Analytical study of Yang-Mills theory from first principles by a massive expansion

Fabio Siringo

Department of Physics and Astronomy
University of Catania, Italy

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PREAMBLE

What is "perturbative" and what is not?

It depends on the Expansion Point

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$\Longrightarrow$ The exact YM theory must predict a mass

Any variational ansatz for $\Delta_0(p)$ works well provided it is *massive* (F.S., 2014,2015): two-step approach

Outline

- Gaussian Effective Potential and mass generation
- Massive expansion around the best vacuum

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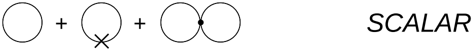
Thermal effects as a check of physical consistency:

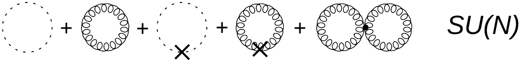
- Analytic properties at finite T
- Gaussian Free Energy and deconfinement at finite T

$$\mathcal{L} = \left[\frac{1}{2}\phi\left(-\partial^2 - m^2\right)\phi\right] - \left[\frac{\lambda}{4!}\phi^4 - m^2\phi^2\right]$$

1^{st} Order Effect. Potential = Vac. Energy by a Gaussian Funct.

[J. M. Cornwall, R. Jackiw and E. Tomboulis (1974); P.M. Stevenson (1985); J.M. Cornwall (1982)]





$$\frac{\delta V}{\delta m^2} = 0 \quad \Longrightarrow \quad \begin{cases} m^2 = 8\pi^2 \alpha\, J(m) \\ \Sigma^{(1)} = 0 \quad \rightarrow \quad \text{Self consist. pole} \end{cases}$$

where $\alpha = \lambda/(16\pi^2)$ and

$$J(m) = \int \frac{d^4 p}{(2\pi)^4}\, \Delta_0(p) = \int \frac{d^4 p}{(2\pi)^4}\, \frac{1}{p^2 + m^2} = \text{?}$$

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$$J(m) = \begin{cases} \frac{m^2}{16\pi^2} \log \frac{m^2}{\Lambda^2} + C & \text{(derivate and integrate back)} \\ -\frac{m^2}{16\pi^2} \left[\frac{2}{\epsilon} + \log \frac{\mu^2}{m^2} \right] = \frac{m^2}{16\pi^2} \log \frac{m^2}{\Lambda_\epsilon^2} & \\ \frac{m^2}{16\pi^2} \log \frac{m^2}{\Lambda_{IR}^2} & \text{(UV finite by subtraction of DR zeros)} \end{cases}$$

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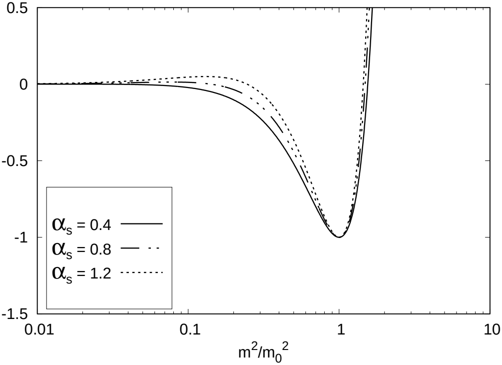
$$\frac{1}{p^2+m^2} \rightarrow \frac{1}{p^2+m^2} - \frac{1}{p^2} + \frac{m^2}{p^4} \quad \text{(leading and sub-leading)}$$

$$V(m) = \frac{m^4}{128\pi^2}\left[\alpha\left(\log\frac{m^2}{m_0^2}\right)^2 + 2\log\frac{m^2}{m_0^2} - 1\right]$$



From the gap eq.:

$$\Lambda_{IR} = m_0 \exp(-1/\alpha)$$

The vacuum energy does not depend on Λ_{IR} and α:

$$V(m_0) = -\frac{m_0^4}{128\pi^2} < 0$$

$\Lambda_{IR} \to 0$ Triviality?

Gaussian Effective Potential (GEP)

Renormalized Effective Potential in units of the best mass m_0

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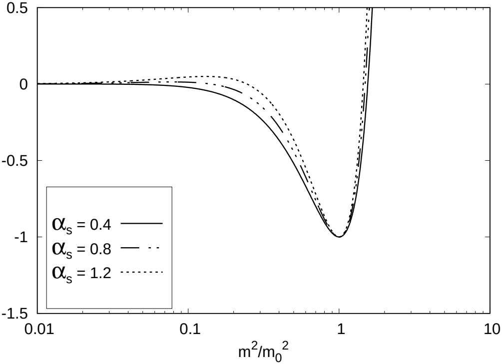

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$\Lambda_{IR} \to 0$ Triviality?

