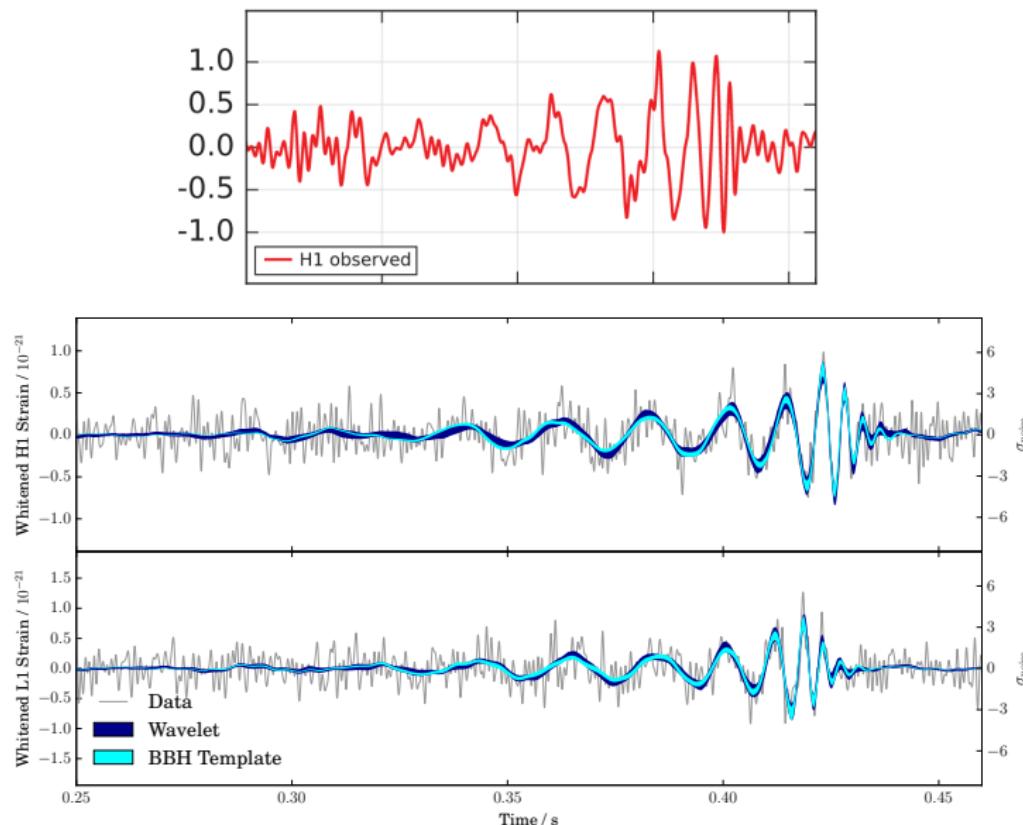
Gravitation et Cosmologie (GReCO)
Institut d'Astrophysique de Paris

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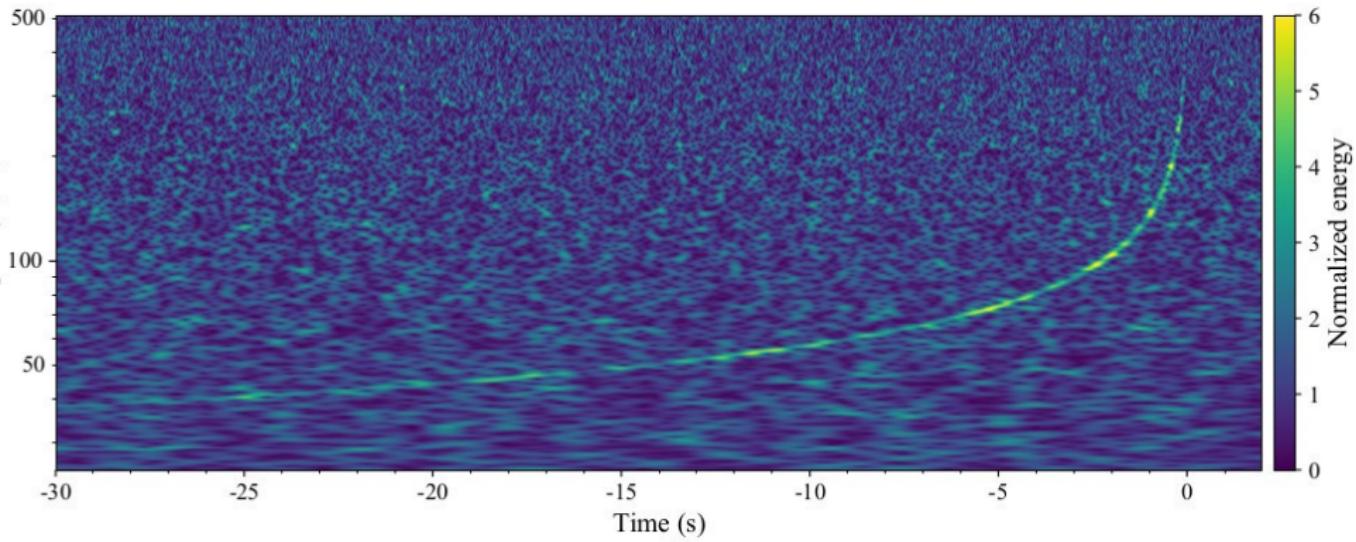
Binary black-hole event GW150914

[LIGO/Virgo collaboration 2016]

Hanford, Washington (H1)



Binary neutron star event GW170817 [LIGO/Virgo 2017]

