Nambu-Goldstone Bosons:
from Composite Higgs to Scattering Amplitudes

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In 1960, Gell-Mann and Levy introduced the nonlinear sigma model (nlsm) as a “toy model” for pion-nucleon interactions

The Axial Vector Current in Beta Decay (*).

M. GELL-MANN (**)

Collège de France and Ecole Normale Supérieure - Paris (***)

M. LÉVY

Faculte des Sciences, Orsay, and Ecole Normale Supérieure - Paris (***)

(ricevuto il 19 Febbraio 1960)

$$\mathscr{L}_3 = \left\{ -\bar{N}[\gamma\,\partial - g_0(\sigma' + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5)]\,N - \frac{(\partial\boldsymbol{\pi})^2}{2} - \frac{(\partial\sigma')^2}{2} - \frac{\mu_0^2}{2f_0}\,\sigma' \right\}$$

$$\sigma' = -\sqrt{\frac{1}{4f_0^2} - \pi^2} = -\frac{1}{2f_0}\sqrt{1 - 4f_0^2\pi^2}$$

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The symmetry perspective is very powerful, and has led to important insights in many areas of theoretical physics, including condensed matter, statistical, particle, and mathematical physics.

This viewpoint also makes it clear that nlsm is about the infrared structure of the theory in the presence of nonlinearly realized symmetries, in particular the nontrivial space of degenerate vacua!

More generally, the Infrared structure of QFT's is an even older subject in physics.

- In both QED and Gravity, scattering amplitudes with one soft gauge boson factorize universally:

$$M_{n+1}(k_1, \cdots, k_n; q) = (S^{(0)} + S^{(1)} + \cdots)M_n(k_1, \cdots, k_n)$$

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QED:

$$S^{(0)} \equiv \sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q}\,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} e_i \frac{\varepsilon_{q\mu} q_\nu J_i^{\mu\nu}}{k_i \cdot q}\,,$$

Gravity:

$$S^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q}\,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q}\,,$$

$$J_i^{\mu\nu} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right)$$

The universality of these expressions implies they must follow from some general prime principles.

Indeed, the leading soft factors of both QED and Gravity follow from the "on-shell gauge invariance":

$$q_\mu M_n^{\mu}(k_1, \cdots, k_{n-1}; q) = 0$$
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$$\sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q} \xrightarrow{\varepsilon_q^\mu \to \varepsilon_q^\mu + q^\mu} \sum_{i=1}^{n} e_i = 0$$

$$\sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q} \xrightarrow{\varepsilon^{\mu\nu} \to \varepsilon^{\mu\nu} + q^\mu \Lambda^\nu} \sum_{i=1}^{n} k_i^\mu = 0$$

- For scalar theories with a global symmetry, degenerate vacua lead to nonlinear sigma models and Nambu-Goldstone bosons,

In this case, the “single soft” limit of Goldstone scattering amplitudes famously exhibits the “Adler’s zero”:

$$\lim_{\tau\to 0} M_{n+1}(k_1,\cdots,k_n;\tau q) = 0$$

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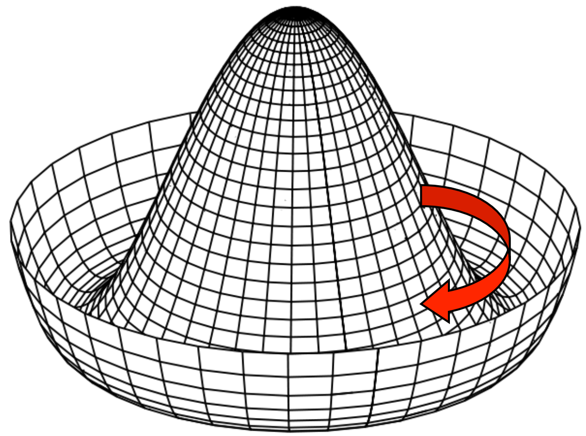
These are statements we knew from half a century ago:

Bloch and Nordsieck (1937), F.E. Low (1954+…), Gell-Mann and Goldberger (1954), Weinberg (1965+…), Adler (1965+…) and etc.

The Adler's zero is a direct consequence of the presence of nontrivial vacua.

Recall the different vacua are related by a rotation in the broken direction:

$$e^{i\theta}|\theta_0\rangle = |\theta_0 + \theta\rangle$$



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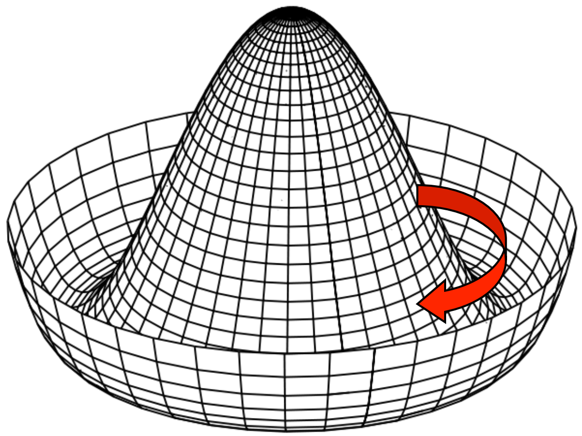


$$e^{i\theta}|\theta_0\rangle = |\theta_0 + \theta\rangle$$

Excitations along the broken direction gives the Goldstone boson,

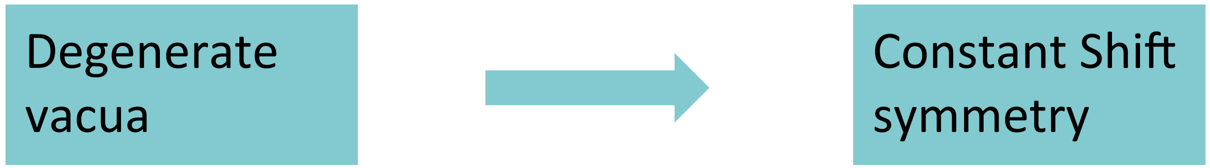
$$e^{i(\rho(x)+\theta)}|\theta_0\rangle = e^{i\rho(x)}|\theta_0 + \theta\rangle$$

But the physics is invariant whether one chooses $|\theta_0\rangle$ or $|\theta_0 + \theta\rangle$

NLSM possesses a constant "shift symmetry"!

The Adler's zero is a direct consequence of the presence of nontrivial vacua.