Gluon mass generation: the same identical result for SU(N) Yang-Mills Theory in any covariant ξ-gauge if $\alpha = 9N\alpha_s/(8\pi)$

Add and subtract a transverse mass term in the exact Faddev-Popov Lagrangian in ξ-gauge:

$$\begin{cases} \Delta_0^{\mu\nu}(p) = \dfrac{1}{p^2+m^2}\, t^{\mu\nu}(p) + \dfrac{\xi}{p^2}\, \ell^{\mu\nu}(p) \quad \text{(free propagator)} \\ \qquad\qquad\qquad\qquad\qquad\qquad \nwarrow \text{Exact since } \Pi^L = 0 \\ \delta\Gamma^{\mu\nu} = -m^2\, t^{\mu\nu}(p) \quad \text{(2-point vertex)} \end{cases}$$

$\Longrightarrow$ Gauge invariant GEP and mass generation

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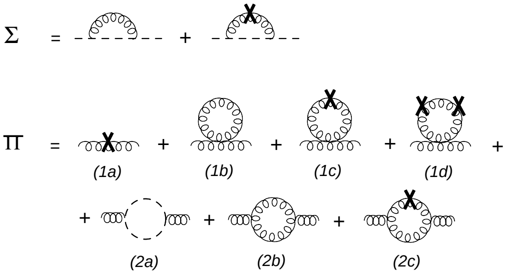


- The pole shift cancels at tree level
- All spurious diverging mass terms cancel without counterterms and/or parameters
- Standard UV behavior

Ignoring RG effects, setting $\alpha \sim N\alpha_s$

$$\Sigma(p) = \alpha\Sigma^{(1)}(p) + \alpha^2\Sigma^{(2)}(p,N) + \cdots$$

$$\Sigma^{(1)} = -p^2\, F(p^2/m^2); \qquad \frac{\Sigma(p)}{\alpha p^2} = -F(p^2/m^2) + \mathcal{O}(\alpha)$$

$$\Delta(p) = \frac{Z}{p^2 - \Sigma(p)} = \frac{J(p)}{p^2}$$

Setting $\quad Z = z\,(1 + \alpha\delta Z) \quad$ (one-loop):

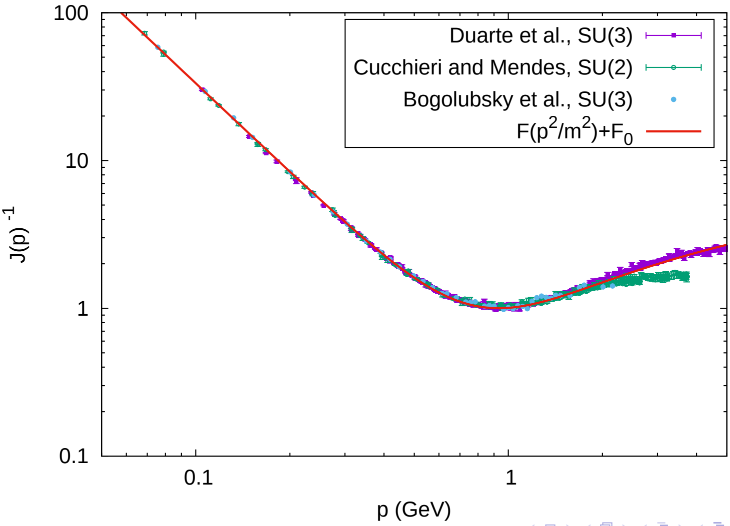
$$z\,J(p)^{-1} = 1 + \alpha\left[F(p^2/m^2) - \delta Z\right] + \mathcal{O}(\alpha^2)$$

$$z\,J(p)^{-1} = 1 + \alpha\left[F(p^2/m^2) - F(\mu^2/m^2)\right] + \mathcal{O}(\alpha^2)$$

Must exist x, y, z:

$$\boxed{z\,J(p/x)^{-1} + y = F(p^2/m^2) + F_0 + \mathcal{O}(\alpha)}$$

UNIVERSAL SCALING
GLUON INVERSE DRESSING FUNCTION (Landau gauge $\xi = 0$)



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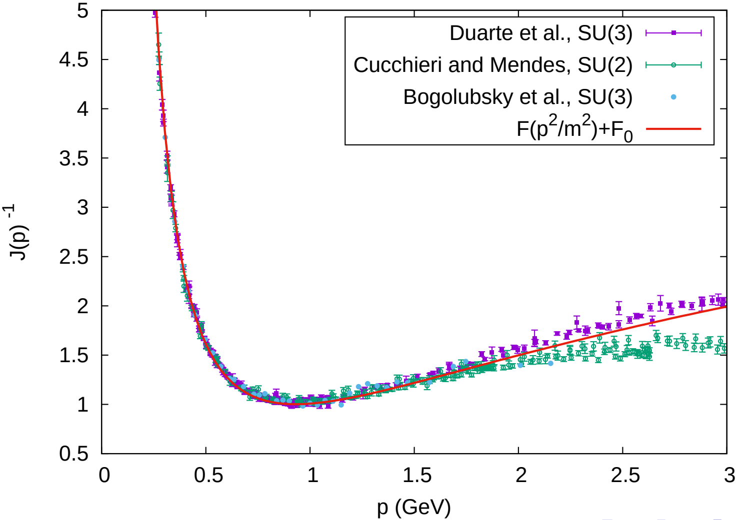
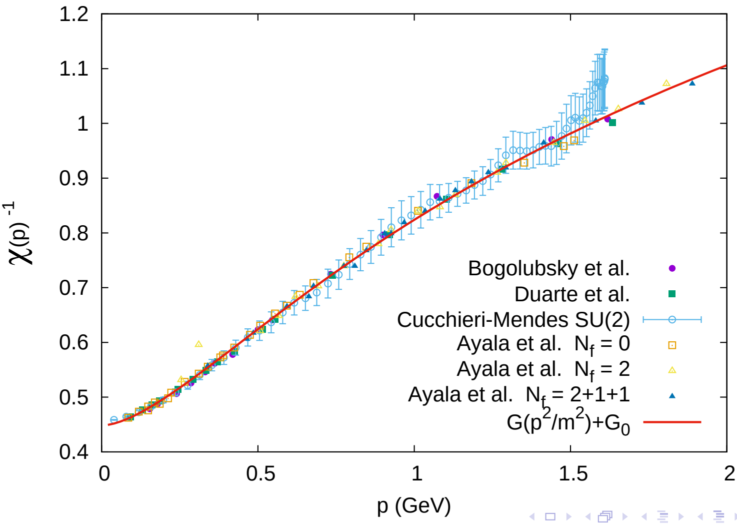