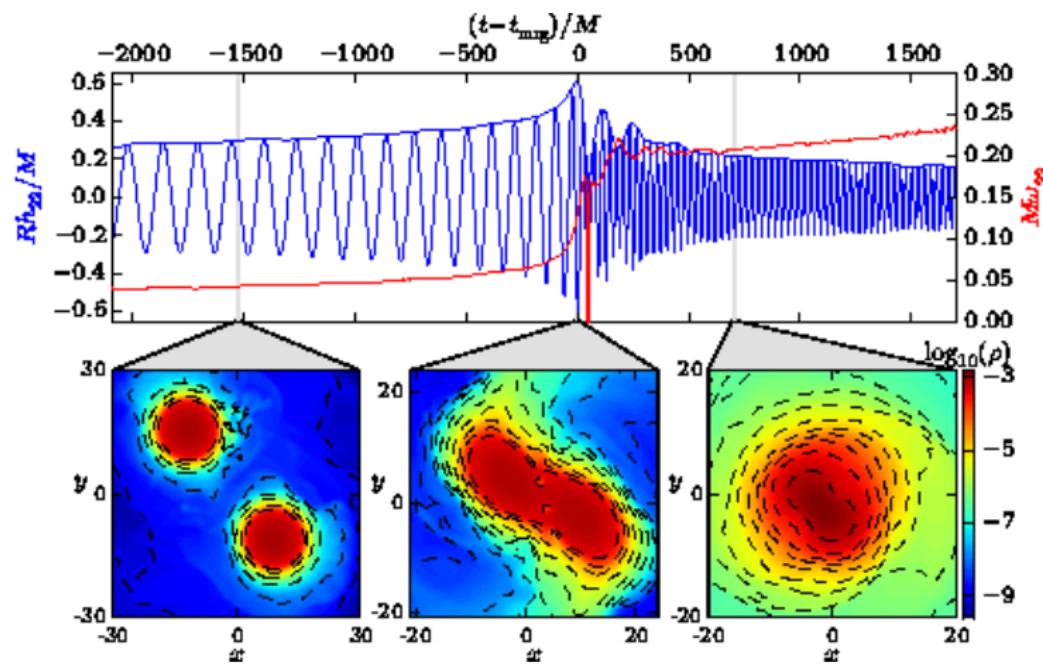


- The signal is observed during ~ 100 s and ~ 3000 cycles and is the loudest gravitational-wave signal yet observed with a **combined SNR of 32.4**
- The advent of **multi-messenger Astronomy** with the concomitant discovery of a short gamma ray burst and an optical kilonova

[see the talk by Marie-Anne Bizouard]

Post-merger waveform of neutron star binaries

[Shibata et al., Rezzolla et al. 1990-2010s]



100 years of gravitational radiation [Einstein 1916]

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DOC. 32 INTEGRATION OF FIELD EQUATIONS

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. EINSTEIN.



Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \quad (1)$$

definierten Größen $\gamma_{\mu\nu}$, welche linearen orthogonalen Transformationen gegenüber Tensoreharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$.

Wir werden zeigen, daß diese $\gamma_{\mu\nu}$ in analoger Weise berechnet werden können wie die retardierten Potentiale der Elektrodynamik.

small perturbation of
Minkowski's metric

Wave-like solutions in metric theories of gravity

- ① Small perturbation of the metric around flat space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1$$

- ② Restrict attention to metric theories of gravity admitting wave-like solutions propagating at the speed of light: $c_g = 1$. Far from the sources the waves are almost planar, hence

$$\square h_{\mu\nu} = 0 \iff h_{\mu\nu} = h_{\mu\nu}(t - z)$$

- ③ From the linearized Bianchi's identity obtain

$$\square R_{\mu\nu\rho\sigma} = 0 \iff R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}(t - z)$$

showing that GWs have an **invariant, coordinate-independent meaning**

- ④ The six components R_{0i0j} (where $i, j = x, y, z$) represent **six independent components** (polarization modes)
- ⑤ In GR $R_{\mu\nu} = 0$ hence there are only **two independent polarization modes**

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

$$4\pi R^2 \bar{G} = \frac{\kappa}{40\pi} \left[\sum_{\mu\nu} \bar{J}_{\mu\nu} - \frac{1}{3} \left(\sum_{\mu} \bar{J}_{\mu\mu} \right)^2 \right].$$

① Einstein quadrupole formula

$$\left(\frac{dE}{dt} \right)^{\text{GW}} = \frac{G}{5c^5} \left\{ \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3} + \mathcal{O} \left(\frac{v}{c} \right)^2 \right\}$$

② Amplitude quadrupole formula

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 D} \left\{ \frac{d^2 Q_{ij}}{dt^2} \left(t - \frac{D}{c} \right) + \mathcal{O} \left(\frac{v}{c} \right) \right\}^{\text{TT}} + \mathcal{O} \left(\frac{1}{D^2} \right)$$

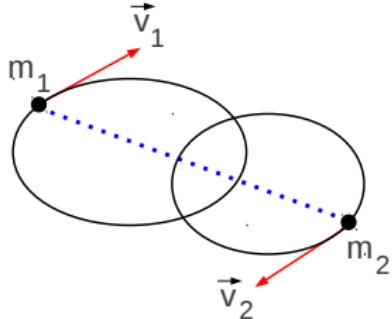
③ Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\text{reac}} = -\frac{2G}{5c^5} \rho x^j \frac{d^5 Q_{ij}}{dt^5} + \mathcal{O} \left(\frac{v}{c} \right)^7$$

which is a $2.5\text{PN} \sim (v/c)^5$ effect in the source's equations of motion

