Poincaré-covariant description of 2- and 3-body bound states from functional approaches to QFT

Mesons and Baryons in QCD from Dyson-Schwinger–Bethe-Salpeter equations and from Dynamical Hadronisation within the FRG

Reinhard Alkofer

Institute of Physics, University of Graz

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Outline

Disclaimer: Confinement seems to make things easy but it is the hard problem!

What are bound states?
A system of several or many constituents which stay together. **?**

- ☹ Even if not disturbed from outside ∃ dramatic changes, *e.g.*, fate of solar system . . . , excited atoms . . .

Bound states: $M_B = \sqrt{P^{tot}_\mu P^{tot\,\mu}} < \sum_i m_i$. **?**

- ☺ Works well in classical, relativistically invariant mechanics.
- ☺ For not-to-strong fields in QED. (⚠In general no relativistically inv. QM!)

Does not work in

- ☹ General Relativity.
- ☹ Quantum *Gauge* Field Theory in confining and/or Higgs phase.

For the purpose of this talk all hadrons are bound states . . .
. . . but the hard problem remains:
Confinement ⇒ **only** singularities from hadrons/glueballs in $S_{H's \to H's}$!?

Bound states in Quantum Theory

In standard course on
QM: (i) Map 2-particle problem to potential problem, and
(ii) solve Schrödinger equation.

Fails for $N \geq 3$ constituents & thus for QFT!

QM potential problem:
Transmission amplitudes $S(E)$ of scattering states: Poles @ iE_B.
(NB: Require for $E = iE_B$ non-vanishing wave function without incident wave,
see, *e.g.*, F. Schwabl, QM, Sect. 3.7.1.)

Can be generalized to all Quantum Theories:
Search for **poles in** analytic continuation of suitable S-matrix elements,
resp., **Green functions**.

Bound states in Quantum Theory

Bound state properties only from **non-perturbative** calculations:

- Finite polynomial in *coupling* $\times$ *loop integrals*:
  No change in analytic structure!
- A least resummation is needed.
  QM example: Born series $\rightarrow$ Lippmann-Schwinger integral eq.

Nevertheless,
fundamental difference between **barely and deeply bound states**!
Main example:
Dichotomic nature of pion as $\bar{q}q$ bound state and Goldstone boson:

- Chiral limit: Goldstone theorem guarantees masslessness, *i.e.*,
  quark self-energies = binding energy.
- Highly collective state with arbitrarily high $(\bar{q}q)^n$ components.

Bound states in QCD

Where to look for the **nucleon in QCD?**
Free propagation of lowest three-quark bound state:

Six-quark Green function!

Calculating it requires either

- to employ a lattice (*i.e.*, give up Poincaré invariance)
- to use Monte-Carlo algorithms (*i.e.*, use a statistical method)
- to run programs on supercomputers

or

- to fix a gauge (*i.e.*, sacrifice gauge invariance)
- to truncate equations in a way which is verified *á posteriori*
- to perform a lot of (computer) algebra

Method 1: Numerical, partly excellent, results for hadron properties!
NB: Based on extrapolations $a \to 0, V \to \infty$ & $m_\pi \to 0$!
Method 2: Qualitative insight!
E.g. relation of observables to confinement, DχSB, axial anomaly, ...

Bound states in QCD

Recent review: G. Eichmann, H. Sanchis Alepuz, R. Williams, C. S. Fischer, RA, Prog. Part. Nucl. Phys. ***91*** *(2016) 1 [arXiv:1606.09602].*

QCD correlation functions contribute to the understanding of

- ★ confinement of gluons, quarks, and colored composites.
- ★ DχSB, *i.e.*, generation of the quarks' ***constituent masses*** and ***chirality-changing quark-gluon interactions.***
- ★ $U_A(1)$ anomaly and topological properties.

Functional Methods

(Exact Renorm. Group, Dyson-Schwinger eqs., *n*PI methods, . . .):
Input into hadron phenomenology via **QCD bound state eqs.**.

- Bethe-Salpeter equations for **mesons**
  form factors, decays, reactions, ...
- covariant Faddeev equations for **baryons**
  nucleon form factors, Compton scattering, meson production, ...

Inhomogeneous DSE (or FRG eq. or . . .) to homogeneous BSE:

- Assume bound state s.t. on-shell momentum $P_{os} = -M_B^2$:

$$G^{(4)}\left(p, p', P_{os}\right) = \frac{-i}{(2\pi)^4} \frac{\chi\left(p, P_{os}\right) \bar{\chi}\left(p', P_{os}\right)}{2\omega\left(P_{os}^0 - \omega + i\epsilon\right)} + \text{reg. terms},$$

$\omega := \sqrt{\mathbf{P}_{os}^2 + M_B^2}$, $\chi\left(x_1, x_2, P\right) = \langle 0 \,|\, T\Phi\left(x_1\right)\Phi\left(x_2\right) |\, P\rangle$ BS ampl.

