# Small parameters in infrared Quantum Chromodynamics

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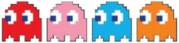
# Landau gauge Euclidean QCD Lagrangian

- Landau gauge Euclidean QCD Lagrangian

$$\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i (\gamma_\mu D_\mu + M_i) \psi_i + \underbrace{i h^a \partial_\mu A^a_\mu}_{\text{Landau gauge}} + \underbrace{\partial_\mu \bar{c}^a (D_\mu c)^a}_{\text{Ghosts}}.$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$
$$D_\mu \psi = \partial_\mu \psi - i g A^a_\mu t^a \psi$$

$$(D_\mu c)^a = \partial_\mu c^a + g f^{abc} A^b_\mu c^c.$$



# Perturbation theory

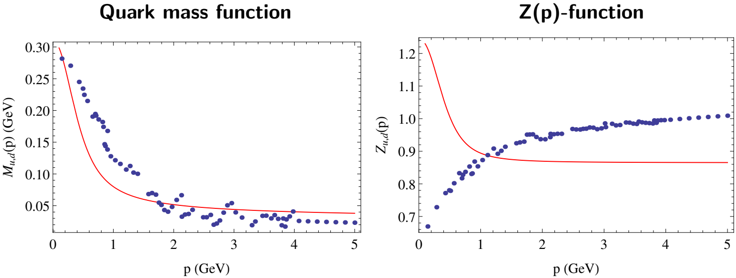
- Standard perturbation theory:
  - Assymptotic freedom
  - Landau pole in the infrared
- Lattice simulations in the infrared:
  - moderate coupling constant e.g. [*Blossier et al., Phys.Rev. D85, 2012*]
  - massive gluon propagator e.g. [*A. Cucchieri, A. Maas and T. Mendes, Phys.Lett. D77, 2008*]
- Massive deformation of Landau gauge QCD Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{inv}} + ih^a \partial_\mu A_\mu^a + \partial_\mu \bar{c}^a (D_\mu c)^a + \mathbf{\frac{m^2}{2} A_\mu^a A_\mu^a}$$

[*Curci-Ferrari (1975)*]

# Perturbation theory within massive Landau gauge Lagrangian

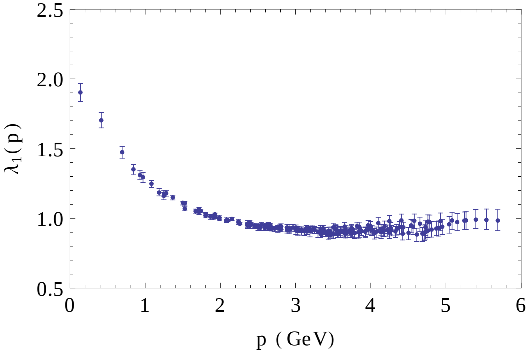
- Yang-Mills one loop calculation gives very accurate results for two and three correlation funcions. see M. Tissiers talk
- However, in the **quark sector** results are not as good as Yang-Mills ones.




The points are lattice data of [*Bowman et al, Phys.Rev. D70 (2004)*] [*M. Peláez, M. Tissier, N. Wschebor, Phys. Rev D90 (2014)*].

# Quark-Gluon coupling VS Ghost-Gluon coupling

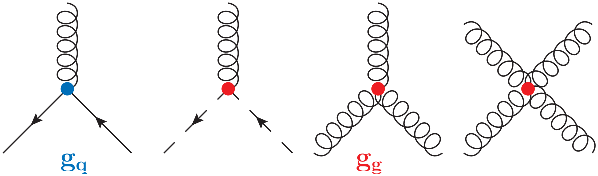
- Quark-gluon coupling constant not too small.
  $g_q(\mu) = g_g(\mu)\lambda_1(\mu)$




Data from [*Skullerud et al. JHEP* **0304**, *047 (2003)*]

# Quark-Gluon coupling VS Ghost-Gluon coupling

- As the quark-gluon $g_q$ and YM $g_g$ running coupling constants are different in the infrared, we treat them separetly,



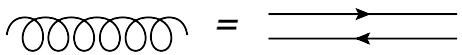

- $g_g$ is considered as small parameter. Yang-Mills sector can be studied perturbately in the infrared.
- $g_q$ is not a small parameter.