Application to compact binaries

[Peters & Mathews 1963; Peters 1964]



$$\left\{ \begin{array}{l} a \text{ semi-major axis of relative orbit} \\ e \text{ eccentricity of relative orbit} \\ \omega = \frac{2\pi}{P} \text{ orbital frequency} \end{array} \right.$$

$$M = m_1 + m_2$$
$$\mu = \frac{m_1 m_2}{M}$$

$$\nu = \frac{\mu}{M} \quad 0 < \nu \leq \frac{1}{4}$$

Averaged energy and angular momentum balance equations

$$\langle \frac{dE}{dt} \rangle = -\langle \mathcal{F}^{\text{GW}} \rangle \quad \langle \frac{dJ_i}{dt} \rangle = -\langle \mathcal{G}_i^{\text{GW}} \rangle$$

are applied to a Keplerian orbit (using Kepler's law $GM = \omega^2 a^3$)

$$\langle \frac{dP}{dt} \rangle = -\frac{192\pi}{5c^5} \nu \left(\frac{2\pi GM}{P} \right)^{5/3} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1 - e^2)^{7/2}}$$
$$\langle \frac{de}{dt} \rangle = -\frac{608\pi}{15c^5} \nu \frac{e}{P} \left(\frac{2\pi GM}{P} \right)^{5/3} \frac{1 + \frac{121}{304}e^2}{(1 - e^2)^{5/2}}$$

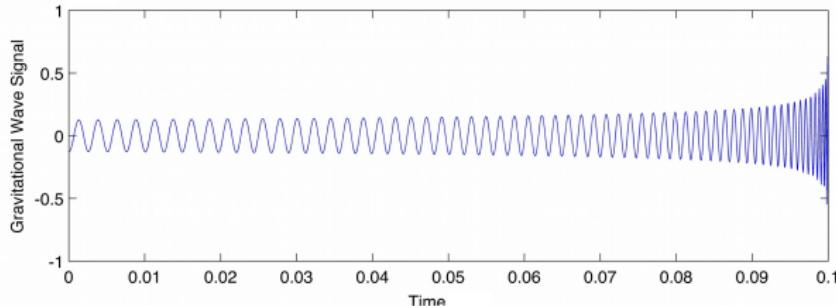
Orbital phase evolution of compact binaries

[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

- ① Compact binaries are circularized when they enter the detector's bandwidth
- ② The amplitude and phase evolution follow an adiabatic chirp in time

$$a(t) = \left(\frac{256}{5} \frac{G^3 M^3 \nu}{c^5} (t_c - t) \right)^{1/4}$$
$$\phi(t) = \phi_c - \frac{1}{32\nu} \left(\frac{256}{5} \frac{c^3 \nu}{GM} (t_c - t) \right)^{5/8}$$

- ③ The amplitude and orbital frequency diverge at the instant of coalescence t_c since the approximation breaks down



Why inspiralling binaries require high PN modelling

[Caltech "3mn paper" 1992; Blanchet & Schäfer 1993]

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PHYSICAL REVIEW LETTERS

17 MAY 1993

The Last Three Minutes: Issues in Gravitational-Wave Measurements of Coalescing Compact Binaries

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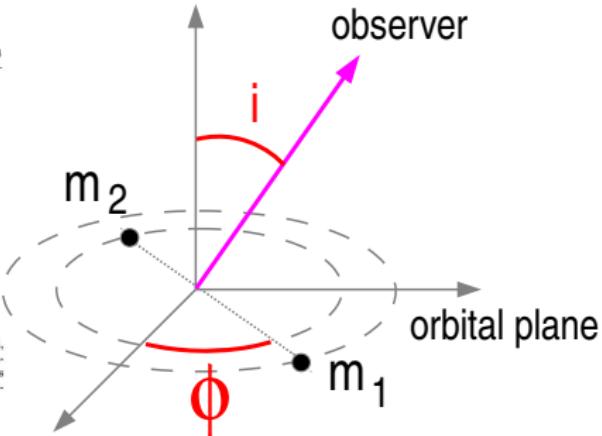
(Received 24 August 1992)

Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy $\ll 10^{-6}$ and relativistic effects are large, the wave forms will be far more complex and carry more information than has been expected. Improved wave form modeling is needed as a foundation for extracting the waves' information, but is not necessary for wave detection.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2], and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network

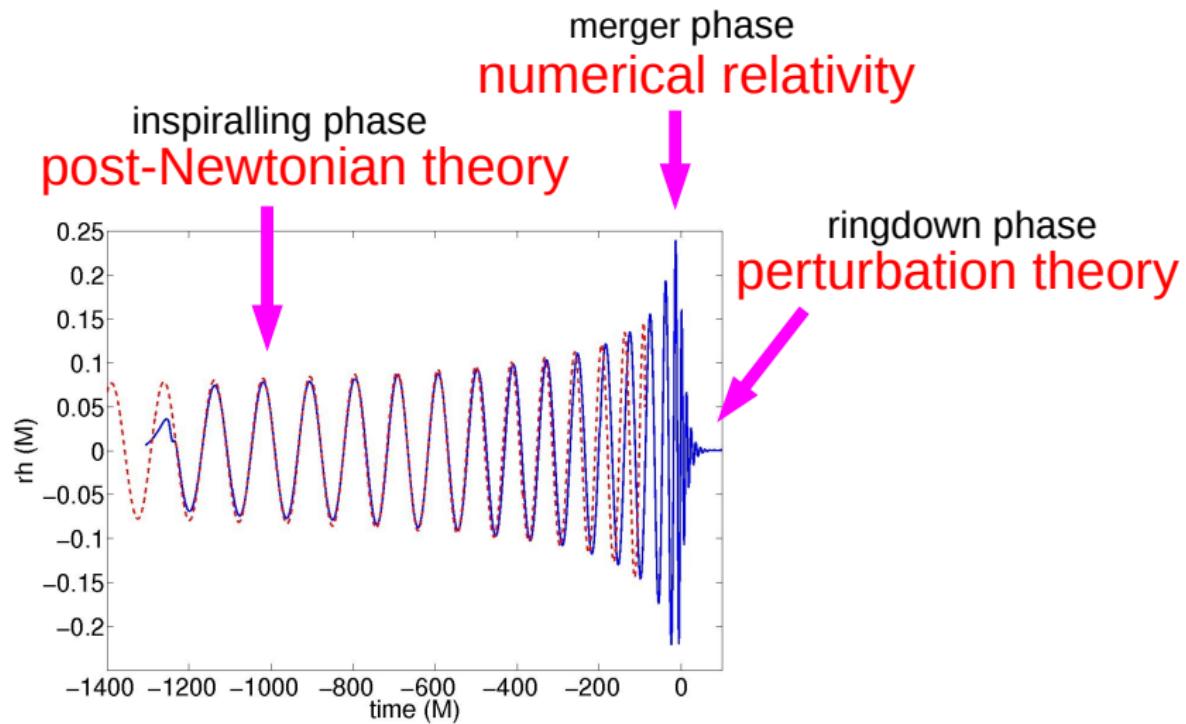
as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. This sensitivity to phase does not mean that accurate templates are needed in searches for the waves (see below). How-