- BS vertex fct. Γ s.t. $\chi\left(p, P\right) = G_1\left(p_1\right) G_2\left(p_2\right) \Gamma\left(p, P\right)$, order $\left(P^0 - \omega\right)^{-1}$ provides

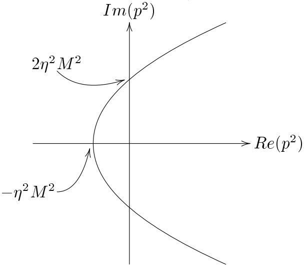
$$\Gamma\left(p; P_{os}\right) = -\int \frac{d^4 p'}{(2\pi)^4} K\left(p, p', P_{os}\right) G_1\left(p_1'\right) G_2\left(p_2'\right) \Gamma\left(p'; P_{os}\right)$$

homog. lin. (eigenvalue) integral equation

- $\mathcal{O}\left(\left(P^0 - \omega\right)^0\right)$: normalization

Bethe-Salpeter Reduction

- drastic simplification:
  4-pt. fct. $\to$ product of 3-pt. fcts., resp, 6-pt. fct. $\to$ 4-pt. fcts.
- symmetry preserving kernels: rainbow-ladder, 3PI–3-loop, ...
- only on-shell quantities for bound states
- ⚠ unphysical (so-called anomalous) solutions:
  negative norm, time-like distances between constitutents, etc.
- analytic continuation needed



Hubbard-Stratonovich Transformations

- (Multi-) gluon exchange: 4-quark (6-quark, etc.) interactions
- Fierz reorder according to meson (baryon) channel
- Introduce meson (baryon) fields via identity
  $\exp\left(\frac{1}{2}\int \bar{q}\Lambda_\alpha q Q^{\alpha\beta}\bar{q}\Lambda_\beta q\right) =$
  $$\frac{1}{\sqrt{\mathrm{Det} 2\pi Q}}\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int \phi_\alpha (Q^{-1})^{\alpha\beta}\phi_\beta - \int \phi_\alpha \bar{q}\Lambda_\alpha q\right)$$
- E.o.m. for composite fields = BS eqs.
- Above cited disadvantages apply . . .

Combine FRG and bosonisation:

H. Gies and C. Wetterich, Phys. Rev. D **65** (2002) 065001 [hep-th/0107221].

- - 4-quark (6-quark, etc.) interactions generated by RG step
  - apply HS transformation
  - Next RG step -> re-bosonise -> RG step -> rebosonise -> . . .
- Disadvantage:
  Analytic continuation in presence of regulator fct. required!
- Advantage: Higher flexibility in choosing truncations. (?)

QCD bound state equations

State-of-the-art for BS eqs.:

- **rainbow-ladder truncation** (= dressed gluon exchange) for mesons **and** baryons in an unified approach.

  NB: Chirality-changing interactions of lesser importance in

  $$0^-, 1^-, \ldots, \frac{1}{2}^+, \frac{3}{2}^+, \ldots \text{ ground states.}$$

  Results include:
  Masses, form factors, decays, Compton scattering, meson prod., ...

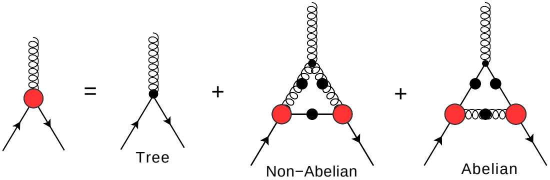
- ... heuristic "beyond rainbow-ladder" calculations ...
- **3PI–3-loop** recent results (masses, decay constants, σ term)

  see, *e.g.*,
  H. Sanchis-Alepuz, R. Williams, Phys.Lett.B **749** (2015) 592 [arXiv:1504.07776];
  J. Phys. Conf. Ser. **631** (2015) 012064 [arXiv:1503.05896];
  R. Williams, C. Fischer, W. Heupel, Phys.Rev.D **93** (2016) 034026 [1512.00455]

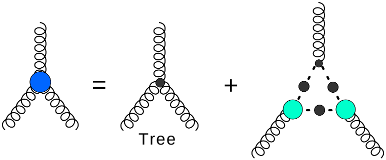
Example for needed input:

R.A., M. Vujinovic, to be published

- Gluon, ghost & quark propagators
- Quark-gluon vertex



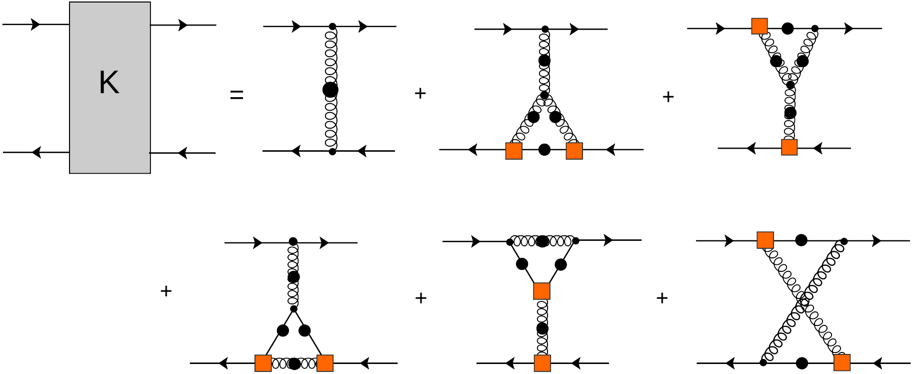
- 3-gluon vertex
(partially self-cons., truncated)



Relativistic 2-Fermion Bound State Equations

Example:

R.A., M. Vujinovic, to be published



Structure of Mesonic Bound State Amplitudes

Mesonic BS amplitudes $\langle 0|q_\alpha \bar{q}_\beta|M_\mathcal{I}\rangle \propto \Phi_{\alpha\beta\mathcal{I}}$:

- scalar and pseudoscalar mesons: 4 tensor structures each
- vector and axialvector mesons: 12 tensor struct. each, 8 transv.
- tensor and higher spin mesons: 8 transverse struct. each

which are functions of two Lorentz-invariant variables.