The usual reasoning:



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The usual reasoning:

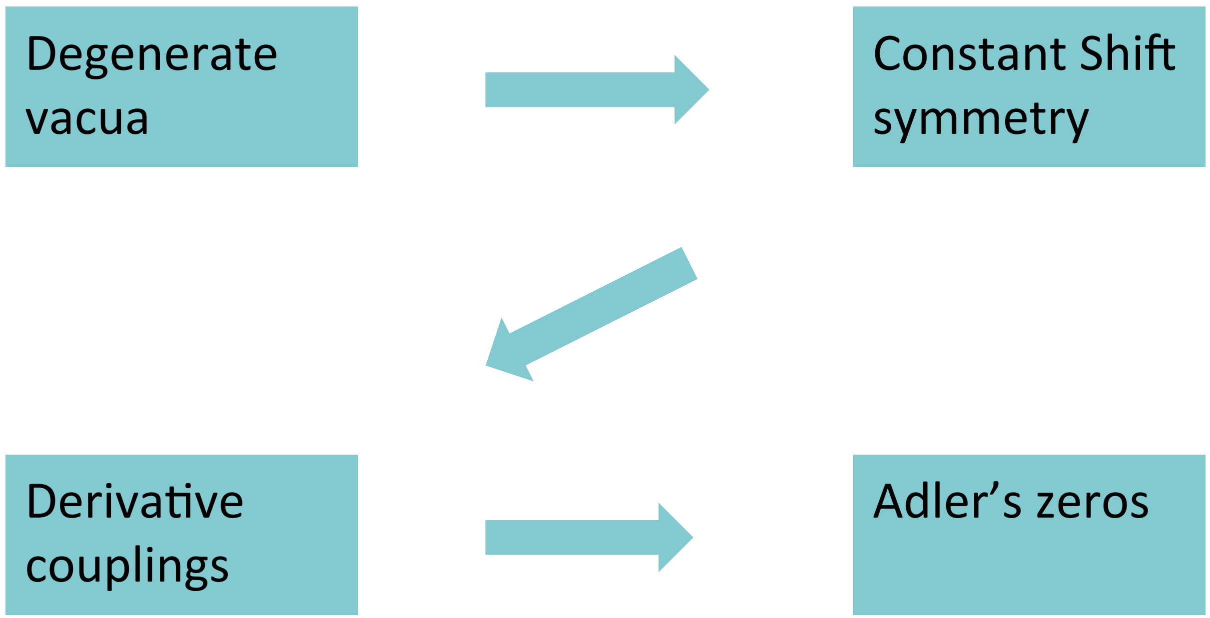
Degenerate vacua

Constant Shift symmetry

Derivative couplings

The Adler's zero is a direct consequence of the presence of nontrivial vacua.

The usual reasoning:



I will report on recent progresses in understanding the infrared structure of the Nambu-Goldstone bosons.

These progresses are driven by two unrelated areas:

- Composite Higgs models where the Higgs arises as a pseudo-Nambu-Goldstone boson.
- Cachazo-He-Yuan (CHY) representations of scattering equations for tree-level on-shell amplitudes.

The naturalness problem has been driving BSM physics for decades.
Two ways to stabilize the Higgs mass using symmetry:

1) bosonic global symmetry ----> higgs as a pseudo Nambu-
Goldstone boson (PNGB)!

2) fermionic global symmetry -----> supersymmetry

- supersymmetric theories are all built upon a minimal lagrangian -- the MSSM:

$$W_{\rm MSSM} = \overline{u}\mathbf{y_u}QH_u - \overline{d}\mathbf{y_d}QH_d - \overline{e}\mathbf{y_e}LH_d + \mu H_u H_d$$

$$\begin{aligned}\mathcal{L}_{\rm soft}^{\rm MSSM} \;=\;& -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B} + {\rm c.c.}\right)\\ & -\left(\widetilde{\overline{u}}\,\mathbf{a_u}\,\widetilde{Q}H_u - \widetilde{\overline{d}}\,\mathbf{a_d}\,\widetilde{Q}H_d - \widetilde{\overline{e}}\,\mathbf{a_e}\,\widetilde{L}H_d + {\rm c.c.}\right)\\ & -\widetilde{Q}^\dagger\,\mathbf{m^2_Q}\,\widetilde{Q} - \widetilde{L}^\dagger\,\mathbf{m^2_L}\,\widetilde{L} - \widetilde{\overline{u}}\,\mathbf{m^2_{\overline{u}}}\,\widetilde{\overline{u}}^\dagger - \widetilde{\overline{d}}\,\mathbf{m^2_{\overline{d}}}\,\widetilde{\overline{d}}^\dagger - \widetilde{\overline{e}}\,\mathbf{m^2_{\overline{e}}}\,\widetilde{\overline{e}}^\dagger\\ & -m^2_{H_u}H_u^*H_u - m^2_{H_d}H_d^*H_d - (bH_uH_d + {\rm c.c.})\,.\end{aligned}$$

This is the minimal lagrangian the makes standard model supersymmetric.

On the other hand, the theory space of a PNGB Higgs looks huge:

$\mathcal{G}$	$\mathcal{H}$	C	N_G	$\mathbf{r}_{\mathcal{H}} = \mathbf{r}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}$ $(\mathbf{r}_{\mathrm{SU}(2)\times\mathrm{U}(1)})$	Ref.
SO(5)	SO(4)	✓	4	$\mathbf{4} = (\mathbf{2},\mathbf{2})$	[11]
SU(3) × U(1)	SU(2) × U(1)		5	$\mathbf{2}_{\pm 1/2} + \mathbf{1}_0$	[10, 35]
SU(4)	Sp(4)	✓	5	$\mathbf{5} = (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$	[29, 47, 64]
SU(4)	$[\mathrm{SU}(2)]^2 \times \mathrm{U}(1)$	✓*	8	$(\mathbf{2},\mathbf{2})_{\pm 2} = 2 \cdot (\mathbf{2},\mathbf{2})$	[65]
SO(7)	SO(6)	✓	6	$\mathbf{6} = 2 \cdot (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$	–
SO(7)	G_2	✓*	7	$\mathbf{7} = (\mathbf{1},\mathbf{3}) + (\mathbf{2},\mathbf{2})$	[66]
SO(7)	SO(5) × U(1)	✓*	10	$\mathbf{10}_0 = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{2},\mathbf{2})$	–
SO(7)	$[\mathrm{SU}(2)]^3$	✓*	12	$(\mathbf{2},\mathbf{2},\mathbf{3}) = 3 \cdot (\mathbf{2},\mathbf{2})$	–
Sp(6)	Sp(4) × SU(2)	✓	8	$(\mathbf{4},\mathbf{2}) = 2 \cdot (\mathbf{2},\mathbf{2})$	[65]
SU(5)	SU(4) × U(1)	✓*	8	$\mathbf{4}_{-5} + \bar{\mathbf{4}}_{+5} = 2 \cdot (\mathbf{2},\mathbf{2})$	[67]
SU(5)	SO(5)	✓*	14	$\mathbf{14} = (\mathbf{3},\mathbf{3}) + (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1})$	[9, 47, 49]
SO(8)	SO(7)	✓	7	$\mathbf{7} = 3 \cdot (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$	–
SO(9)	SO(8)	✓	8	$\mathbf{8} = 2 \cdot (\mathbf{2},\mathbf{2})$	[67]
SO(9)	SO(5) × SO(4)	✓*	20	$(\mathbf{5},\mathbf{4}) = (\mathbf{2},\mathbf{2}) + (\mathbf{1}+\mathbf{3},\mathbf{1}+\mathbf{3})$	[34]
$[\mathrm{SU}(3)]^2$	SU(3)		8	$\mathbf{8} = \mathbf{1}_0 + \mathbf{2}_{\pm 1/2} + \mathbf{3}_0$	[8]
$[\mathrm{SO}(5)]^2$	SO(5)	✓*	10	$\mathbf{10} = (\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1}) + (\mathbf{2},\mathbf{2})$	[32]
SU(4) × U(1)	SU(3) × U(1)		7	$\mathbf{3}_{-1/3} + \bar{\mathbf{3}}_{+1/3} + \mathbf{1}_0 = 3 \cdot \mathbf{1}_0 + \mathbf{2}_{\pm 1/2}$	[35, 41]
SU(6)	Sp(6)	✓*	14	$\mathbf{14} = 2 \cdot (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{3}) + 3 \cdot (\mathbf{1},\mathbf{1})$	[30, 47]
$[\mathrm{SO}(6)]^2$	SO(6)	✓*	15	$\mathbf{15} = (\mathbf{1},\mathbf{1}) + 2 \cdot (\mathbf{2},\mathbf{2}) + (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$	[36]