Denoting by $G(s)$ the ghost universal function ($F(s) \to G(s)$)

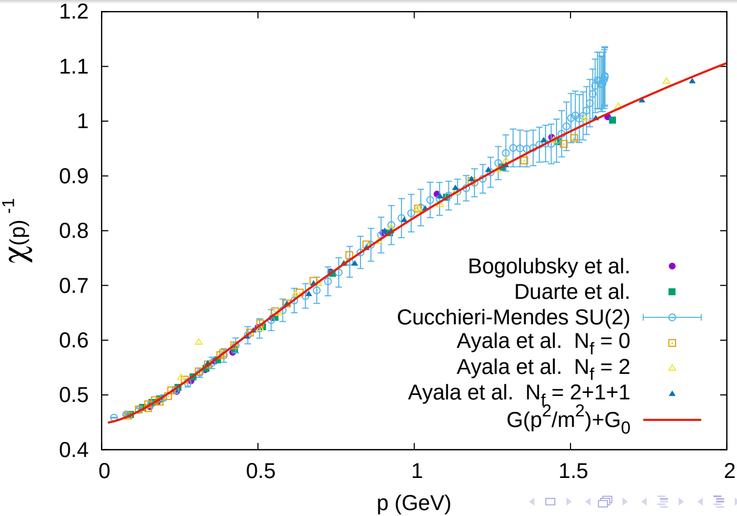


UNIVERSAL SCALING

GHOST INVERSE DRESSING FUNCTION (Landau gauge $\xi = 0$)

The ghost universal function is just

$$G(s) = \frac{1}{12}\left[2 + \frac{1}{s} - 2s\log s + \frac{1}{s^2}(1+s)^2(2s-1)\log(1+s)\right]$$



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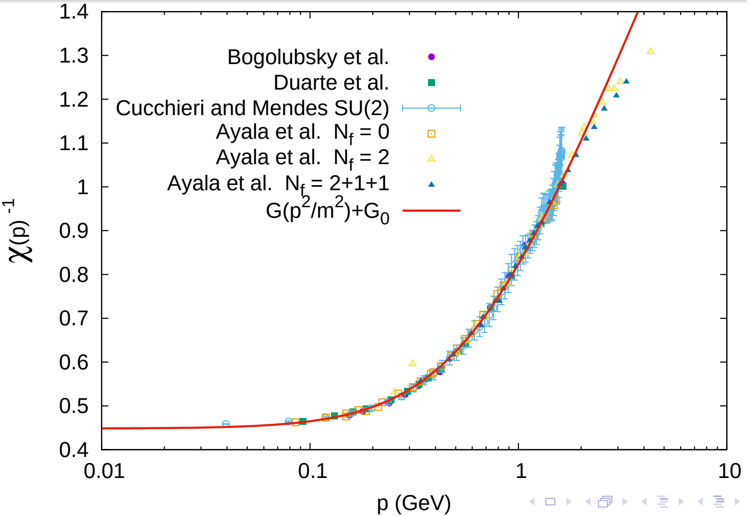


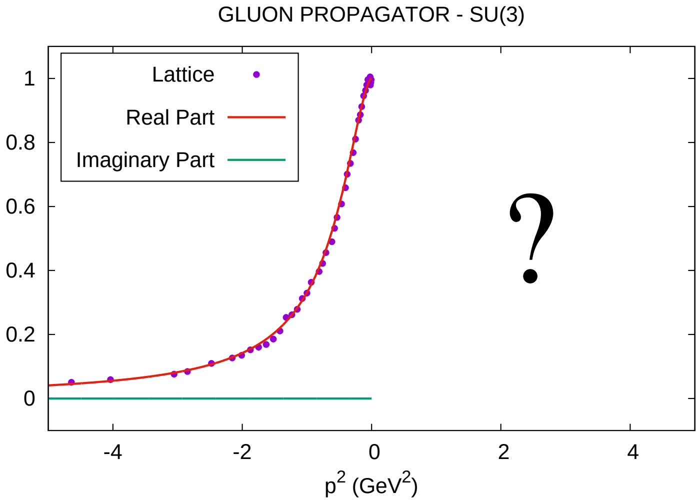
TABLE of OPTIMIZED RENORMALIZATION CONSTANTS: $z\,J(p/x)^{-1} + y = F(p^2/m^2) + F_0$

arXiv:1607.02040

Data set	N	N_f	x	y	z	y'	z'
Bogolubsky et al.	3	0	1	0	3.33	0	1.57
Duarte et al.	3	0	1.1	-0.146	2.65	0.097	1.08
Cucchieri-Mendes	2	0	0.858	-0.254	1.69	0.196	1.09
Ayala et al.	3	0	0.933	-	-	0.045	1.17
Ayala et al.	3	2	1.04	-	-	0.045	1.28
Ayala et al.	3	4	1.04	-	-	0.045	1.28