C. H. Llewellyn-Smith, Annals Phys. **53** (1969) 521.

Facts about the decomposition:

- Independent of any truncation of the bound state equation.
- Only Poincaré covariance and parity invariance exploited.
- It includes all possible internal spin and orbital angular momenta.

Numerical results for meson masses

Light scalar meson in (unquenched) QCD from the 3PI effective action:

	RL	2PI - 3L	3PI - 3L
$m_{0^{++}}/f_\pi$	6.96	5.05	10.5 ± 1.0

	$m_{1^{--}}/f_\pi$	$m_{1^{++}}/f_\pi$
3PI-3L	7.0	12.4 ± 1.0

R. Williams, C.S. Fischer, W. Heupel, Phys.Rev. D**93** (2016) 034026

Light scalar meson in SU(2) gauge theory with two light flavours:

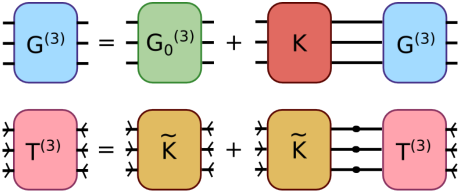
	1PI	3PI-type
$m_{0^{++}}/f_{PS}$ †	5.0 ± 0.1	5.1 ± 0.1

	$m_{1^{--}}/f_\pi$	$m_{1^{++}}/f_\pi$
3PI-type	8.1 ± 0.2	10.9 ± 0.2

R.A., M. Vujinovic, to be published

† with non-Abelian diagram only: 5.7 ± 0.1 and 5.4 ± 0.1

Dyson-Schwinger eq. for 6-point fct. $\Longrightarrow$ 3-body bound state eq.:



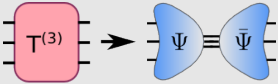
BOUND STATE:

Pole in $G^{(3)}$

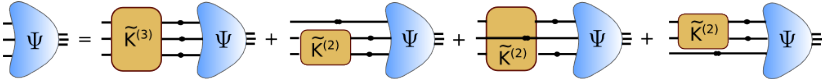
or (equiv.) for $P^2 = -M_B^2$

Pole in $T^{(3)}$

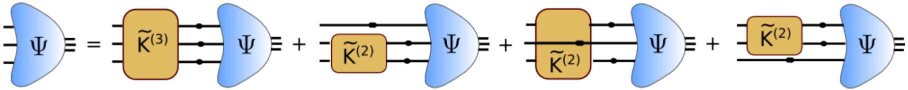
bound state amplitudes:



covariant 3-body bound state eq. (*cf.*, Bethe-Salpeter for 2-body BS)**:**



3-body bound state eq.:



NB: With 3-particle-irreducible interactions $\tilde{K}^{(3)}$ neglected:
Poincaré-covariant Faddeev equation.

Elements needed for bound state equation:

- Tensor structures (color, flavor, Lorentz / Dirac) of the BS ampl.
- Full quark propagators for *complex* arguments
- Interaction kernels $K_{2,3}$

Needed for coupling to e.m. current:

- Full quark-photon vertex

Structure of Hadronic Bound State Amplitudes

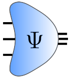
Mesonic BS amplitudes $\langle 0|q_\alpha \bar{q}_\beta|M_\mathcal{I}\rangle \propto \Phi_{\alpha\beta\mathcal{I}}$:

- scalar and pseudoscalar mesons: 4 tensor structures each
- vector and axialvector mesons: 12 tensor struct. each, 8 transv.
- tensor and higher spin mesons: 8 transverse struct. each

which are functions of two Lorentz-invariant variables.

C. H. Llewellyn-Smith, Annals Phys. **53** (1969) 521.

Baryonic BS amplitudes



$\equiv \sim \langle 0|q_\alpha q_\beta q_\gamma|B_\mathcal{I}\rangle \propto \Psi_{\alpha\beta\delta\mathcal{I}}$ (with multi-indices $\alpha = \{x, D, c, f, \ldots\}$)

and $\mathcal{I}$ baryon (multi-)index $\Longrightarrow$ baryon quantum numbers

C. Carimalo, J. Math. Phys. **34** (1993) 4930.

For a solution with all tensor components:

G. Eichmann, RA, A. Krassnigg, D. Nicmorus, PRL **104** (2010) 201601

Structure of Baryonic Bound State Amplitudes

Facts about the decomposition:

- Independent of any truncation of the bound state equation.
- Only Poincaré covariance and parity invariance exploited.
- It includes all possible internal spin and orbital angular momenta.
- For positive-parity, positive-energy (particle) baryons it consists of

spin-$\frac{1}{2}$ particle: 64 elements

	# elements
s-wave	8
p-wave	36
d-wave	20

G. Eichmann et al., PRL 104 (2010) 201601

spin-$\frac{3}{2}$ particle: 128 elements

s-wave	4
p-wave	36
d-wave	60
f-wave	28

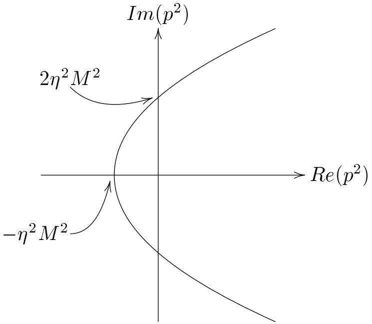
H. Sanchis Alepuz et al. PRD 84 (2011) 096003

Each tensor structure is multiplied by a function of five Lorentz-invariant variables!

NB: Quark angular momentum contribute $\sim$ 40 - 50 % of proton spin!