$$\phi(t) = \phi_0 - \underbrace{\frac{1}{\nu} \left(\frac{GM\omega}{c^3} \right)^{-5/3}}_{\text{quadrupole formalism}} \left\{ 1 + \underbrace{\frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \cdots}_{\text{needs to be computed with 3PN precision at least}} + \frac{3\text{PN}}{c^6} + \cdots \right\}$$

Here 3PN means 5.5PN as a radiation reaction effect !

The gravitational chirp of compact binaries



The GW templates of compact binaries

- ① In principle, the templates are obtained by matching together:
 - A **high-order 3.5PN waveform** for the inspiral [Blanchet *et al.* 1998, 2002, 2004]
 - A **highly accurate numerical waveform** for the merger and ringdown [Pretorius 2005; Baker *et al.* 2006; Campanelli *et al.* 2006]
- ② In practice, for **black hole binaries** (such as GW150914), effective methods that interpolate between the PN and NR play a key role in the data analysis
 - **Hybrid inspiral-merger-ringdown (IMR)** waveforms [Ajith *et al.* 2011] are constructed by matching the PN and NR waveforms in a time interval through an intermediate phenomenological phase
 - **Effective-one-body (EOB)** waveforms [Buonanno & Damour 1998] are based on resummation techniques extending the domain of validity of the PN approximation beyond the inspiral phase
- ③ In the case of **neutron star binaries** (such as GW170817), the masses are smaller and the templates are entirely **based on the 3.5PN waveform**

Methods to compute PN equations of motion

- ① ADM Hamiltonian canonical formalism [Ohta *et al.* 1973; Schäfer 1985]
 - ② EOM in harmonic coordinates [Damour & Deruelle 1985; Blanchet & Faye 1998, 2000]
 - ③ Extended fluid balls [Grishchuk & Kopeikin 1986]
 - ④ Surface-integral approach [Itoh, Futamase & Asada 2000]
 - ⑤ Effective-field theory (EFT) [Goldberger & Rothstein 2006; Foffa & Sturani 2011]
-
- EOM derived in a general frame for arbitrary orbits
 - Dimensional regularization is applied for UV divergences¹
 - Radiation-reaction dissipative effects added separately by matching
 - Spin effects can be computed within a pole-dipole approximation
 - Tidal effects incorporated at leading 5PN and sub-leading 6PN orders

¹Except in the surface-integral approach

Methods to compute PN radiation field

- ① Multipolar-post-Minkowskian (MPM) & PN [Blanchet-Damour-Iyer 1986, ..., 1998]
 - ② Direct iteration of the relaxed field equations (DIRE) [Will-Wiseman-Pati 1996, ...]
 - ③ Effective-field theory (EFT) [Hari Dass & Soni 1982; Goldberger & Ross 2010]
-
- Involves a machinery of tails and related non-linear effects
 - Uses dimensional regularization to treat point-particle singularities
 - Phase evolution relies on balance equations valid in adiabatic approximation
 - Spin effects are incorporated within a pole-dipole approximation
 - Provides polarization waveforms for DA & spin-weighted spherical harmonics decomposition for NR

Einstein field equations as a “Problème bien posé”

- Start with the GR action for the metric $g_{\mu\nu}$ with the matter term

$$S_{\text{GR}} = \underbrace{\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R}_{\text{Einstein-Hilbert action}} + \underbrace{S_m[g_{\mu\nu}, \Psi]}_{\text{matter fields}}$$

- Add the harmonic coordinates gauge-fixing term (where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}$)

$$S_{\text{GR}} = \frac{c^3}{16\pi G} \int d^4x \left(\sqrt{-g} R - \underbrace{\frac{1}{2} \mathfrak{g}_{\alpha\beta} \partial_\mu \mathfrak{g}^{\alpha\mu} \partial_\nu \mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}} \right) + S_m$$

- Obtain a well-posed system of equations [Choquet-Bruhat 1952]

$$\begin{aligned} \mathfrak{g}^{\mu\nu} \partial_{\mu\nu}^2 \mathfrak{g}^{\alpha\beta} &= \frac{16\pi G}{c^4} |g| T^{\alpha\beta} + \underbrace{\Sigma^{\alpha\beta}[\mathfrak{g}, \partial \mathfrak{g}]}_{\text{non-linear source term}} \\ \underbrace{\partial_\mu \mathfrak{g}^{\alpha\mu}}_{\text{harmonic-gauge condition}} &= 0 \end{aligned}$$