# Large $N_c$ limit

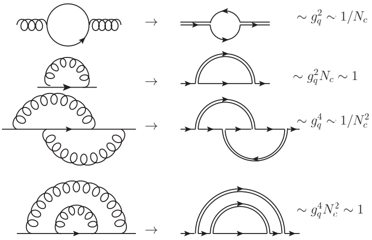
- Large $N_c$ limit shows the same general features of QCD.

  [*G. 't Hooft, Nucl. Phys. B* **75**, *461 (1974). Witten, Nucl. Phys. B* **160**, *57 (1979)]*

In the large $N_c$ limit, gluon propagators can be replaced by double color lines and



$$g_q \sim 1/\sqrt{N_c}$$

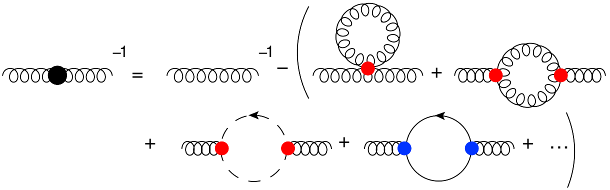


# Organizing the systematic expansion:

- How to implement the systematic expansion, $\ell$-order improved expansion:
  - We write all diagrams until $\ell$-loops
  - We count the powers of $g_g$ and $1/N_c$
  - We also add higher loop order diagrams with the same powers of $g_g$ and $1/N_c$.

# Gluon propagator

- At first order in this double expansion, the gluon propagator is dominated by its tree level form.



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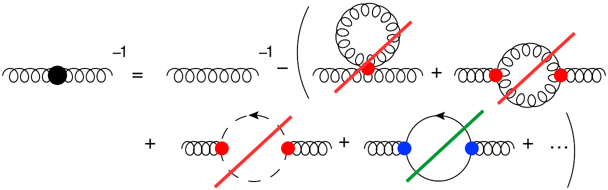
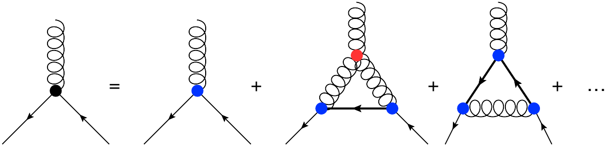


# Quark-Gluon vertex

- At first order in this double expansion, the quark-gluon vertex is also dominated by its tree level form.



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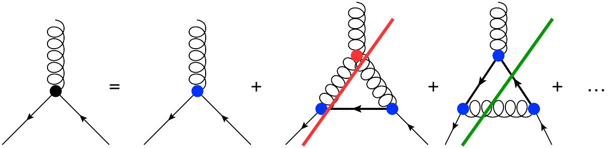
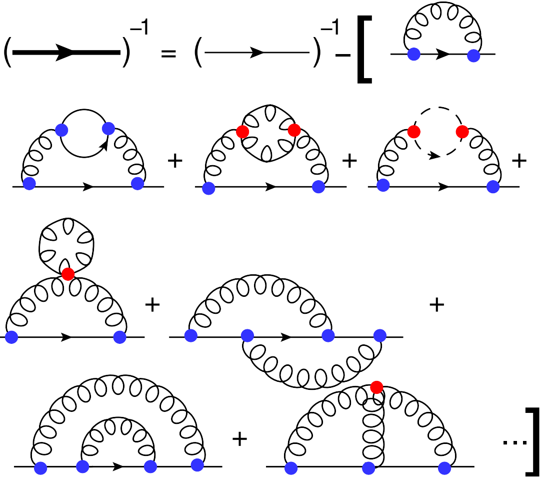
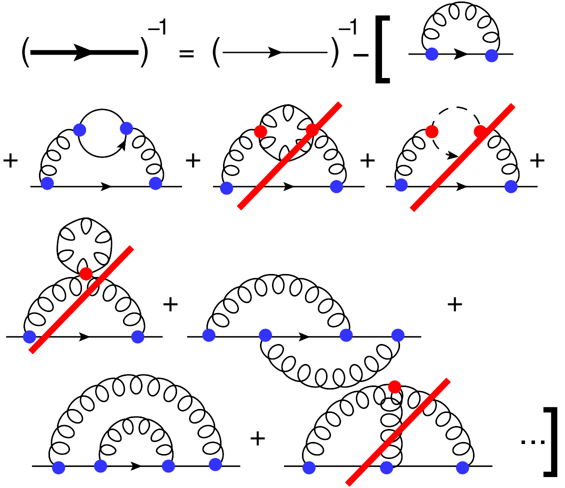