Table 1: Symmetry breaking patterns $\mathcal{G} \to \mathcal{H}$ for Lie groups. The third column denotes whether the breaking pattern incorporates custodial symmetry. The fourth column gives the dimension N_G of the coset, while the fifth contains the representations of the GB's under $\mathcal{H}$ and $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ (or simply $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ if there is no custodial symmetry). In case of more than two SU(2)'s in $\mathcal{H}$ and several different possible decompositions we quote the one with largest number of bi-doublets.

Bellazzni, Csaki and Serre:1401.2457

Construction of effective Lagrangians for composite Higgs bosons relies on the CCWZ formalism:

Structure of Phenomenological Lagrangians. I*

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and

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(Received 13 June 1968)

Structure of Phenomenological Lagrangians. II*

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- CCWZ is a very geometrical approach:

standard forms, which are described in detail. The mathematical problem is equivalent to that of finding all (nonlinear) realizations of a (compact, connected, semisimple) Lie group which become linear when restricted to a given subgroup. The relation between linear representations and nonlinear realizations is

- To crank the machinery, one decides on a nonlinearly realized group G, and a subgroup H of G that is linearly realized.

We say G is the broken group and H the unbroken group:

$$\xi = e^{i\Pi/f}, \quad \Pi = \pi^a X^a,$$

$$g\,\xi = \xi'\, U(g,\xi)\ , \quad U(g,\xi) \in H$$

- The "pions" are the coordinates on the coset manifold G/H, and the action of the full group G on pions is complicated and nonlinear!

CCWZ thus looked for objects that have "simple" transformation properties under the action of G.

These are contained in the Cartan-Maurer one-form:

$$\xi^\dagger \partial_\mu \xi \;=\; i\mathcal{D}_\mu^a X^a + i\mathcal{E}_\mu^i T^i \equiv i\mathcal{D}_\mu + i\mathcal{E}_\mu$$

They are the "Goldstone covariant derivative" and the "associated gauge field",

$$\mathcal{D}_\mu \to U\mathcal{D}_\mu U^{-1}\ , \quad \mathcal{E}_\mu \to U\mathcal{E}_\mu U^{-1} - (\partial_\mu U)U^{-1}$$

upon which the complete effective lagrangian can be built (apart from the topological terms)

$$\mathcal{L}_{eff} = \frac{f^2}{2}\mathrm{Tr}\mathcal{D}_\mu\mathcal{D}^\mu + \cdots$$

In this fashion, CCWZ circumvents the problem of working out how the pions transform under the broken G:

$$\xi = e^{i\Pi/f}, \qquad g\,\xi = \xi'\,U(g,\xi)$$

$$\Pi' = \Pi'(\Pi, g)$$

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In this fashion, CCWZ circumvents the problem of working out how the pions transform under the broken G:

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CCWZ is extremely powerful, but it adopts a “top-down” perspective, which requires knowing ahead of time what the broken group “G” is in the UV.

- In composite Higgs models, CCWZ is often matched to the SILH lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SILH}} = & \boxed{\frac{c_H}{2f^2}\partial^\mu (H^\dagger H)\partial_\mu (H^\dagger H) + \frac{c_T}{2f^2}\left(H^\dagger \overleftrightarrow{D}^\mu H\right)\left(H^\dagger \overleftrightarrow{D}_\mu H\right)} \\
& - \frac{c_6 \lambda}{f^2}(H^\dagger H)^3 + \left(\frac{c_y y_f}{f^2} H^\dagger H \bar{f}_L H f_R + \mathrm{h.c.}\right) \\
& + \frac{i c_W g}{2m_\rho^2}\left(H^\dagger \sigma^a \overleftrightarrow{D}_\mu H\right)(D_\nu W^{\mu\nu})^a + \frac{i c_B g'}{2m_\rho^2}\left(H^\dagger \overleftrightarrow{D}_\mu H\right)(\partial_\nu B^{\mu\nu}) \\
& + \frac{i c_{HW} g}{16\pi^2 f^2}(D^\mu H)^\dagger \sigma^a (D^\nu H) W^a_{\mu\nu} + \frac{i c_{HB} g'}{16\pi^2 f^2}(D^\mu H)^\dagger (D^\nu H) B_{\mu\nu} \\
& + \frac{c_\gamma g'^2}{16\pi^2 f^2}\frac{g^2}{g_\rho^2} H^\dagger H B_{\mu\nu} B^{\mu\nu} + \frac{c_g g_s^2}{16\pi^2 f^2}\frac{y_t^2}{g_\rho^2} H^\dagger H G_{\mu\nu} G^{\mu\nu},
\end{aligned}
$$

nlsm contribution to SILH coefficients for some of the composite Higgs models:

- SU(5)/SO(5) Littlest Higgs: $c_H^{(\sigma)} = \frac{1}{4}\ ,\quad c_T^{(\sigma)} = -\frac{1}{16}$

- SO(5)/SO(4) minimal composite Higgs (MCHM): $c_H^{(\sigma)} = 1\ ,\quad c_T^{(\sigma)} = 0$

- SO(9)/SO(5)xSO(4) littlest Higgs: $c_H^{(\sigma)} = \frac{1}{12}\ ,\quad c_T^{(\sigma)} = 0$

- $\frac{SU(5)}{SO(5)} \times \frac{[SU(2)\times U(1)]_L \times [SU(2)\times U(1)]_R}{[SU(2)\times U(1)]_V}$ T-parity: $c_H^{(\sigma)} = \frac{1}{6}\ ,\quad c_T^{(\sigma)} = 0$

c_T is dictated by the custodial symmetry. However, c_H is different for different coset.