Post-Minkowskian expansion [e.g. Bertotti & Plebanski 1960]

- Appropriate for **weakly self-gravitating** isolated matter sources

$$\gamma_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \quad \left\{ \begin{array}{l} M \text{ mass of source} \\ a \text{ size of source} \end{array} \right.$$

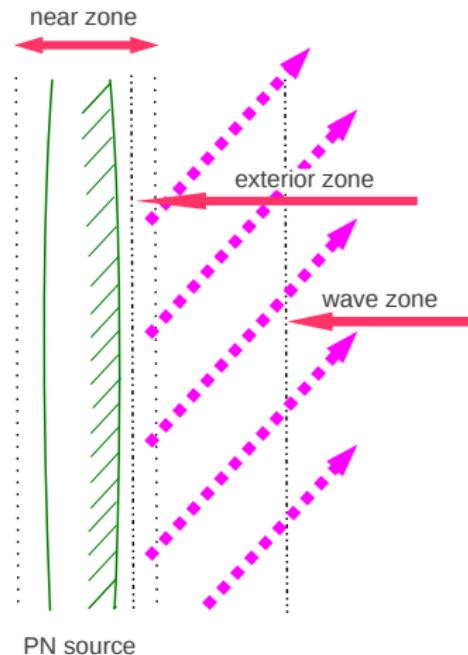
$$g^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n h_{(n)}^{\alpha\beta}}_{G \text{ labels the PM expansion}}$$

$$\square h_{(n)}^{\alpha\beta} = \frac{16\pi G}{c^4} |g| T_{(n)}^{\alpha\beta} + \overbrace{\Lambda_{(n)}^{\alpha\beta}[h_{(1)}, \dots, h_{(n-1)}]}^{\text{know from previous iterations}}$$
$$\partial_\mu h_{(n)}^{\alpha\mu} = 0$$

- Very difficult approximation to implement in practice for general sources at high post-Minkowskian orders

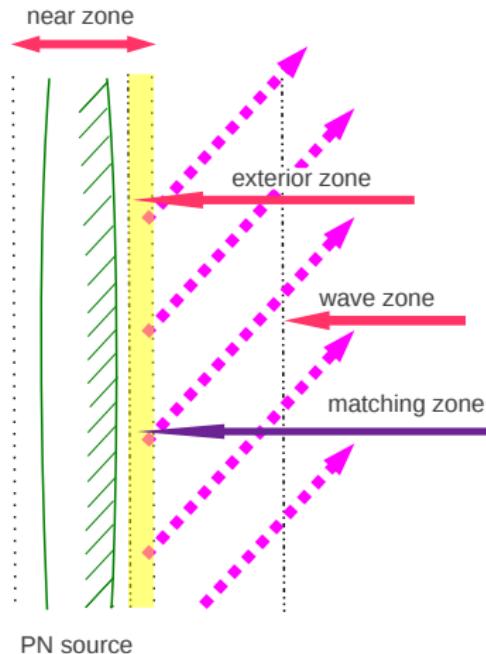
The MPM-PN formalism [BDI; Blanchet 1995, ...]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



The MPM-PN formalism [BDI; Blanchet 1995, ...]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



$$\overbrace{\mathcal{M}(h^{\alpha\beta})}^{\text{matching equation}} = \mathcal{M}(\bar{h}^{\alpha\beta})$$

The MPM-PN formalism [BDI; Blanchet 1995, ...]

- ① Radiative multipole moments observed at infinity from the source (\mathcal{J}_+)

$$U_L(T - R/c), \quad V_L(T - R/c)$$

- ② Source multipole moments describe a specific matter system

$$I_L(t), \quad J_L(t)$$

- The relations between the radiative moments and the source moments are obtained by the MPM algorithm
- The expressions of the source moments in terms of the source parameters follow from the matching to the PN source
- The radiation reaction effects in the PN solution are also obtained

The source multipole moments [Blanchet 1998]

$$I_L(t) = \text{PF} \int d^3x \int_{-1}^1 dz \left\{ \delta_\ell \hat{x}_L \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1} \hat{x}_{iL} \Sigma_i^{(1)} + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2} \hat{x}_{ijL} \Sigma_{ij}^{(2)} \right\} \Big|_{(\mathbf{x}, t+zr/c)}$$

$$J_L(t) = \text{PF} \int d^3x \int_{-1}^1 dz \epsilon_{ab\langle i_\ell} \left\{ \delta_\ell \hat{x}_{L-1\rangle a} \Sigma_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)} \delta_{\ell+1} \hat{x}_{L-1\rangle ac} \Sigma_{bc}^{(1)} \right\} \Big|_{(\mathbf{x}, t+zr/c)}$$

- Σ , Σ_i and Σ_{ij} are the matter currents defined from the PN expansion of the components of the source's stress-energy pseudo tensor
- The **FP** procedure means the Hadamard "Partie Finie" and plays the role of an **IR regularization** in the multipole moments