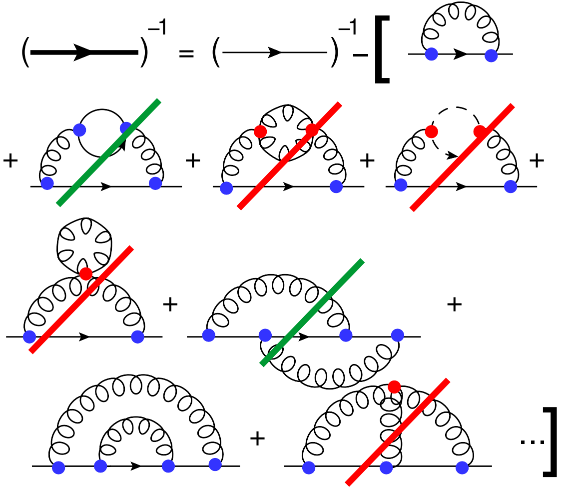
# Quark propagator



# Quark propagator

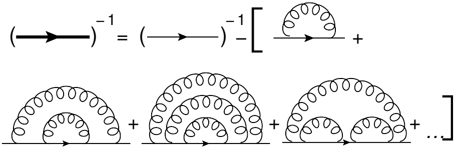


# Quark propagator

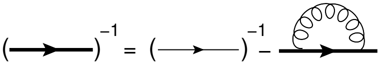


# Rainbow equation

- Only Rainbow diagrams survive



- They can be resummed in:



which is the well-known Rainbow approximation for the quark propagator.

see e.g. [*Johnson et al, PRB (1964). Maskawa, PTP (1975). Atkinson et al, PRD (1988). Miransky et al, PRC (2004).* ]

[Maris et al, IJMP (2003). Roberts et al,EPJST (2007). Eichman et al, PRC (2008). ]

# Rainbow equation

- Rainbow equation represents a system of two coupled integral equations for the two scalar functions of the quark propagator.

$$S(p) = \left[-iA(p)\not{p} + B(p)\right]^{-1} = i\tilde{A}(p)\not{p} + \tilde{B}(p)\,,$$

where

$$\tilde{A}(p) = \frac{A(p)}{A^2(p)p^2 + B^2(p)}\,,$$
$$\tilde{B}(p) = \frac{B(p)}{A^2(p)p^2 + B^2(p)}\,,$$

- It is well-known that Rainbow resummation reproduces correctly the phenomenology of Spontaneous Chiral Symmetry Breaking.
- Let us stress that the main point here is that the Rainbow approximation is justified when considering $g_g$ and $1/N_c$ as small parameters.

# Renormalization

- We introduce the renormalization factors: $A^a_{\mu,\Lambda} = \sqrt{Z_A} A^a_\mu$, $\psi_\Lambda = \sqrt{Z_\psi}\psi$, $m^2_\Lambda = Z_{m^2} m^2$, $M_\Lambda = Z_M M$ and $g_{q,\Lambda} = Z_{g_q} g_q$.
- Renormalization condition: $S^{-1}(p=\mu_0, \mu_0) = -i\not{\mu}_0 + M(\mu_0)$
- The renormalized equations take the form:

$$A(p,\mu_0) = Z_\psi - Z_{g_q}^2 Z_\psi^2 Z_A g_q^2(\mu_0) C_F \int_{|q|<\Lambda} \frac{f(q,p)\tilde{A}(q,\mu_0)}{Z_A[(p+q)^2 + Z_{m^2} m^2(\mu_0)]},$$

$$B(p,\mu_0) = Z_\psi Z_M M(\mu_0) + Z_{g_q}^2 Z_\psi^2 Z_A g_q^2(\mu_0) C_F \int_{|q|<\Lambda} \frac{(d-1)\tilde{B}(q,\mu_0)}{Z_A[(p+q)^2 + Z_{m^2} m^2(\mu_0)]},$$

$$f(q,p) \equiv \frac{2p^2q^2 + 3(p^2+q^2)(p\cdot q) + 4(p\cdot q)^2}{p^2(q+p)^2}$$