Table: Scaling constants x, y, z (gluon) and y', z' (ghost). The constant shifts $F_0 = -1.05$, $G_0 = 0.24$ and the mass $m = 0.73$ GeV are optimized by requiring that $x = 1$ and $y = y' = 0$ for the lattice data of Bogolubsky et al. (2009)

ANALYTIC CONTINUATION
arXiv:1605.07357



Lattice data are from Bogolubsky et al. (2009)

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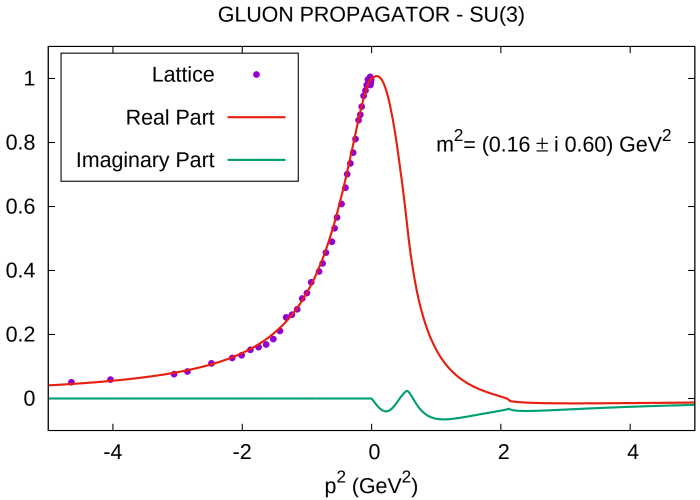

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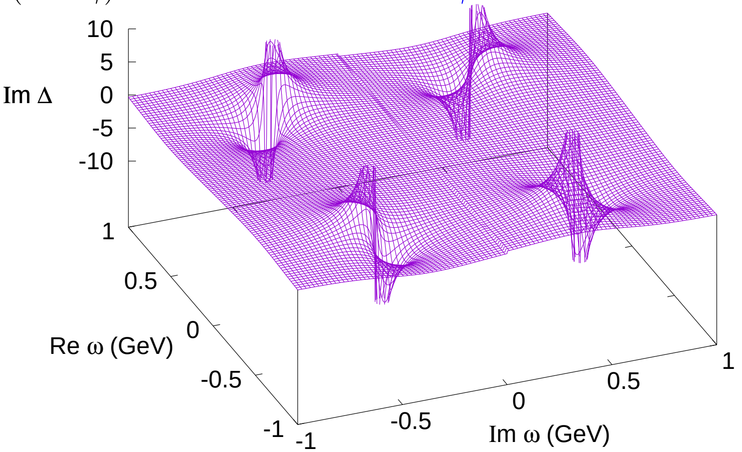
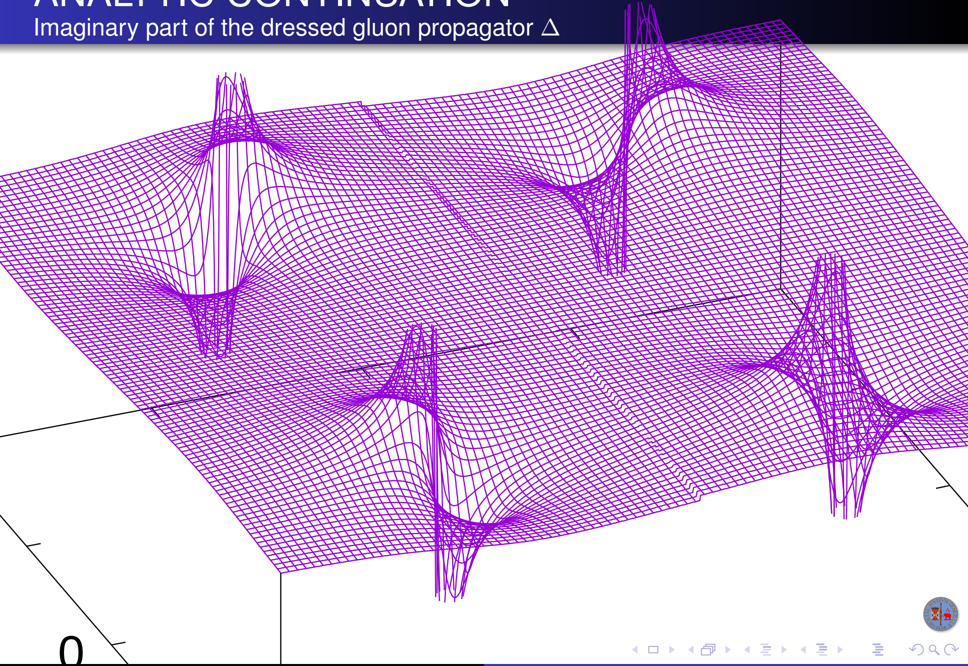
In the long wave-length limit $p^2 = \omega^2 - \mathbf{k}^2 \to \omega^2$ the poles are at $\omega = \pm(m \pm i\gamma)$ where $m = 0.63$ GeV and $\gamma = 0.48$ GeV.



(Using $m_0 = 0.73$ GeV and $F_0 = -1.05$ in the Landau gauge)

ANALYTIC CONTINUATION

Imaginary part of the dressed gluon propagator Δ



No violation of unitarity and casuality (Stingl, 1996)

$$\Delta(\omega) \approx \sum_{\pm} R_{\pm} \left[\frac{1}{\omega - (m \pm i\gamma)} - \frac{1}{\omega + (m \pm i\gamma)} \right]$$

$$\Delta(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \Delta(\omega) e^{i\omega t} \approx -2e^{-\gamma|t|}\, |R|\, \sin\left(m|t| + \phi\right)$$

where $R_{\pm} = |R| e^{\pm i\phi}$

- Short-lived quasigluons with lifetime $\tau = 1/\gamma$ (canceled from asymptotic states)
- During its short life the quasigluon $\approx$ eigenstate with energy m
- Mass and damping rate are physical or artefacts of the expansion?