The radiative quadrupole moment

[Marchand, Blanchet & Faye 2016]

$$\begin{aligned}
 U_{ij}(t) = & I_{ij}^{(2)}(t) + \underbrace{\frac{G\textcolor{red}{M}}{c^3} \int_0^{+\infty} d\tau I_{ij}^{(4)}(t-\tau) \left[2 \ln \left(\frac{\tau}{2\tau_0} \right) + \frac{11}{6} \right]}_{\text{1.5PN tail integral}} \\
 & + \frac{G}{c^5} \left\{ \underbrace{-\frac{2}{7} \int_0^{+\infty} d\tau I_{a < i}^{(3)} I_{j > a}^{(3)}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms} \right\} \\
 & + \underbrace{\frac{G^2 \textcolor{red}{M}^2}{c^6} \int_0^{+\infty} d\tau I_{ij}^{(5)}(t-\tau) \left[2 \ln^2 \left(\frac{\tau}{2\tau_0} \right) + \frac{57}{35} \ln \left(\frac{\tau}{2\tau_0} \right) + \frac{124627}{22050} \right]}_{\text{3PN tail-of-tail integral}} \\
 & + \underbrace{\frac{G^3 \textcolor{red}{M}^3}{c^9} \int_0^{+\infty} d\tau I_{ij}^{(6)}(t-\tau) \left[\frac{4}{3} \ln^3 \left(\frac{\tau}{2\tau_0} \right) + \dots + \frac{129268}{33075} + \frac{428}{315} \pi^2 \right]}_{\text{4.5PN tail-of-tail-of-tail integral}} \\
 & + \mathcal{O} \left(\frac{1}{c^{10}} \right)
 \end{aligned}$$

Phasing formula of inspiralling compact binaries

[BDIWW 1995; B 1996, 1998; BIJ 2002, BFIJ 2002; BDEI 2006]

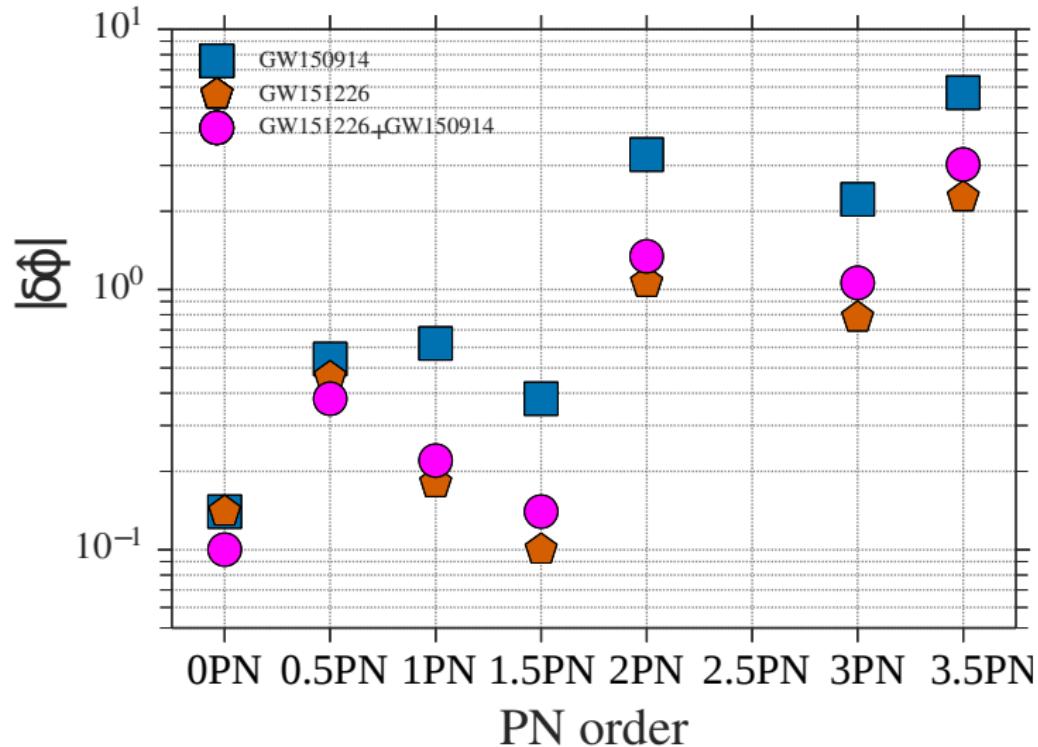
$$\phi(\omega) = \phi_0 - \frac{1}{32\nu} \left(\frac{GM\omega}{c^3} \right)^{-5/3} \left\{ 1 + \overbrace{\left(\frac{3715}{1008} + \frac{55}{12}\nu \right) \left(\frac{GM\omega}{c^3} \right)^{2/3}}^{1\text{PN}} - \overbrace{10\pi \left(\frac{GM\omega}{c^3} \right)}^{1.5\text{PN (tail)}} + \overbrace{\left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2 \right) \left(\frac{GM\omega}{c^3} \right)^{4/3}}^{2\text{PN}} + \dots \right\}$$