Bound state masses require time-like momenta!

All (non-perturbative) approaches to QFT employ Euclidean momenta:
Connection to the world of real particles requires analytic continuation!



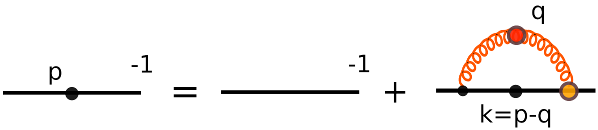
In bound state eqs.:

- Knowledge of the quark propagator inside parabolic region required.
- Parabola limited by nearest quark singularities: $M < 2m_q(3m_q)$ for mesons (baryons)
- ground states unaffected by singularities.

- Lattice: Values for real $p^2 \geq 0$ only.
- Dyson-Schwinger / ERG eqs.: complex p^2 accessible.

Quark Propagator and Rainbow Truncation

Dyson-Schwinger eq. for Quark Propagator:



$$S^{-1}(p) = Z_2 S_0^{-1} + g^2 Z_{1f} \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k) \Gamma^\nu(k,p;q) D_{\mu\nu}(q)$$

Rainbow truncation

Projection onto tree-level tensor γ_μ, restrict momentum dependence

$$Z_{1f}\frac{g^2}{4\pi} D_{\mu\nu}(q)\Gamma_\nu(k,p;q) \to \begin{cases} Z_{1f}\frac{g^2}{4\pi} T_{\mu\nu}(q) \frac{Z(q^2)}{q^2}\left(Z_{1f} + \Lambda(q^2)\right)\gamma_\nu \\ =: Z_2^2 T_{\mu\nu}(q) \frac{\alpha_{eff}(q^2)}{q^2}\gamma_\nu \end{cases}$$

Interaction Kernels and Rainbow-Ladder Truncation

- Truncation of the quark-gluon vertex in the quark DSE.
- The BSE interaction kernel must be truncated accordingly.
- **Physical requirement: Chiral symmetry**
  axial WT id. relates quark DSE and bound-state eq. kernel.

Ladder truncation

$q\bar{q}$ kernel compatible with rainbow truncation and axial WT id.:

$$K^{q\bar{q}} = 4\pi Z_2^2 \frac{\alpha_{eff}(q^2)}{q^2} T_{\mu\nu}(q)\gamma^\mu \otimes \gamma^\nu$$

Together constitute the DSE/BSE **Rainbow-Ladder truncation.**

Note: the truncation can and should be systematically improved!

H. Sanchis-Alepuz, C. S. Fischer and S. Kubrak, Phys. Lett. B **733** (2014) 151;
H. Sanchis-Alepuz and R. Williams, Phys. Lett. B **749** (2015) 592.

Interaction Kernels and Rainbow-Ladder Truncation

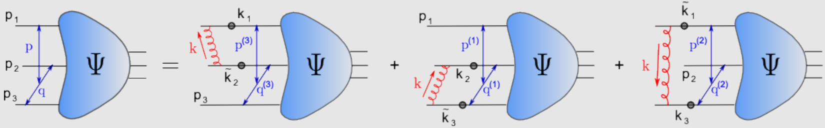
Rainbow-Ladder truncated three-body BSE:

- Previous studies used successfully the quark-diquark ansatz (reduction to a two-body problem).
- pNRQCD: 3-body contribution $\sim$ 25 MeV for heavy baryons.

> Supported by this, **the three-body irreducible kernel $K^{(3)}$ is neglected** (Faddeev approximation).

- Quark-quark interaction $K^{(2)}$: **same as quark-antiquark truncated kernel.** (!Different color factor!)

Rainbow-Ladder truncated covariant Faddeev equation



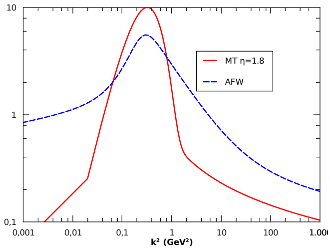
Models for effective interaction:

Maris-Tandy model (Maris & Tandy PRC60 1999)

$$\alpha(k^2) = \alpha_{IR}(k^2; \Lambda, \eta) + \alpha_{UV}(k^2)$$

- Purely phenomenological model.
- Λ fitted to f_π.
- Ground-state pseudoscalar properties *almost* insensitive to η around 1.8

Describes very succesfully hadron properties.



DSE motivated model (R.A.,C.S. Fischer,R. Williams EPJ A38 2008)

$$\alpha(k^2; \Lambda_S, \Lambda_B, \Lambda_{IR}, \Lambda_{YM})$$

- DSE-based in the deep IR.
- Designed to give correct masses of π, ρ and η' *($U_A(1)$ anomaly!)*.
- 4 energy scales! Fitted to π, K and η'.

Interaction Kernels and Rainbow-Ladder Truncation

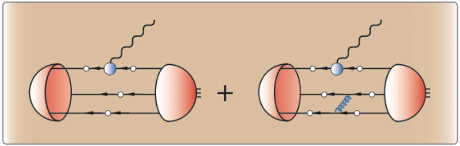
Note: The resulting qq-interaction is chirality-conserving, flavour-blind and current-quark mass independent.

Beyond Rainbow-Ladder

- "Corrections beyond-RL" refers to corrections to the effective coupling but also to additional structures beyond vector-vector interaction.
- They can induce a different momentum dependence of the interaction.
- They can also **induce a quark-mass and quark-flavour dependence of the interaction**
- Question: how important are beyond-RL effects?