- The extension to finite T is straightforward (but tedious!)
- Lucky enough: many explicit details by Reinosa, Serreau, Tissier and Wschebor (2014)
- New crossed graphs by a simple derivative
- Set $m_0 = 0.73$ GeV as for $T = 0$

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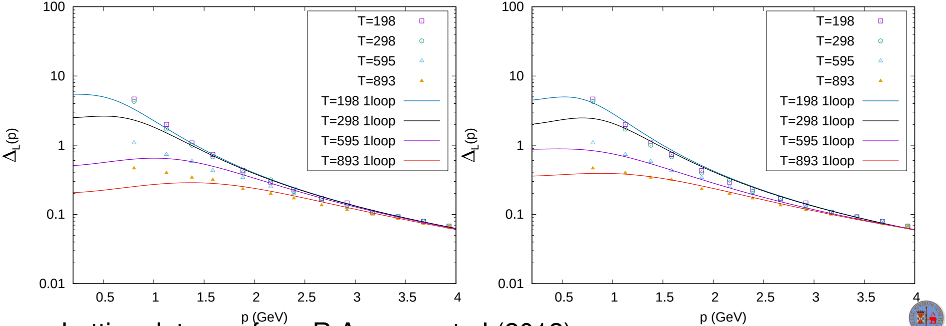
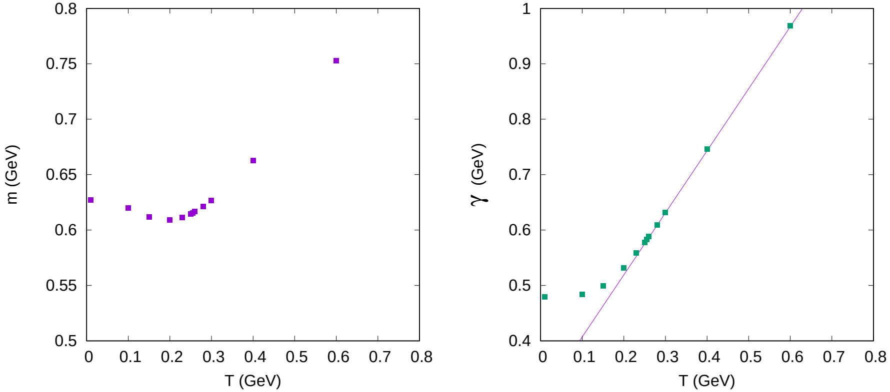


Lattice data are from R.Aouane et al.(2012)

In the limit $\mathbf{k} \to 0$ the pole $\omega = \pm(m \pm i\gamma)$ is the same for Δ_L, Δ_T.
Using $m_0 = 0.73$ GeV and $F_0 = -1.05$ (optimal at $T = 0$):

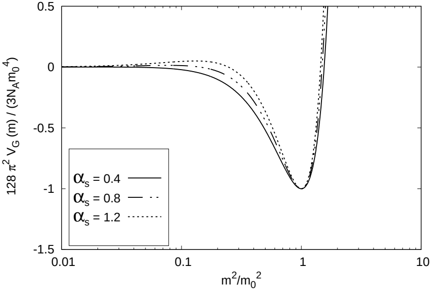


The line is the fit $\gamma = \gamma_0 + bT$ with $\gamma_0 = 0.295$ GeV and $b = 1.12$.
(Hard thermal loops: $\gamma/T = 3.3\alpha_s$)

Gaussian Free Energy

At finite temperature the GEP becomes predicitive!

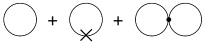
$$V_G(m) = \frac{3N_A m^4}{128\pi^2}\left[\alpha\left(\log\frac{m^2}{m_0^2}\right)^2 + 2\log\frac{m^2}{m_0^2} - 1\right] \quad\Longrightarrow\quad \mathcal{F}_G(T,m)$$



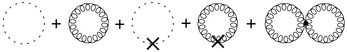
$$m_0 \Longrightarrow m(T)$$

such that $m(0) = m_0$

Just add the thermal part to the same vacuum graphs:

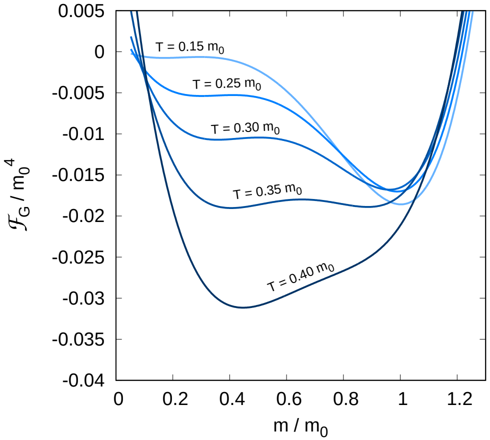


SCALAR

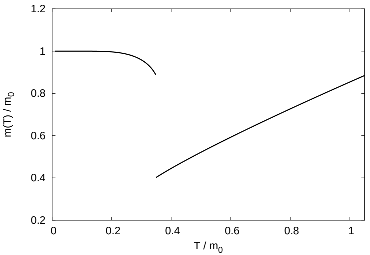


SU(N)