The phase evolution is currently known up to 3.5PN order

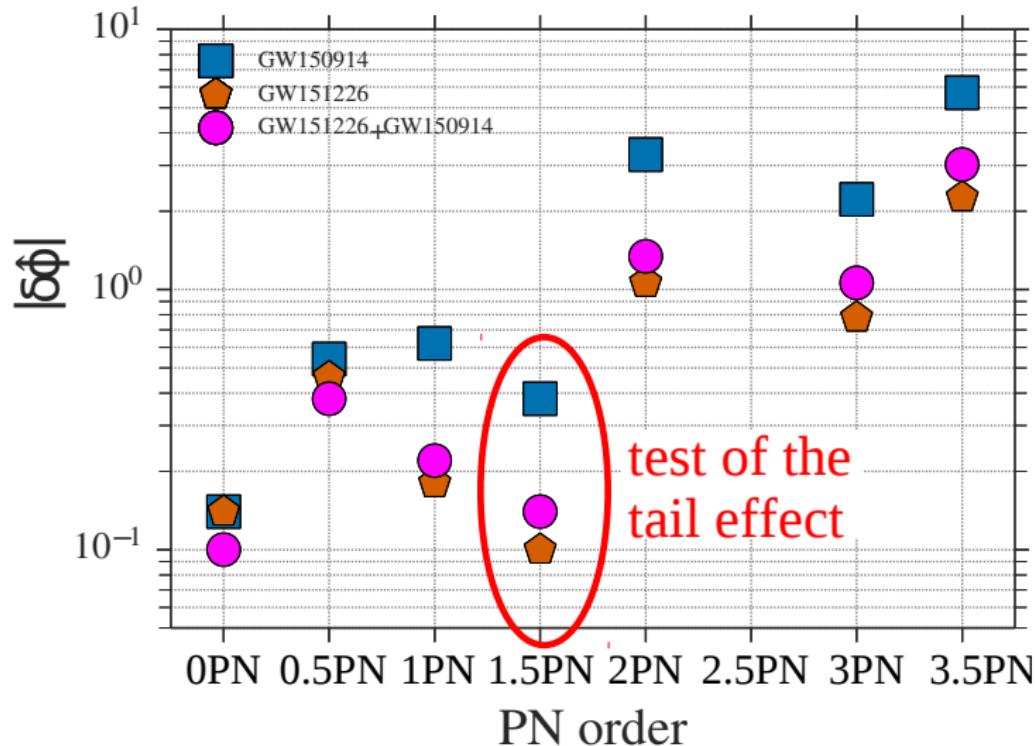
Typical coefficient in the 3.5PN phasing formula

$$\frac{12348611926451}{18776862720}$$

Measurement of PN parameters [LIGO/VIRGO 2016]

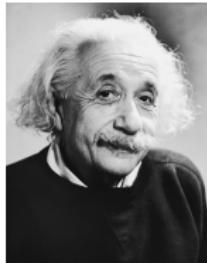


Measuring GW tails [Blanchet & Sathyaprakash 1994, 1995]



The 1PN equations of motion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]



$$\begin{aligned} \frac{d^2\mathbf{r}_A}{dt^2} = & - \sum_{B \neq A} \frac{Gm_B}{r_{AB}^2} \mathbf{n}_{AB} \left[1 - 4 \sum_{C \neq A} \frac{Gm_C}{c^2 r_{AC}} - \sum_{D \neq B} \frac{Gm_D}{c^2 r_{BD}} \left(1 - \frac{\mathbf{r}_{AB} \cdot \mathbf{r}_{BD}}{r_{BD}^2} \right) \right. \\ & \left. + \frac{1}{c^2} \left(\mathbf{v}_A^2 + 2\mathbf{v}_B^2 - 4\mathbf{v}_A \cdot \mathbf{v}_B - \frac{3}{2} (\mathbf{v}_B \cdot \mathbf{n}_{AB})^2 \right) \right] \\ & + \sum_{B \neq A} \frac{Gm_B}{c^2 r_{AB}^2} \mathbf{v}_{AB} [\mathbf{n}_{AB} \cdot (3\mathbf{v}_B - 4\mathbf{v}_A)] - \frac{7}{2} \sum_{B \neq A} \sum_{D \neq B} \frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3} \mathbf{n}_{BD} \end{aligned}$$

4PN: state-of-the-art on equations of motion

$$\frac{dv_1^i}{dt} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \underbrace{\left(\frac{1}{c^2} \left\{ \left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \dots \right] n_{12}^i + \dots \right\} \right)}_{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4} [\dots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5} [\dots]}_{\text{2.5PN radiation reaction}} + \underbrace{\frac{1}{c^6} [\dots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7} [\dots]}_{\text{3.5PN radiation reaction}} + \underbrace{\frac{1}{c^8} [\dots]}_{\text{4PN conservative \& radiation tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc]	Fokker Lagrangian
	[Foffa & Sturani 2012, 2013] (partial results)	Effective field theory

Fokker action of N particles [Fokker 1929]



- ① Gauge-fixed Einstein-Hilbert action for N point particles

$$S_{\text{g.f.}} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[R - \underbrace{\frac{1}{2} g_{\mu\nu} \Gamma^\mu \Gamma^\nu}_{\text{Gauge-fixing term}} \right] - \sum_A \underbrace{m_A c^2 \int dt \sqrt{-(g_{\mu\nu})_A v_A^\mu v_A^\nu / c^2}}_{N \text{ point particles}}$$

- ② Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

$$g_{\mu\nu}(\mathbf{x}, t) \longrightarrow \bar{g}_{\mu\nu}(\mathbf{x}; \mathbf{y}_B(t), \mathbf{v}_B(t), \dots)$$

- ③ The PN equations of motion of the N particles (self-gravitating system) are

$$\boxed{\frac{\delta S_F}{\delta \mathbf{y}_A} \equiv \frac{\partial L_F}{\partial \mathbf{y}_A} - \frac{d}{dt} \left(\frac{\partial L_F}{\partial \mathbf{v}_A} \right) + \dots = 0}$$

- ④ The Fokker action is equivalent to the effective action used by the EFT

Gravitational wave tail effect at the 4PN order

[Blanchet & Damour 1988; Blanchet 1993, 1997; Foffa & Sturani 2011; Galley et al. 2016]