Coupling of E.M. Current and Quark-Photon Vertex

Electromagnetic current in the three-body approach:



by "gauging of equations"

M. Oettel, M. Pichowsky and L. von Smekal, Eur. Phys. J. A **8** (2000) 251; A. N. Kvinikhidze and B. Blankleider, Phys. Rev. C **60** (1999) 044003.

Impulse appr. + Coupling to spectator q + Coupling to 2-q kernel not present in RL appr. + Coupling to 3-q kernel not present in Faddeev appr.

Additional Input: Quark-Photon Vertex

Coupling of E.M. Current and Quark-Photon Vertex

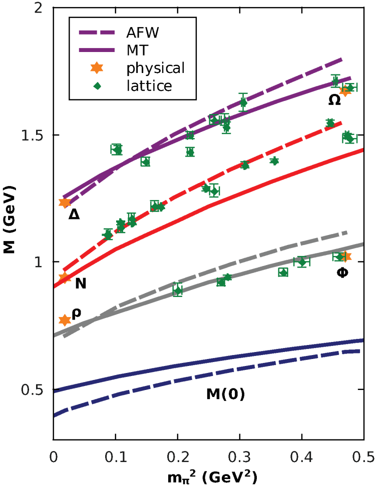
Quark-Photon Vertex:

- Vector WT id. determines vertex up to purely transverse parts:
  "Gauge" part (Ball-Chiu vertex)
  completely specified by dressed quark propagator.
- Can be straightforwardly calculated in Rainbow-Ladder appr.:
  - important for renormalizibility (Curtis-Pennington term),
  - anomalous magnetic moment,
  - contains ρ meson pole!

The latter property is important to obtain the correct physics!

All elements specified to calculate baryon amplitudes and properties:
Use computer with sufficient RAM ($\sim$ tens of GB) and run for a few hours . . .

Some Selected Results for Baryons



PoS QNP2012 (2012) 112

- Both models designed to reproduce correctly DχSB and pion properties within RL. **They capture beyond-RL effects at this quark-mass.**
- This behaviour extends to other light states (ρ, N, Δ), one gets a good description.
- Both interactions similar at intermediate momentum region: **$\sim 0.5 - 1$ GeV is the relevant momentum region for DχSB & ground-state hadron props.**
- Slight differences at larger current masses, however, **qualitative model indep.**

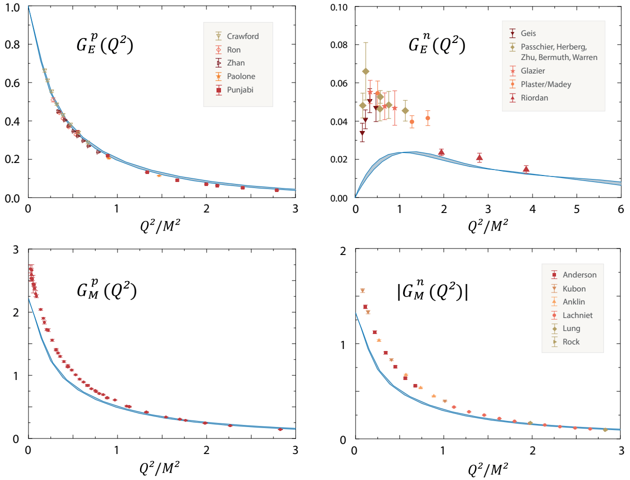
Some Selected Results for Baryons

Nucleon electromagnetic form factors

Nucleon em. FFs
vs. momentum transfer
Eichmann, PRD 84 (2011)

- Good agreement with recent **data** at large Q^2
- Good agreement with **lattice** at large quark masses
- **Missing pion cloud** below ~2 GeV^2, in chiral region

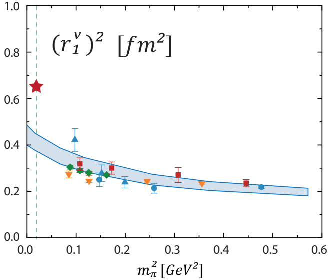
~ **nucleon quark core** without pion effects



Some Selected Results for Baryons

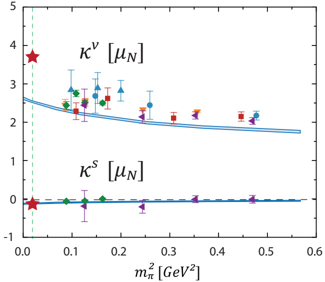
Nucleon electromagnetic form factors

Nucleon charge radii:
isovector (p-n) Dirac (F1) radius



- **Pion-cloud effects** missing in chiral region (⇒ divergence!), agreement with lattice at larger quark masses.

Nucleon magnetic moments:
isovector (p-n), isoscalar (p+n)



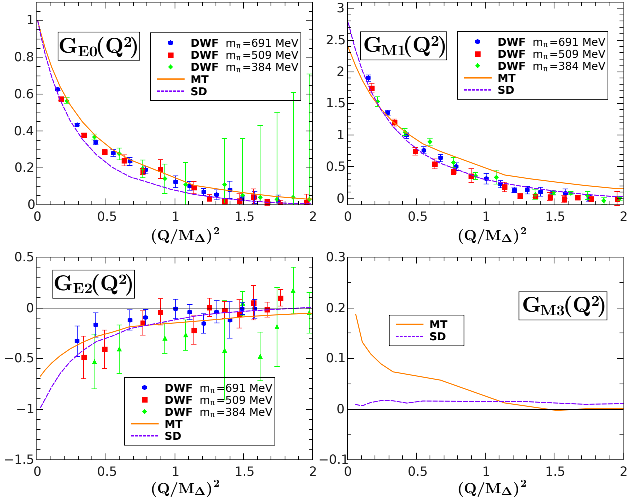
Eichmann, PRD 84 (2011)