G.Comitini and F.S., arXiv:1707.06935



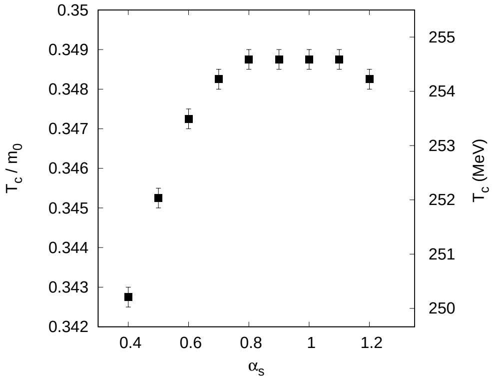
$m(T)$ in units of m_0



here $\alpha_s = 0.9$ but...

not too much sensitive to α_s in the range $0.4 < \alpha_s < 1.2$

G.Comitini and F.S., arXiv:1707.06935



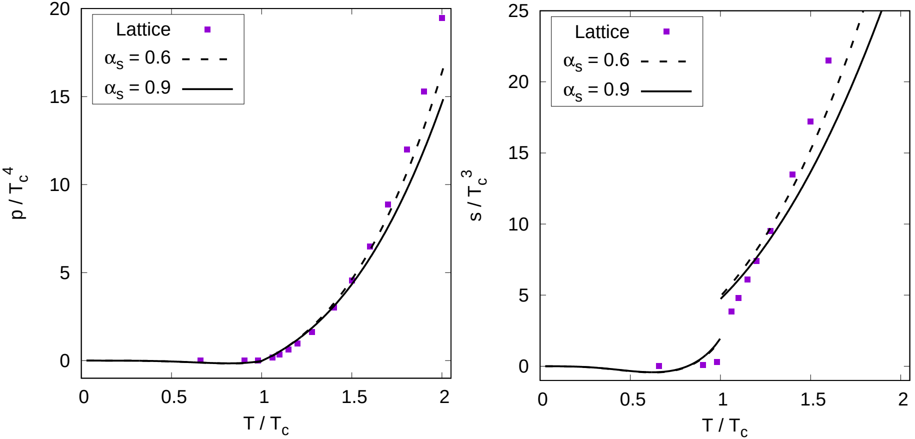
$$\Lambda_{IR} = m_0 \exp(-1/\alpha)$$

T_c stationary at $\alpha_s \approx 0.9$

↖ ($m_0 = 0.73$ GeV)

Compares well with the lattice value $T_c = 270$ MeV found by P.J. Silva et al. (2014).

$$p = -\left[\mathcal{F}_G(T, m(T)) - \mathcal{F}_G(0, m_0)\right] \qquad s = -\frac{\partial}{\partial T}\mathcal{F}_G(T, m(T))$$



Not a fit! (no free parameters)

Lattice data are from L. Giusti and M. Pepe (2017).

An incomplete wish list:

- A consistent criterion for optimization?
- Are the poles gauge invariant?
- What about higher loops? (A general criterion for truncation?)
- RG matching with UV limit
- Gribov copies (expected to be relevant deep in the IR)
- Bound states (BS equation)

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What we have done so far...

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THANK YOU

BACKUP SLIDES

Averaging over free-boson fields and using Jensen inequality

$$\mathcal{F}_{exact} = \mathcal{F}_0^A - T\log\left\langle e^{S_{int}^A}\,\mathrm{Det}\mathcal{M}_{FP}(A)\right\rangle_0 \le \mathcal{F}_1^A + \mathcal{F}^{gh}$$

where $\mathcal{F}^{gh} = -T\left\langle \log\mathrm{Det}\mathcal{M}_{FP}(A)\right\rangle_0 \ne \mathcal{F}^{gh}_{1Loop}$.
In any linear covariant gauge $\mathcal{M}_{FP}(A) = \mathcal{G}_0^{-1} + \delta\mathcal{M}(A)$

$$\mathcal{F}^{gh} = T\left[\mathrm{Tr}\log\mathcal{G}_0\right] + \frac{T}{2}\left\langle \mathrm{Tr}\left[\mathcal{G}_0\delta\mathcal{M}(A)\mathcal{G}_0\delta\mathcal{M}(A)\right]\right\rangle_0 + \cdots = \mathcal{F}^{gh}_{1Loop} + \mathcal{F}^{gh}_{2Loop} +$$

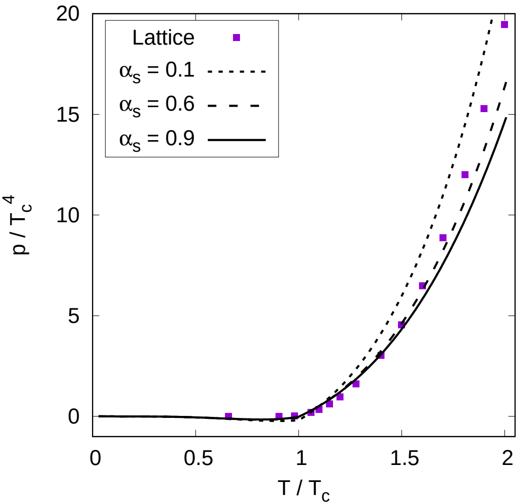
where $\mathcal{F}^{gh}_{2Loop} \sim \alpha\int\mathcal{G}_0\Delta_m\mathcal{G}_0$, etc., so that