So in CCWZ each symmetry breaking pattern gives a seemingly different effective Lagrangian!

Each time a young hot shot comes up with a new composite Higgs model, we need to work out the experimental consequences all over again.

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This begs the question:

Are there universal predictions of a composite Higgs boson that are independent of the symmetry-breaking pattern?

To this end, let's recall that nlsm is all about the presence of non-trivial vacua:

Goldstone bosons are long-range degrees of freedom that connect different vacua!

It then seems a little odd that their interactions would care about the broken group G in the UV!

The IR perspective was pursued vigorously in the context of pions in the '60s by Adler, Nambu, Goldstone, Weinberg, etc.

This body of work was collectively known as "soft pion theorems," although a significant part of them does not depend on the particular symmetry breaking pattern!

- one particularly important "soft-pion" theorem is the Adler's zero condition:

on-shell scattering amplitudes of Goldstone bosons must vanish in the limit the momentum of one Goldstone boson is taken off-shell and soft.

- often this is over-simplified as saying "the Goldstone boson is derivatively coupled."

it is an over-simplification because it doesn't do justice to the full power of the Adler's zero condition.

I would advocate promoting Adler's zero condition to be the **defining property** of Goldstone bosons:

Nambu-Goldstone bosons are defined by the Adler's zeros and their transformation property under the unbroken group in the IR.

for now assume only one flavor of Goldstone boson and consider 4-pt scattering amplitudes, written in terms of the Mandelstam variables.

- Adler's zero condition forbids a constant term!

$$\mathcal{A}(\pi\pi \to \pi\pi) = c_1 s + c_2 t + c_3 u + \mathcal{O}(p^4)$$

- Bose symmetry implies $c_1 = c_2 = c_3$!

$$\mathcal{A}(\pi\pi \to \pi\pi) = \mathcal{O}(p^4)$$

the argument can be generalized to n-pt amplitudes to show that $O(p^2)$ term always vanishes!

$$\begin{aligned} \mathcal{A}(\pi\pi\cdots \;\to\; \pi\pi\cdots) &= c(p_1 + p_2 + \cdots + p_n)^2 + \mathcal{O}(p^4) \\ &= \mathcal{O}(p^4) \end{aligned}$$

- the simplest lagrangian satisfying these properties is the familiar one:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \mathcal{O}(\partial^4)$$

- as is well known, L_0 can be obtained by requiring that there is a constant "shift symmetry" acting on pion:

$$\pi \to \pi + \epsilon$$

- the derivative of pion has simpler transformation under the broken symmetry:

$$\partial_\mu \pi \to \partial_\mu \pi$$

$\partial_\mu \pi$ is the building block of the effective lagrangian!

we have learned a simple yet powerful statement that is universal in nlsm:

For any coset G/H, self-interactions among Goldstones of the same flavor are fixed by Adler's zero condition and Bose symmetry, and must have the form:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \mathcal{O}(\partial^4)$$

The goal --

Construct an effective Lagrangian satisfying the following two properties:

- The Lagrangian for Goldstone bosons of the same flavor reduces to

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- The Lagrangian for Goldstone bosons of the same flavor reduces to

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 when all other flavors are turned off.

- Invariance under the unbroken group H in the IR is preserved.

When there are multiple flavors of Goldstones, higher order terms appear in the shift symmetry.

- let’s consider two flavors of goldstones transforming as a complex scalar under an unbroken U(1):

$$\phi = (\pi_1 + i\pi_2)/\sqrt{2} \quad \rightarrow \quad e^{i\alpha}\,\phi$$

- nonlinear shift symmetry at NLO can be written as

$$\phi \mapsto \phi' = \phi + \epsilon - \frac{c_1}{f^2}(\phi^*\epsilon)\phi - \frac{c_2}{f^2}(\epsilon^*\phi)\phi$$

- when we turn off one of the two flavors , we must return to the single flavor case, $\pi_i \to \pi_i + \varepsilon_i$,

$$\phi \mapsto \phi' = \phi + \epsilon - \frac{c_1}{f^2}(\phi^*\epsilon - \epsilon^*\phi)\phi$$

This is the generalization of constant shift symmetry:

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The question:

What is the Lagrangian that is invariant under the generalized shift symmetry?

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The question:

What is the Lagrangian that is invariant under the generalized shift symmetry?

We can do it by brute force, or we can try to be a little more clever…

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Let's look for objects that have simple transformation properties under the general shift symmetry:

$$\mathcal{D}_\mu \phi \mapsto \mathcal{D}_\mu \phi' = e^{i\alpha\, u(\phi,\epsilon)/f}\ \mathcal{D}_\mu \phi$$

Then the effective Lagrangian can be built straightforwardly.

- By demanding the Adler’s zero and invariant under the unbroken U(1), we can write down

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - \frac{d_1}{f^2}(\phi\,\partial_\mu \phi^* - \partial_\mu \phi\ \phi^*)\phi$$

- the form is again fixed by reducing to the single flavor case:

$$\mathcal{D}_\mu \phi\big|_{\pi_{1,2}=0} = \partial_\mu \phi$$

When all is said and done, the leading two-derivative Lagrangian can be obtained:

$$
\begin{aligned}
\mathcal{L}^{(2)} &= \mathcal{D}_\mu \phi^* \mathcal{D}^\mu \phi \\
&= \partial_\mu \phi^* \partial^\mu \phi - \frac{c_1}{f^2} |\partial_\mu \phi^* \ \phi - \partial_\mu \phi \ \phi^*|^2 + \mathcal{O}(1/f^4)
\end{aligned}
$$

which is invariant under

$$
\phi \mapsto \phi' = \phi + \epsilon - \frac{c_1}{f^2} (\phi^* \epsilon - \epsilon^* \phi) \phi
$$

The surprise is this procedure can be continued order-by-order in 1/f:

$$\mathcal{D}_\mu \phi = \partial_\mu \phi + \phi \frac{\partial_\mu \phi^* \, \phi - \partial_\mu \phi \, \phi^*}{2|\phi|^2} \left(1 - \frac{\tilde{f}}{|\phi|} \sin \frac{|\phi|}{\tilde{f}} \right)$$

$$\mathcal{L}^{(2)} \; = \; \mathcal{D}_\mu \phi \mathcal{D}^\mu \phi = \partial_\mu \phi^* \partial^\mu \phi - \frac{|\partial_\mu \phi^* \, \phi - \partial_\mu \phi \, \phi^*|^2}{4|\phi|^2} \left(1 - \frac{\tilde{f}^2}{|\phi|^2} \sin^2 \frac{|\phi|}{\tilde{f}} \right)$$

Low: 1412.2145
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- We managed to derive the effective Lagrangian without referring to any UV coset!