- **But:** pion-cloud **cancels** in κ^s ⟺ **quark core**

Exp: $\kappa^s = -0.12$
Calc: $\kappa^s = -0.12(1)$ **!!**

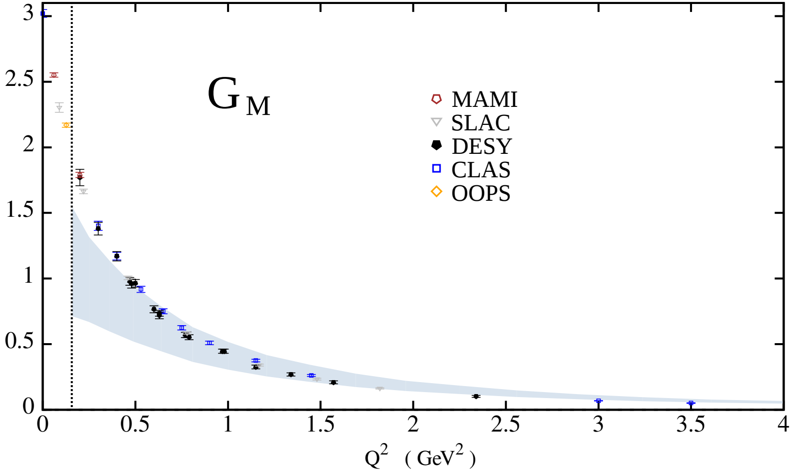
Δ electromagnetic form factors

H. Sanchis-Alepuz *et al.*, Phys. Rev. D **87** (2013) 095015 [arXiv:1302.6048 [hep-ph]].

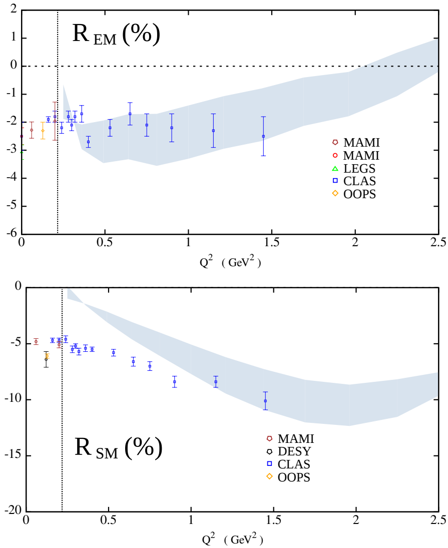


- G_{E2} and G_{M3}: Deviation from sphericity!
- Important: Difference to quark-diquark model in G_{E2} and G_{M3}.
- Large G_{E2} for small Q^2!
- "Small" G_{M3}?

$\Delta \to N\gamma$ Electromagnetic Transition Form Factors



Magnetic f.f.: Large Q^2 good, at small Q^2 missing pion cloud effects?!



$$R_{EM} = -\frac{G_E^*}{G_M^*}$$

- Deformations of N and Δ!
- Non-rel. quark model: sub-lead. d-wave?
- Relativistically: 4-spinors with lower components!

Leading order: p-wave!

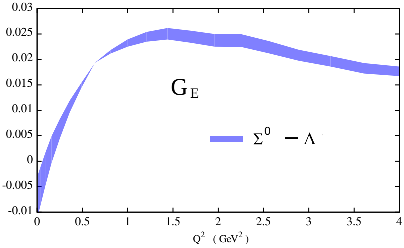
- Inherent to the approach!

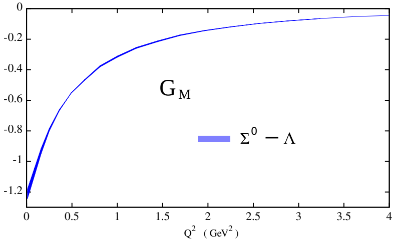
$$R_{SM} = -\frac{M_N^2}{M_\Delta^2}\sqrt{\lambda_+\lambda_-}\frac{G_C^*}{G_M^*}$$

$\Sigma^0 - \Lambda$ transition

- Only octet-octet transition
- PANDA (FAIR): also time-like transition f.f.
- Considerable theoretical interest
- Related to low-energy constants,

  Our results: $\left.\frac{dG_E}{dQ^2}\right|_{Q^2=0} = 0.053..0.073\,, \qquad \left.\frac{dG_M}{dQ^2}\right|_{Q^2=0} = 1.93..1.75\,.$





Expt.: $|\mu_{\Sigma^0\Lambda}| = (1.61 \pm 0.08)\ \mu_N$

Summary

Hadrons from QCD bound state equations:

- QCD bound state equations:
  Unified approach to mesons and baryons feasible!
- So far:
  In beyond rainbow-ladder appr. meson observables,
  in RL appr. octet / decuplet masses, (e.m., axial, . . .) form factors.
  full octet/decuplet (*i.e.*, incl. hyperons): H. Sanchis-Alepuz *et al.*, in preparation
- Under consideration:
  In rainbow-ladder appr. 2-photon processes as,
  e.g., nucleon Compton scattering;
  G. Eichmann and C. S. Fischer, Phys. Rev. D **87** (2013) 036006
  [arXiv:1212.1761 [hep-ph]] .
  and hadronic contribution to light-light scattering and thus $(g-2)_\mu$
  T. Goecke, C. S. Fischer and R. Williams, Phys. Rev. D **87** (2013) 034013.