$$\boxed{\mathcal{F}_G = \mathcal{F}_1^A + \mathcal{F}^{gh}_{1Loop} \ge \mathcal{F}_{exact} - \delta\mathcal{F}} \quad \text{where} \quad \delta\mathcal{F} = \mathcal{F}^{gh} - \mathcal{F}^{gh}_{1Loop}$$

and by Jensen inequality again:

$$\mathcal{F}^{gh} \ge -T\left[\mathrm{Tr}\log\left\langle\mathcal{M}_{FP}(A)\right\rangle_0\right] = T\left[\mathrm{Tr}\log\mathcal{G}_0\right] = \mathcal{F}^{gh}_{1Loop}$$

so that $\boxed{\delta\mathcal{F} \ge 0}$



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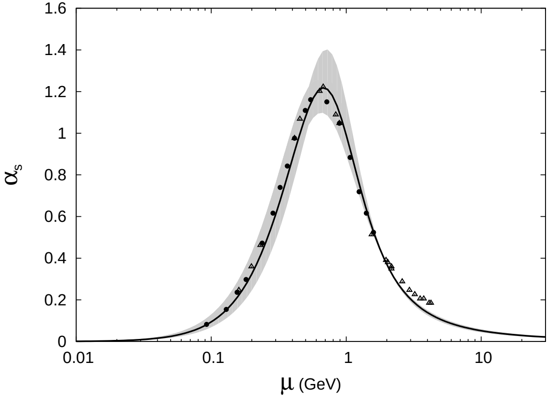
Running Coupling
Pure Yang-Mills SU(3)

RG invariant product (Landau Gauge – MOM-Taylor scheme):

$$\alpha_s(\mu) = \alpha_s(\mu_0)\frac{J(\mu)\chi(\mu)^2}{J(\mu_0)\chi(\mu_0)^2}$$

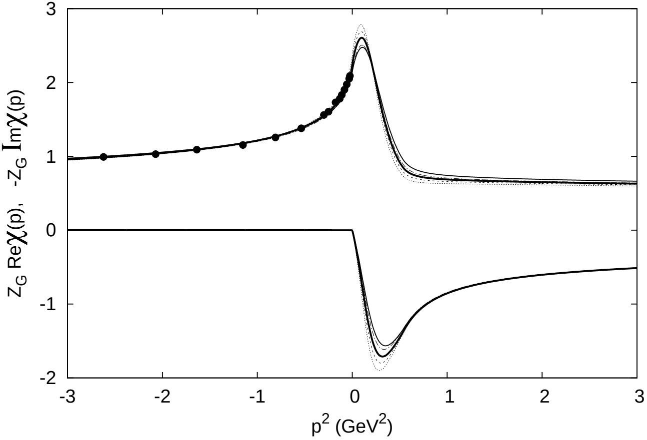
What if $\delta F_0 = \delta G_0 = \pm 25\%$?



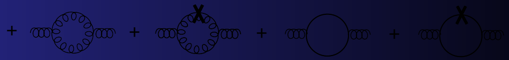
$\mu_0 = 2$ GeV, $\alpha_s = 0.37$, data of Bogolubsky et al.(2009).

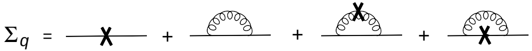
ANALYTIC CONTINUATION

Ghost dressing function: $\mathcal{G}(p^2) = \frac{\chi(p^2)}{p^2}$



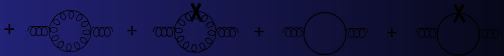
Lattice data are from Bogolubsky et al. (2009)

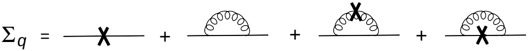




- The counterterm $\delta\Gamma = -M$ cancels the mass at tree-level
- A massive propagator from *loops* $\to S(p) = \frac{Z(p)}{\not{p} - M(p)}$
- A new parameter $x = M/m$

but



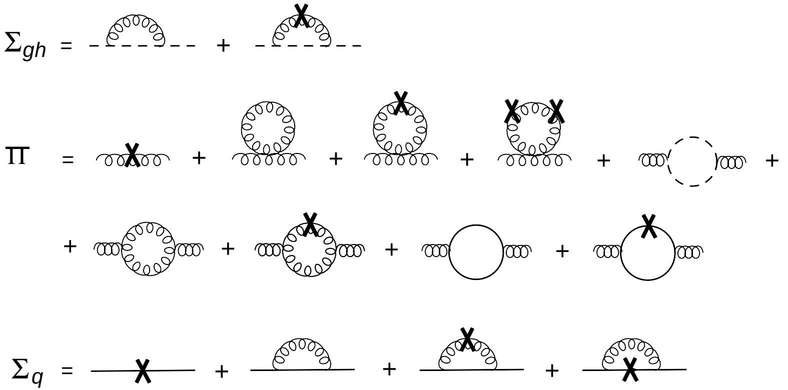


- The counterterm $\delta\Gamma = -M$ cancels the mass at tree-level
- A massive propagator from *loops* $\to S(p) = \frac{Z(p)}{\not{p} - M(p)}$
- A new parameter $x = M/m$