- There is only one undetermined parameter in the end, which corresponds to the overall normalization of f:

$$\tilde{f} = f / \sqrt{6 c_1}$$

Low: 1412.2145
Low: 1412.2146

- the sign of c_1 is not fixed:

  a positive sign implies a compact G/H (suppression), while a negative sign implies a non-compact G/H (enhancement).

- if UV completion is a concern, $c_1 > 0$ and the sign of the dim-6 operator is negative.

$$\tilde{f} = f/\sqrt{6c_1}$$

- one could introduce another object that transforms non-homogeneously like a gauge field:

$$\mathcal{E}_\mu \mapsto e^{-iu}\mathcal{E}_\mu e^{iu} - ie^{-iu}\partial_\mu e^{iu} = \mathcal{E}_\mu + \partial_\mu u(\phi, \epsilon)$$

$$\mathcal{E}_\mu = \frac{i}{\alpha}\frac{\partial_\mu\phi^*\,\phi - \partial_\mu\phi\,\phi^*}{|\phi|^2}\sin^2\frac{|\phi|}{2\tilde{f}}$$

- the non-homogeneous term also allows us to couple Matters to the Goldstone, much like the nucleon coupling to pions:

$$(\partial_\mu \Phi^* - i\mathcal{E}_\mu \Phi^*)(\partial^\mu \Phi + i\mathcal{E}^\mu \Phi)$$

$$i\bar{\psi}\not{\partial}\psi + \bar{\psi}\not{\mathcal{E}}\psi$$

Let's pause for a moment and reflect on what's happened…

We derived the two-derivative lagrangian for a complex Goldstone boson charged under an unbroken U(1):

$$\mathcal{L}^{(2)} = \mathcal{D}_\mu\phi\mathcal{D}^\mu\phi = \partial_\mu\phi^*\partial^\mu\phi - \frac{|\partial_\mu\phi^*\,\phi - \partial_\mu\phi\,\phi^*|^2}{4|\phi|^2}\left(1 - \frac{\tilde{f}^2}{|\phi|^2}\sin^2\frac{|\phi|}{\tilde{f}}\right)$$

The only assumptions are

1. The Adler's zero condition.
2. There exists an unbroken U(1).

As such, this is the universal lagrangian among all nlsm's containing a complex Goldstone!

We can check against the universality using explicit examples:

$$SU(2)/U(1) \to |\partial_\mu \phi|^2 - \frac{1}{3f^2}\,|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*|^2 + \frac{8}{45f^4}\,|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*|^2\,|\phi|^2 - \frac{16}{315f^6}\,|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*|^2\,|\phi|^4 + \cdots \ , \tag{3.18}$$

$$SU(5)/SO(5) \to |\partial_\mu \Phi|^2 - \frac{1}{48f^2}\,|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*|^2 + \frac{1}{1440f^4}\,|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*|^2\,|\Phi|^2 - \frac{1}{80640f^6}\,|\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*|^2\,|\Phi|^4 + \cdots \ , \tag{3.19}$$

At the first glance the two Lagrangians do not look the same…

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At the first glance the two Lagrangians do not look the same…

But upon f -> 4f in SU(2)/U(1) case, the two become identical!

This approach can be generalized to a general unbroken group H in the IR.

We assume a set of scalars furnishing a linear representation under a simple Lie group H:

$$\pi^a(x) \to \pi^a(x) + i\alpha^i (T^i)_{ab} \pi^b(x)$$

It is convenient to choose a basis where all generators are purely imaginary (and hence anti-symmetric!)

$$(T^i)^T = -T^i \text{ and } (T^i)^* = -T^i$$

This choice will simplify the analysis and makes the correspondence with CCWZ more transparent.

Requiring that

- Adler's zero is preserved.
- Unbroken H-invariance is respected.

We can derive the most general shift symmetry to all orders in 1/f, without ever referring to a coset G/H:

$$\mathcal{T}_{ab} = (T^i)_{ar}(T^i)_{sb}\pi^r\pi^s \qquad F_1(\mathcal{T}) = \sqrt{\mathcal{T}}\cot\mathcal{T}$$

$$\pi^{a\,\prime} = \pi^a + [F_1(\mathcal{T})]_{ab}\,\varepsilon^b$$

Low and Yin: 1709.08639

The Lagrangian invariant under the shift symmetry is

$$\mathcal{L} = \frac{1}{2}[F_2(\mathcal{T})^2]_{ab}\ \partial_\mu \pi^a \partial^\mu \pi^b \qquad\qquad F_2(\mathcal{T}) = \frac{\sin\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}}$$

This is the universal Lagrangian for all Nambu-Goldstone bosons (arising from a symmetric coset.)

Just like the MSSM, the symmetry-preserving part of the lagrangian for composite Higgs models is universal among all symmetric coset G/H!

Using this approach, we are working out universal predictions of a composite Higgs boson in the couplings of the 125 GeV Higgs!

Da Liu, IL and Zhewei Yin: in progress

The shift symmetry approach turned out to be useful in understanding a mystery in the soft limit of Goldstone scattering amplitudes.

The shift symmetry approach turned out to be useful in understanding a mystery in the soft limit of Goldstone scattering amplitudes.

In the amplitude community, many soft limits involving massless particles have been computed in recent years:

- the single and double soft limits for *massless* particles,
- the leading, subleading, subsubleading orders,
- in dimensionality $\neq 4$,
- Yang-Mills, Gravity, NLSM, supersymmetric theories, string theories and other more exotic theories.