Outlook

- Even in ground state form factors beyond rainbow-ladder effects at small Q^2!
  There: Likely hadronic (pionic) effects!
- Chirality-violating BRL effects ("spin-flip interactions"):
  - scalar and axialvector mesons,
  - excited mesons and baryons (spectrum)
  - exotic mesons and baryons (tetraquarks, pentaquarks, . . .)
  - . . .
- Systematic approach: Include knowledge on quark-gluon vertex!

- To be used:

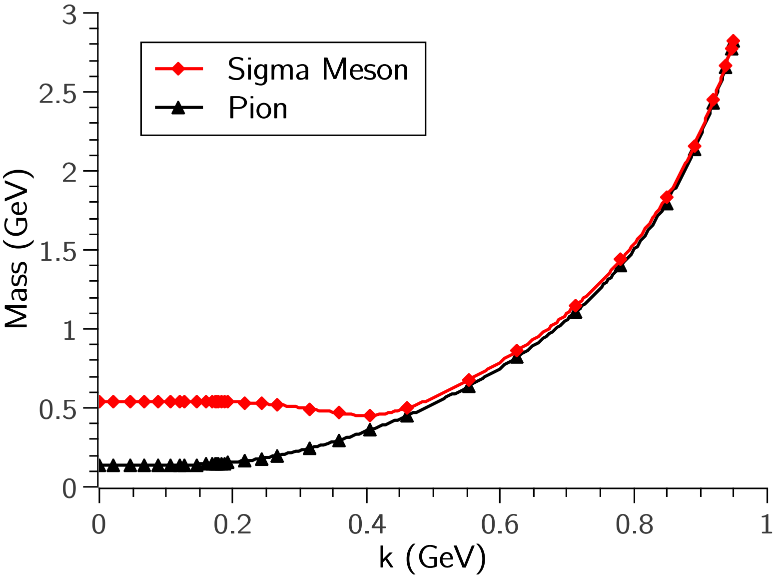
• Bethe-Salpeter /covariant Faddeev eq.
with kernels at consistent 3PI level.

• Dynamical hadronization in the
Exact Renormalization Group approach.

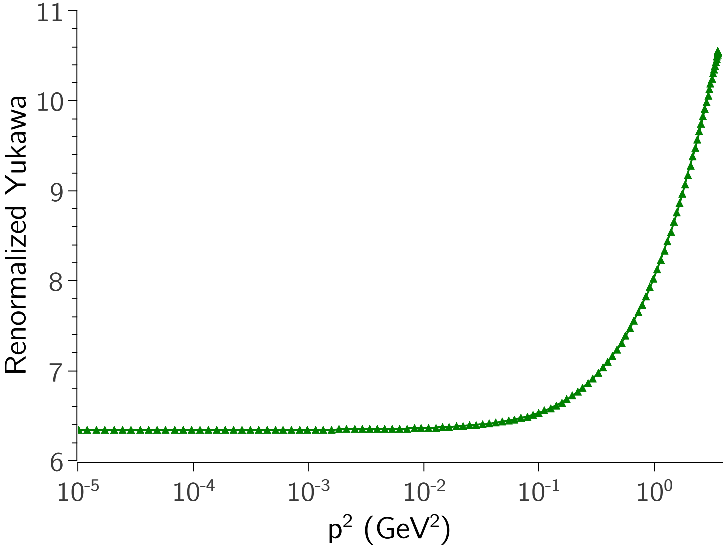
W. Mian, M. Mitter, J. Paris-Lopez, N. Wink, *et al.*, in preparation

- Quark-meson model (NB: 1st step towards QCD)
- Wetterich eq. + Dynamical Bosonisation
- Truncation: LPA' incl.
  momentum-dep. wave-fcts., masses and Yukawa interactions

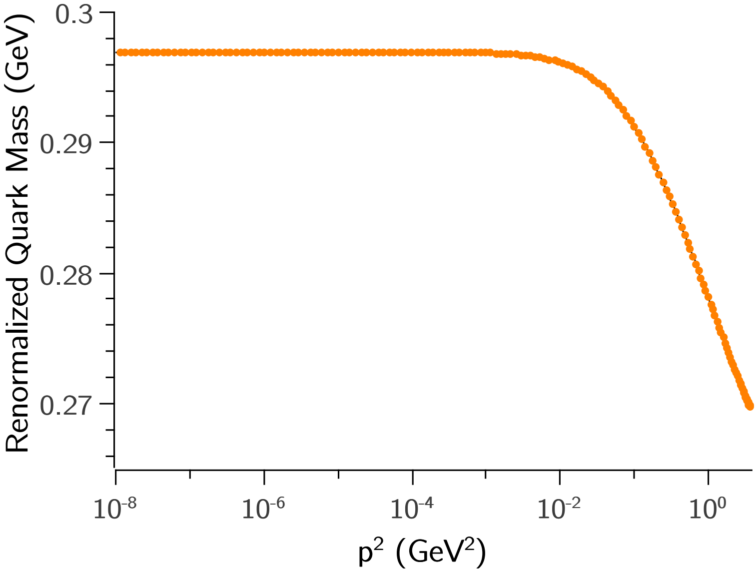
First results from Dynamical Hadronization