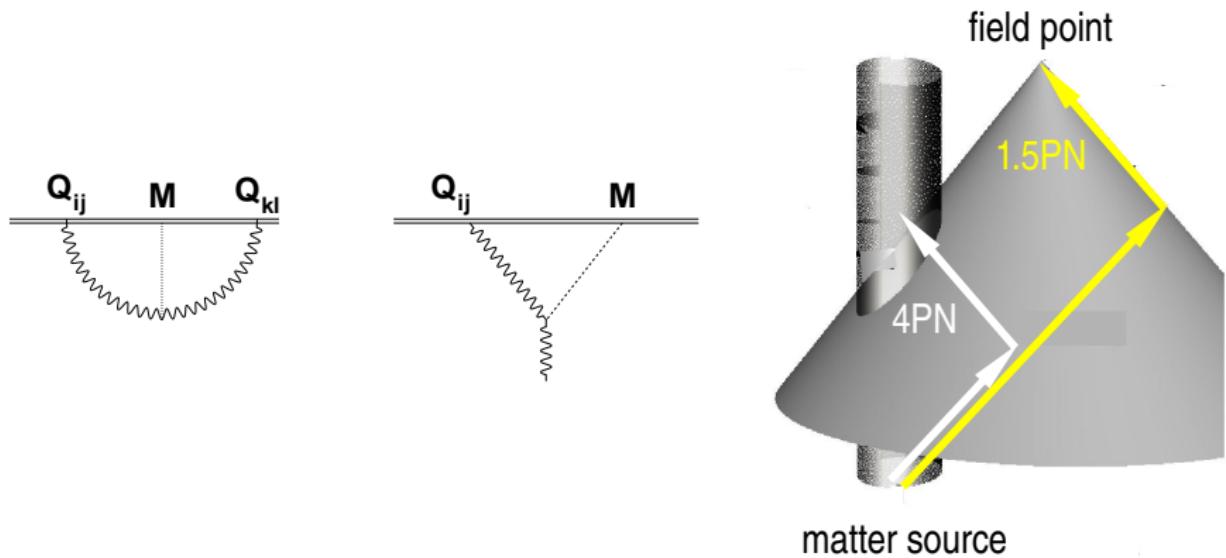
- ① At the 4PN order appear gravitational wave tails in the local (near-zone) dynamics of the source
- ② This leads to a **non-local-in-time** contribution in the Fokker action

$$S_F^{\text{tail}} = \frac{G^2 M}{5c^8} \text{PF} \iint \frac{dt dt'}{|t - t'|} Q_{ij}^{(3)}(t) Q_{ij}^{(3)}(t')$$

- ③ The tail effect implies the appearance of **IR divergences** in the Fokker action at the 4PN order
- ④ It corresponds to specific Feynman diagrams in the EFT

Gravitational wave tail effect at the 4PN order

[Blanchet & Damour 1988; Blanchet 1993, 1997; Foffa & Sturani 2011; Galley et al. 2016]



$$S_F^{\text{tail}} = \frac{G^2 M}{5c^8} \text{PF} \iint \frac{dt dt'}{|t - t'|} Q_{ij}^{(3)}(t) Q_{ij}^{(3)}(t')$$

Problem of the IR divergences

- ① The tail effect implies the appearance of **IR divergences** in the Fokker action at the 4PN order
- ② Our initial calculation of the Fokker action was based on the Hadamard regularization to treat the IR divergences (**FP** procedure when $B \rightarrow 0$)
- ③ However computing the conserved energy and periastron advance for circular orbits we found it does not agree with GSF calculations
- ④ The problem was due to the HR and conjectured that a different IR regularization would give (modulo shifts)

$$L = L^{\text{HR}} + \underbrace{\frac{G^4 m m_1^2 m_2^2}{c^8 r_{12}^4} \left(\delta_1 (n_{12} v_{12})^2 + \delta_2 v_{12}^2 \right)}_{\text{two ambiguity parameters } \delta_1 \text{ and } \delta_2}$$

- ⑤ Matching with GSF results for the energy and periastron advance uniquely fixes the two ambiguity parameters and we are in complete agreement with the results from the Hamiltonian formalism [DJS]

Dimensional regularization of the IR divergences

- The Hadamard regularization of IR divergences reads

$$I_{\mathcal{R}}^{\text{HR}} = \underset{B=0}{\text{FP}} \int_{r>\mathcal{R}} d^3 \mathbf{x} \left(\frac{r}{r_0} \right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I_{\mathcal{R}}^{\text{DR}} = \int_{r>\mathcal{R}} \frac{d^d \mathbf{x}}{\ell_0^{d-3}} F^{(\mathbf{d})}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d - 3$)

$$\boxed{\mathcal{D}I = \sum_q \underbrace{\left[\frac{1}{(q-1)\varepsilon} - \ln \left(\frac{r_0}{\ell_0} \right) \right]}_{\text{IR pole}} \int d\Omega_{2+\varepsilon} \varphi_{3,q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon)}$$

Ambiguity-free completion of the 4PN EOM

[Marchand, Bernard, Blanchet & Faye 2017]

- ① The tail effect contains a UV pole which cancels the IR pole coming from the instantaneous part of the action

$$S_F^{\text{tail}} = \frac{2G^2 M}{5c^8} \int_{-\infty}^{+\infty} Q_{ij}^{(3)}(t) \int_0^{+\infty} d\tau \left[\ln \left(\frac{c\sqrt{\bar{q}}\tau}{2\ell_0} \right) \underbrace{-\frac{1}{2\varepsilon}}_{\text{UV pole}} + \frac{41}{60} \right] Q_{ij}^{(4)}(t - \tau)$$

- ② For the tail effect we are in complete agreement with the EFT calculation based on a single Feynman diagram [Galley, Leibovich, Porto & Ross 2011]
- ③ Adding up all contributions we obtain the conjectured form of the ambiguity terms with the correct values of the ambiguity parameters δ_1 and δ_2
- ④ The lack of a consistent matching in the ADM Hamiltonian formalism [DJS] forces this formalism to be plagued by one ambiguity parameter