Schematically, they all look something like

$$M_{n+1}(k_1, \cdots, k_n; q) = (S^{(0)} + S^{(1)} + \cdots) M_n(k_1, \cdots, k_n)$$

$$M_{n+2}(k_1, \cdots, k_n; q_1, q_2) = (S_d^{(0)} + S_d^{(1)} + \cdots) M_n(k_1, \cdots, k_n)$$

But here lies a puzzle:

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But here lies a puzzle:

For NLSM it is well-known that only even-point amplitudes exist—

$$M_{2n}(k_1, \cdots, k_{2n-1}; q) = (S^{(0)} + \overset{?}{S^{(1)}} + \cdots)\overset{?}{M_{2n-1}}(k_1, \cdots, k_{2n-1})$$

Is $S^{(1)}$ also zero? And if not, what is M_{2n-1} ??

(For half a century, these questions were never studied!)

Surprisingly, this question was answered only two years ago in a very elegant yet obscure fashion, by using the so-called CHY formulation of scattering equations.

Surprisingly, this question was answered only two years ago in a very elegant yet obscure fashion, by using the so-called CHY formulation of scattering equations.

Cachazo, He, and Yuan proposed a compact and elegant formula for tree-level n-point scattering amplitudes of massless particles (spin-0, spin-1, spin-2).

The proposal contains two parts:

- the kinematics

- the dynamics

(Cachazo, He, Yuan:1306.6575, 1306.2962,1307.2199,1309.0885)

Kinematics –

Scattering of *n* massless particles in an arbitrary dimension involves *n* null vectors satisfying total momentum conservation:

$$\{k_1^\mu, \ldots, k_n^\mu \mid \sum_{a=1}^{n} k_a^\mu = 0,\ k_1^2 = \ldots = k_n^2 = 0\}$$

CHY proposed a map from the null light-cone to the Riemann sphere with n-punctures:

$$k_a^\mu = \frac{1}{2\pi i} \oint_{|z-\sigma_a|=\epsilon} dz \frac{p^\mu(z)}{\prod_{b=1}^{n}(z-\sigma_b)}$$

$\{\sigma_1, \cdots, \sigma_n\}$ is the location of the punctures.

.

It turned out the mapping from CP^1 to the light cone of (complexified) momentum can be achieved by imposing

$$p^2(z) = 0$$

This constraint is embodied in a set of equations called the scattering equation:

$$p(\sigma_n) \cdot p'(\sigma_n) \propto \sum_{a \neq n} \frac{2k_n \cdot k_a}{\sigma_n - \sigma_a} = 0$$

The full CHY proposal looks like:

$$M_n = \int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \bullet$$

The full CHY proposal looks like:

Encode the dynamics

$$M_n = \int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \bullet$$

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$$\bullet = \quad \mathcal{I}_L(\{k,\epsilon,\sigma\})\,\mathcal{I}_R(\{k,\tilde{\epsilon},\sigma\})$$

	$\mathcal{I}_L$	$\mathcal{I}_R$
bi-adjoint scalar	$\mathcal{C}_n(\omega)$	$\mathcal{C}_n(\tilde{\omega})$
Yang-Mills	$\mathcal{C}_n(\omega)$	$\text{Pf}'\Psi_n$
Einstein gravity	$\text{Pf}'\Psi_n$	$\text{Pf}'\tilde{\Psi}_n$
Born-Infeld	$(\text{Pf}'\mathsf{A}_n)^2$	$\text{Pf}'\Psi_n$
Non-linear sigma model	$\mathcal{C}_n(\omega)$	$(\text{Pf}'\mathsf{A}_n)^2$
Yang-Mills-scalar	$\mathcal{C}_n(\omega)$	$\text{Pf}\,\mathsf{X}_n\,\text{Pf}'\mathsf{A}_n$
Einstein-Maxwell-scalar	$\text{Pf}\,\mathsf{X}_n\,\text{Pf}'\mathsf{A}_n$	$\text{Pf}\,\mathsf{X}_n\,\text{Pf}'\mathsf{A}_n$
Dirac-Born-Infeld (scalar)	$(\text{Pf}'\mathsf{A}_n)^2$	$\text{Pf}\,\mathsf{X}_n\,\text{Pf}'\mathsf{A}_n$
special Galileon	$(\text{Pf}'\mathsf{A}_n)^2$	$(\text{Pf}'\mathsf{A}_n)^2$

Cachazo, Cha, Mizera: 1604.03893

The CHY proposal embodies the Color-Kinematic duality:

	BS	NLSM	YM
BS	BS	NLSM	YM
NLSM	NLSM	SG	BI
YM	YM	BI	G

Fig. 1: Multiplication table of QFTs, including bi-adjoint scalar (BS) theory, the nonlinear sigma model (NLSM), Yang-Mills (YM) theory, the special Galileon (SG), Born-Infeld (BI) theory, and gravity (G).

Cheung: 1708.03872

CHY proposal has been checked extensively. In the case of YM it is verified to all orders using BCFW recursion. In the case of NLSM, it's been checked explicitly up to 8-pt analytically and 10-pt numerically.

Using CHY, it's simple and straightforward to derive the subleading single soft limit for NLSM:

$$\boxed{A_n^{\mathrm{NLSM}}(\mathbb{I}_n) = -\tau \sum_{a=2}^{n-2} \hat{s}_{an} A_{n-1}^{\mathrm{NLSM}\oplus\phi^3}(\mathbb{I}_{n-1} \,|\, n-1, a, 1) + \mathcal{O}(\tau^2),}$$

$$A_n^{\mathrm{NLSM}\oplus\phi^3}(\alpha|\beta) = \oint d\mu_n \left(\mathcal{C}_n(\alpha)\right) \left(\mathcal{C}(\beta)\,(\mathrm{Pf}\,\mathsf{A}_{\bar{\beta}})^2\right).$$

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The color factor of the original SU(N) flavor structure

The color factor of an alternative SU(N) flavor structure

$A_n^{\mathrm{NLSM}\oplus\phi^3}$: **the amplitudes of a mysterious theory containing biadjoint cubic scalar interacting with NLSM**

Cachazo, Cha, Mizera: 1604.03893

The statement from CHY:

There is an “extended theory” residing in the subleading single soft limit of Goldstone scattering amplitudes.

The mysterious theory contains cubic biadjoint scalars interacting with the Nambu-Goldstone bosons.

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There is an “extended theory” residing in the subleading single soft limit of Goldstone scattering amplitudes.

The mysterious theory contains cubic biadjoint scalars interacting with the Nambu-Goldstone bosons.

Question:

What the heck is going on?