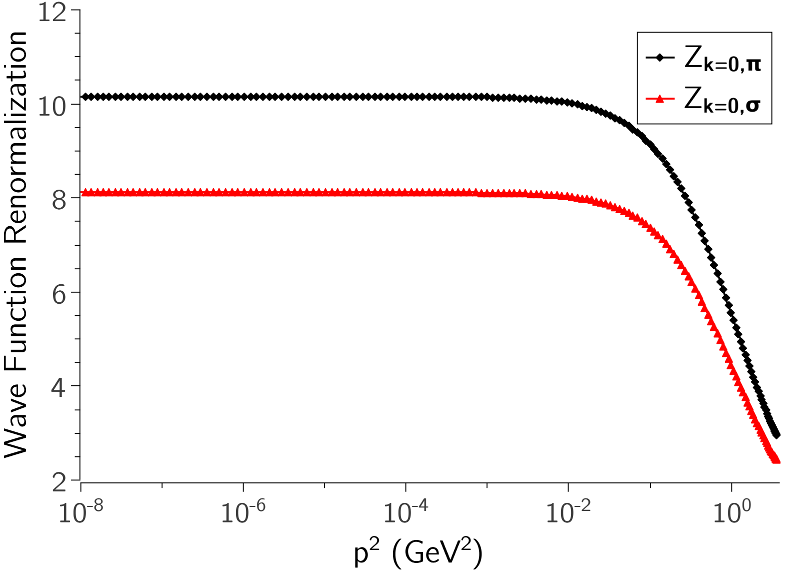
First results from Dynamical Hadronization



First results from Dynamical Hadronization



First results from Dynamical Hadronization



Applying analytical continuation to obtain pole masses (PM) and comparing with curvature masses (CM) $\overline{m}_{k,i}(0)$:

Particle	CM (Input)	PM Padé	PM Schlessinger
Pion	138.22	136.2 ± 0.1	136.4 ± 1.0
Sigma meson	536.37	472 ± 7	479 ± 10

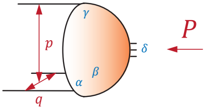
Table: Pole masses vs. curvature masses, all in MeV.

Structure of Baryonic Bound State Amplitudes

s	l	T_{ij}
1/2	0	$\mathbb{1} \otimes \mathbb{1}$ **s waves (8)**
1/2	0	$\gamma^\mu_T \otimes \gamma^\mu_T$
1/2	1	$\mathbb{1} \otimes \frac{1}{2}[\slashed{p}, \slashed{q}]$ **p waves (36)**
1/2	1	$\mathbb{1} \otimes \slashed{p}$
1/2	1	$\mathbb{1} \otimes \slashed{q}$
1/2	1	$\gamma^\mu_T \otimes \gamma^\mu_T \frac{1}{2}[\slashed{p}, \slashed{q}]$
1/2	1	$\gamma^\mu_T \otimes \gamma^\mu_T \slashed{p}$
1/2	1	$\gamma^\mu_T \otimes \gamma^\mu_T \slashed{q}$
3/2	1	$3(\slashed{p} \otimes \slashed{q} - \slashed{q} \otimes \slashed{p}) - \gamma^\mu_T \otimes \gamma^\mu_T [\slashed{p}, \slashed{q}]$
3/2	1	$3\slashed{p} \otimes \mathbb{1} - \gamma^\mu_T \otimes \gamma^\mu_T \slashed{p}$
3/2	1	$3\slashed{q} \otimes \mathbb{1} - \gamma^\mu_T \otimes \gamma^\mu_T \slashed{q}$
3/2	2	$3\slashed{p} \otimes \slashed{p} - \gamma^\mu_T \otimes \gamma^\mu_T$ **d waves (20)**
3/2	2	$\slashed{p} \otimes \slashed{p} + 2\slashed{q} \otimes \slashed{q} - \gamma^\mu_T \otimes \gamma^\mu_T$
3/2	2	$\slashed{p} \otimes \slashed{q} + \slashed{q} \otimes \slashed{p}$
3/2	2	$\slashed{q} \otimes [\slashed{q}, \slashed{p}] - \frac{1}{2}\gamma^\mu_T \otimes [\gamma^\mu_T, \slashed{p}]$
3/2	2	$\slashed{p} \otimes [\slashed{p}, \slashed{q}] - \frac{1}{2}\gamma^\mu_T \otimes [\gamma^\mu_T, \slashed{q}]$

$$\chi(x_1, x_2, x_3) = \langle 0 | \, T \, \psi(x_1) \, \psi(x_2) \, \psi(x_3) \, | N \rangle$$

Momentum space:
Jacobi coordinates p, q, P
$\Rightarrow$ 5 Lorentz invariants
$\Rightarrow$ 64 Dirac basis elements



$$\chi(p, q, P) = \sum_k \; f_k(p^2, q^2, p\cdot q, p\cdot P, q\cdot P) \;\; \textbf{Momentum} \;\; \tau^k_{\alpha\beta\gamma\delta}(p, q, P) \;\; \textbf{Dirac} \;\; \otimes \; \textbf{Flavor} \; \otimes \; \textbf{Color}$$

Complete, orthogonal **Dirac tensor basis**
(partial-wave decomposition in nucleon rest frame):
Eichmann, Alkofer, Krassnigg, Nicmorus, PRL 104 (2010)

$$T_{ij}\,(\Lambda_\pm \gamma_5 C \otimes \Lambda_+)$$
$$(\gamma_5 \otimes \gamma_5)\, T_{ij}\,(\Lambda_\pm \gamma_5 C \otimes \Lambda_+)$$

$$(A \otimes B)_{\alpha\beta\gamma\delta} = A_{\alpha\beta}\, B_{\gamma\delta}$$