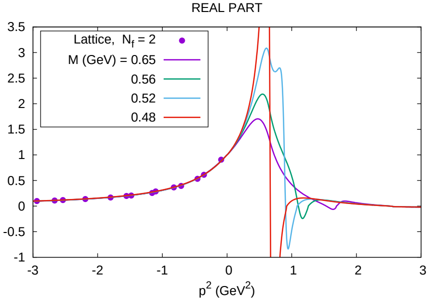
but

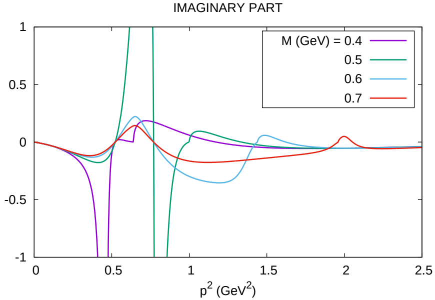
- Agreement not as good as for pure YM theory ($Z(p)$ is decreasing)
- $M(p)$ depends on α_s
- Optimization is not easy without RG corrections!

Yang-Mills → F.S., Nucl. Phys. B **907** 572 (2016)
QCD → F.S., arXiv:1607.02040



Optimized by the Lattice $N_f = 2$, $m = 0.8$ GeV $\quad M$ = ?





Lattice data are for two light quarks, from Ayala et al. (2012)

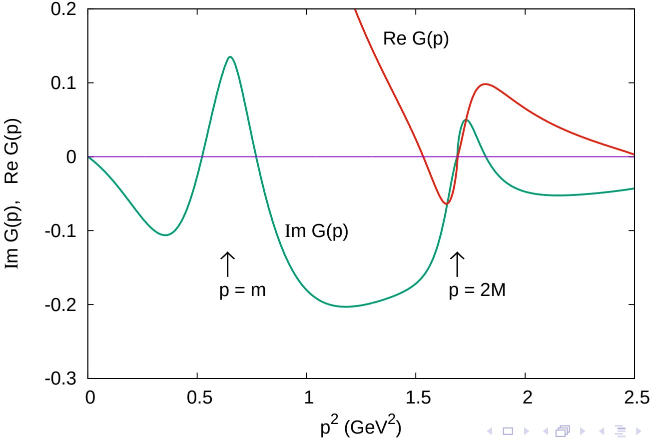
What about poles ?

2 pairs of compex conjugated poles

Optimized by the Lattice:

$m = 0.8$ GeV, $M = 0.65$ GeV

$m_1^2 = (0.54 \pm 0.52i)$ GeV^2, $m_2^2 = (1.69 \pm 0.1i)$ GeV^2



Quark propagator:

$$S(p) = S_p(p^2)\not{p} + S_M(p^2)$$

NO COMPLEX POLES $\Longrightarrow$ Standard Dispersion Relations

$$\rho_M(p^2) = -\frac{1}{\pi}\,\mathrm{Im}\,S_M(p^2)$$
$$\rho_p(p^2) = -\frac{1}{\pi}\,\mathrm{Im}\,S_p(p^2)$$

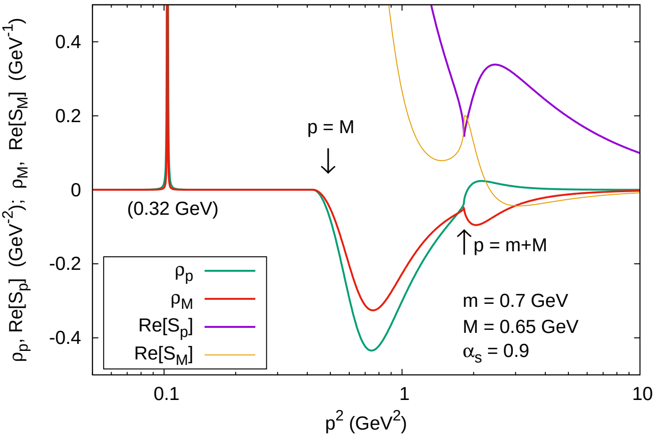
$$S(p) = \int_0^\infty \mathrm{d}q^2 \frac{\rho_p(q^2)\not{p} + \rho_M(q^2)}{p^2 - q^2 + i\varepsilon}.$$

Positivity Conditions:

$\rho_p(p^2) \geq 0,$ $\qquad p\,\rho_p(p^2) - \rho_M(p^2) \geq 0$

CHIRAL QCD

Quark sector: $N_f = 2$, $M = 0.65$ GeV, $m = 0.7$ GeV



Positivity Conditions:

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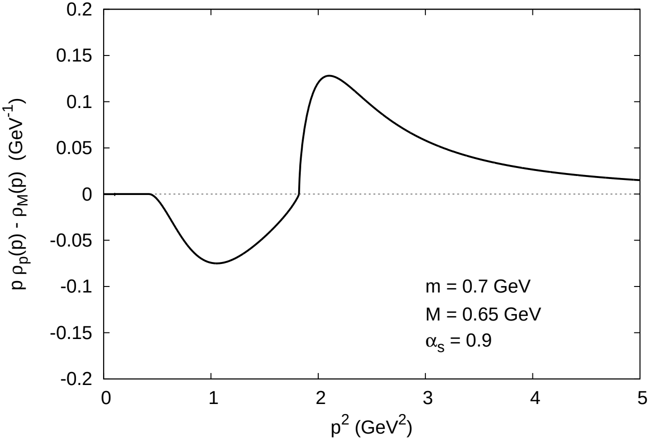
CHIRAL QCD
Quark sector: $N_f = 2$, $M = 0.65$ GeV, $m = 0.7$ GeV



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