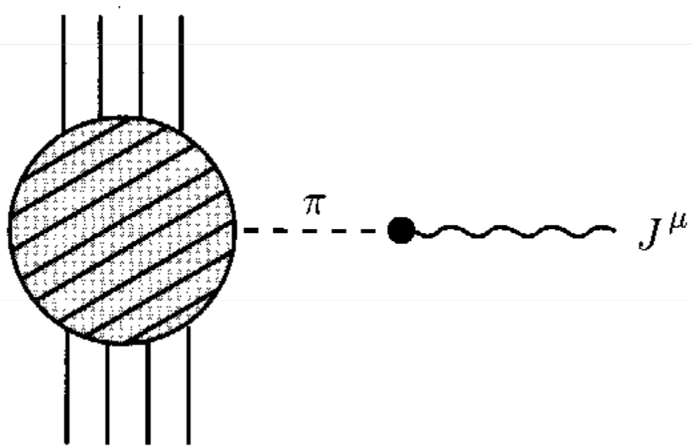
Why didn’t people discover it during all these years?

Recall how Weinberg derives the Adler's zero, starting from:

$$\langle 0|J^a_\mu|\pi^b(x)\rangle \sim \delta^{ab} f_\pi \, p_\mu \, e^{ip\cdot x}$$

which implies there's a one-particle pole in the matrix element

$$\langle f|J^a_\mu|i\rangle \sim$$



$$\to i\frac{f_\pi q^\mu}{q^2} M_{fi} + N^\mu_{fi}$$

Figure from Weinberg, QFT Vol II

Current conservation then implies

$$M_{fi} = \frac{i}{f_\pi} q_\mu N^\mu_{fi} \to 0 \qquad \text{for} \qquad q \to 0$$

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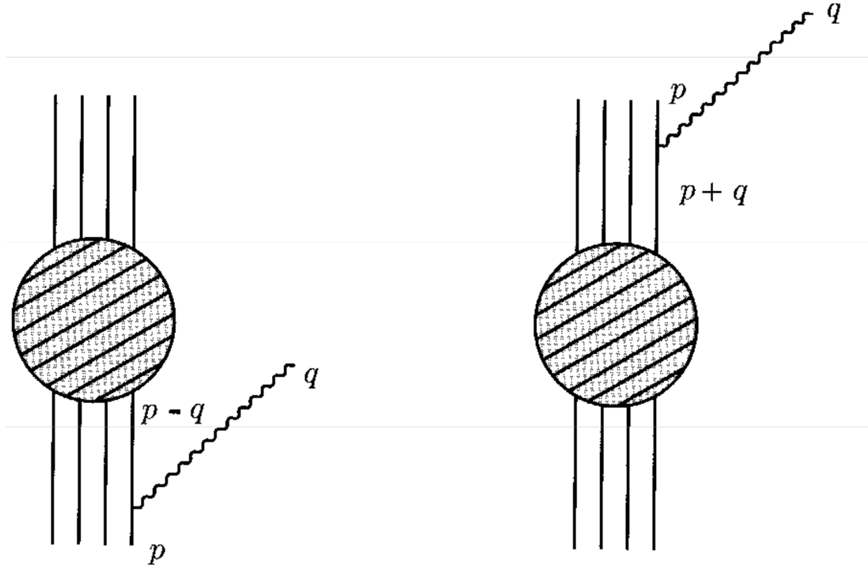
One caveat is if N_{fi} has a pole in q itself, but this can only happens if the current is attached to an external leg:



For pure pion scattering amplitudes, no such insertion exists.

It is not clear how to generalize this argument beyond the leading order in the soft limit.

(Perhaps this is why the subleading soft factor was never computed previously!)

It is not clear how to generalize this argument beyond the leading order in the soft limit.

(Perhaps this is why the subleading soft factor was never computed previously!)

It turned out the shift symmetry approach is ideal for studying the soft limits!

Recall the general effective Lagrangian

$$\mathcal{L} = \frac{1}{2}[F_2(\mathcal{T})^2]_{ab}\ \partial_\mu \pi^a \partial^\mu \pi^b \qquad\qquad F_2(\mathcal{T}) = \frac{\sin\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}}$$

And it is invariant under the shift symmetry

$$\pi^{a\,\prime} = \pi^a + [F_1(\mathcal{T})]_{ab}\ \varepsilon^b \qquad\qquad F_1(\mathcal{T}) = \sqrt{\mathcal{T}}\cot\mathcal{T}$$

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It is natural to ask:

What is the quantum Ward identity resulting from this shift symmetry?

It is not too difficult to work it out

$$\boxed{\begin{aligned} & i\partial_\mu \langle 0| \left\{ [F_4(\mathcal{T})]_{ab}\, \partial^\mu \pi^b \right\}(x) \prod_{i=1}^{n} \pi^{a_i}(x_i)|0\rangle \\ & = \sum_{r=1}^{n} \langle 0|\pi^{a_1}(x_1)\cdots [F_1(\mathcal{T})]_{a_r a}(x_r)\delta^{(4)}(x-x_r)\cdots \pi^{a_n}(x_n)|0\rangle . \end{aligned}}$$

$$F_4(\mathcal{T}) = \frac{\sin\sqrt{\mathcal{T}}\cos\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} \qquad\qquad F_1(\mathcal{T}) = \sqrt{\mathcal{T}}\cot\sqrt{\mathcal{T}}$$

Zhewei Yin and IL: 1709.08639

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Why is this Ward identity interesting?

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$$F_4(\mathcal{T}) = \frac{\sin\sqrt{\mathcal{T}}\cos\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} \qquad\qquad F_1(\mathcal{T}) = \sqrt{\mathcal{T}}\cot\sqrt{\mathcal{T}}$$

Why is this Ward identity interesting?

This is a new representation of on-shell amplitudes of Goldstone bosons, different from the Feynman diagrams.

(Recall the current can be used as an interpolating field of Goldstones.)

Zhewei Yin and IL: 1709.08639

In essence, this is “boostrapping” the NLSM amplitudes from Adler’s zeros:

Starting from a lower point amplitudes, construct the higher point amplitudes such that the Adler’s zero is satisfied, by introducing the necessary higher point vertices.

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Starting from a lower point amplitudes, construct the higher point amplitudes such that the Adler’s zero is satisfied, by introducing the necessary higher point vertices.

In fact, Susskind and Frye constructed the 6-pt and 8-pt NLSM amplitudes this way:

Algebraic Aspects of Pionic Duality Diagrams

LEONARD SUSSKIND* AND GRAHAM FRYE
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033
(Received 9 May 1969)

Certain algebraic aspects are abstracted from the duality principle and are incorporated in a simple model of pion n-point functions. An algorithm for constructing the n-point function in the tree-graph approximation is based on the duality assumption and the Adler condition which states that the amplitudes vanishes if any pion four-momentum vanishes, all others remaining on shell. The resulting amplitudes satisfy the constraints of current algebra and partial conservation of axial-vector current for $n=4$, 6, and 8, and (we conjecture) for all n. In addition, duality specifies a definite form for chiral symmetry breaking.

In the amplitude community, before BCFW recursion, it was customary to look at recursion relations of amplitudes with one leg off-shell.

Semi-on-shell amplitudes:

$$J^{a_1\cdots a_n,a}(p_1,\cdots,p_n) = \langle 0|\pi^a(0)|\pi^{a_1}(p_1)\cdots\pi^{a_n}(p_n)\rangle$$

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Semi-on-shell amplitudes:

$$J^{a_1\cdots a_n,a}(p_1,\cdots,p_n) = \langle 0|\pi^a(0)|\pi^{a_1}(p_1)\cdots\pi^{a_n}(p_n)\rangle$$

The Ward identity now turns into:

$$\begin{aligned} & q^2 J^{a_1\cdots a_n,a}(p_1,\cdots,p_n) \\ = & \sum_{k=1}^{\infty} \frac{(-4)^k}{(2k+1)!} \langle 0|\tilde{O}_k^a(q)|\pi^{a_1}(p_1)\cdots\pi^{a_n}(p_n)\rangle \end{aligned}$$

$$\tilde{O}_k^a(q) = \int d^4x\, e^{-iq\cdot x}\partial_\mu \left\{ \left[\mathcal{T}^k(x)\right]_{ab} \partial^\mu \pi^b(x) \right\}$$

At tree-level, this gives a new recursion relation for semi-on-shell amplitudes.

More importantly, the Ward identity gives the subleading single soft limit without additional work:

$$M^{a_1 \cdots a_{n+1}} \to \frac{1}{\sqrt{Z}} \sum_{k=1}^{\infty} \frac{-(-4)^k}{(2k+1)!}$$

$$\times \, \tau \, \langle 0| \int d^4x \, [\mathcal{T}^k(x)]_{ab} \; i p_{n+1} \cdot \partial \, \pi^b(x) | \pi^{a_1} \cdots \pi^{a_n} \rangle$$

This is valid at the quantum level, and to all orders in 1/f.

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This is valid at the quantum level, and to all orders in 1/f.

To compare with CHY, we only need to go to the tree-level and flavor-ordered amplitudes:

$$M(\mathbb{I}_{n+1}) = \sum_{k=1}^{[n/2]} \frac{-(-4)^k}{(2k+1)! f^{2k}} \sum_{\{l_m\}} \sum_{j=1}^{2k-1} \left[\binom{2k}{j} (-1)^j - 1 \right] p_{n+1} \cdot q_{l_{j+1}}$$
$$\times \prod_{m=1}^{2k+1} J(l_{m-1}+1, \cdots, l_m) \; ,$$

Recall the CHY proposal

$$M_{n+1}^{\mathrm{nl}\sigma\mathrm{m}}(\mathbb{I}_{n+1}) = \tau \sum_{i=2}^{n-1} s_{n+1,i}\ M_n^{\mathrm{nl}\sigma\mathrm{m}\oplus\phi^3}(\mathbb{I}_n|1,n,i) + \mathcal{O}(\tau^2)$$

So comparison with our result would give

$$M_n^{\mathrm{nl}\sigma\mathrm{m}\oplus\phi^3}(\mathbb{I}_n|1,n,i) = \frac{1}{2}\sum_{k=1}^{[n/2]} \frac{-(-4)^k}{(2k+1)!f^{2k}} \sum_{j=1}^{2k-1} \sum_{\substack{\{l_m\} \\ l_j<i\leq l_{j+1}}} \left[\binom{2k}{j}(-1)^j - 1\right] \times \prod_{m=1}^{2k+1} J(l_{m-1}+1,\cdots,l_m)$$

We checked that lower point amplitudes agree on both sides.

What have we learned here?

What have we learned here?

The “extended theory” in CHY arises from the matrix elements of

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In this case, the matrix elements can be interpreted as “amplitudes”:

$$M^{1234} = \tau\, \frac{2}{3f^2} (T^i)_{4r} (T^i)_{sb} \langle 0| \int d^4x\, \pi^r \pi^s\, iq\cdot\partial\, \pi^b |1234\rangle$$

$$M_{n+1}^{\mathrm{nl\sigma m}}(\mathbb{I}_{n+1}) = \tau \sum_{i=2}^{n-1} s_{n+1,i}\; M_n^{\mathrm{nl\sigma m}\oplus\phi^3}(\mathbb{I}_n|1,n,i)$$

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In terms of Feynman diagrams, NLSM has only even-point vertices.

In terms of general nonlinear shift, NLSM has only odd-point vertices in the current corresponding to the shift symmetry.

CHY only provided "amplitudes" for the extended theory,

$$A_4(1^\Sigma, 2^\Sigma, 3^\phi, 4^\phi) = A_4(1^\Sigma, 2^\phi, 3^\Sigma, 4^\phi) = -s_{24}$$

$$A_5(1^\phi, 2^\phi, 3^\phi, 4^\Sigma, 5^\Sigma) = \frac{s_{34} + s_{45}}{s_{12}} + \frac{s_{45} + s_{15}}{s_{23}} - 1$$

$$A_5(1^\phi, 2^\phi, 3^\Sigma, 4^\phi, 5^\Sigma) = \frac{s_{34} + s_{45}}{s_{12}} - 1$$

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$$A_5(1^\phi, 2^\phi, 3^\Sigma, 4^\phi, 5^\Sigma) = \frac{s_{34} + s_{45}}{s_{12}} - 1$$

while we get a little bit more information,

$$V^{\mathrm{nl}\sigma\mathrm{m}\oplus\phi^3}(\mathbb{I}_{2k+1}|1, 2k+1, j) = \frac{i}{2}\frac{-(-4)^k}{(2k+1)!f^{2k}}\left[\binom{2k}{j-1}(-1)^{j-1} - 1\right]$$

But a concrete formulation remains somewhat elusive.

One thing is clear: the "second flavor index" in the extended theory

$$M_{n+1}^{\mathrm{nl}\sigma\mathrm{m}}(\mathbb{I}_{n+1}) = \tau \sum_{i=2}^{n-1} s_{n+1,i}\; M_n^{\mathrm{nl}\sigma\mathrm{m}\oplus\phi^3}(\mathbb{I}_n|1,n,i)$$

is "counting" the derivative!

Is there any further connection with the Color-kinematic duality?

Concluding Remarks:

- The Adler's zero should be taken as the defining property of Nambu-Goldstone bosons.

- Goldstone interactions are universal among a common unbroken group H in the IR.

- All composite Higgs models contain a common universal Lagrangian (the symmetry-preserving part.)

- The soft limits of Goldstone amplitudes, and the infrared structure of NLSM, is much richer than we knew.

We have just begun our exploration!