# Renormalization

- **At the order $g_g^0$ and $1/N_c^0$:**
  - No corrections for the gluon propagator: $Z_A \sim 1$ and $Z_{m^2} \sim 1$.
  - No corrections for the quark-gluon vertex: $Z_{g_q} Z_\psi \sqrt{Z_A} \sim 1$
- The equations can be simplified as

$$A(p,\mu_0) = Z_\psi(\mu_0) - g_q^2(\mu_0) C_F \int_q \tilde{A}(q,\mu_0) \frac{f(q,p)}{(p+q)^2 + m^2(\mu_0)},$$

$$B(p,\mu_0) = M(\mu_0) + g_q^2(\mu_0) C_F (d-1) \int_q \tilde{B}(q,\mu_0) \left( \frac{1}{(p+q)^2 + m^2(\mu_0)} - \frac{1}{(\mu_0+q)^2 + m^2(\mu_0)} \right)$$

# Renormalization Group

- **Renormalization Group** equation for the quark propagator: $(\mu\partial_\mu - \gamma_\psi + \beta_{X_i}\partial_{X_i})S^{-1} = 0$
- The **solution** at different renormalization scales:

$$S^{-1}(p,\mu,X_i(\mu)) = z_\psi(\mu,\mu_0)S^{-1}(p,\mu_0,X_i(\mu_0)),$$

with $\log z_\psi(\mu,\mu_0) = \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_\psi(\mu')$

-

$$\boxed{\begin{array}{c} \mathrm{A}(\mathrm{p},\mu_0) = z_\psi^{-1}(p,\mu_0) \\ \mathrm{B}(\mathrm{p},\mu_0) = z_\psi^{-1}(p,\mu_0)M(p) \\ \mathrm{z}_\psi(p,\mu_0) = Z_\psi(p)/Z_\psi(\mu_0) \end{array}}$$

- In order to avoid large logarithms we take the p-derivative at fixed bare quantities. Defining $u = \cos\theta$ we obtain

$$Z_\psi(p) = 1 + \frac{g_q^2(p) C_F \Omega_{d-1}}{p^2 Z_\psi(p)(2\pi)^d} \int_0^\infty dq\ q^{d-1} \frac{Z_\psi(q)}{q^2 + M^2(q)} \times \int_{-1}^{1} du (1-u^2)^{\frac{d-3}{2}} \frac{2p^2q^2 + 3(p^2+q^2)pqu + 4p^2u^2q^2}{(p^2+2pqu+q^2)(p^2+2pqu+q^2+m_0^2)},$$

$$-\gamma_\psi(p)M(p) + pM'(p) = -(d-1)\frac{g_q^2(p) C_F \Omega_{d-1}}{Z_\psi(p)(2\pi)^d} \int_0^\infty dq\ \ q^{d-1} \frac{Z_\psi(q)M(q)}{q^2+M^2(q)} \times \int_{-1}^{1} du (1-u^2)^{\frac{d-3}{2}} \frac{2p^2 + 2puq}{(p^2+2qup+q^2+m_0^2)^2},$$

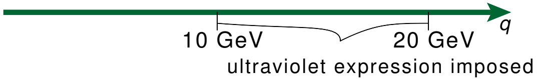
where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$

# Ultraviolet behaviour ($p \gg m_0$):

- $Z_\psi(\mu) \to 1$
- Two possible solutions for the running mass $M(\mu)$:
  - Massive solution: $\boxed{M(\mu) = M_0 \left[1 + 2\beta_0 g_0^2 \ln\left(\frac{\mu}{\mu_0}\right)\right]^{-\frac{3C_f}{16\pi^2\beta_0}}}$
  - $S\chi SB$ solution: $\boxed{M(\mu) = \frac{1}{\mu^2 \ln(\mu)^{\frac{-3C_f}{16\pi^2\beta_0}+1}}}$

with $\beta_0 = \frac{1}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$

[Fischer and Alkofer PRD **67** (2003). Atkinson et al, PRD **37**(1988). Miransky, PL **165B** (1985). Aguilar et al, PRD **83** (2011). ]




- In the second region the values of $Z_\psi(q)$ and $M(q)$ are replaced by their ultraviolet expressions:

$$Z_\psi^{\mathrm{UV}}(q) = 1$$

$$M^{\mathrm{UV}}(q) = b_0 \left( \ln \frac{q^2 + m_0^2}{m_0^2} \right)^{\alpha} + \frac{b_2}{q^2} \left( \ln \frac{q^2 + m_0^2}{m_0^2} \right)^{-(\alpha+1)}$$

where the exponent $\alpha = -\frac{3C_f}{16\pi^2\beta_0}$.

- The coefficients $b_0$ and $b_2$ are chosen in order to make $M(p)$ **continuous and differentiable** (so they are not free parameters).
- We use the **ultraviolet one-loop running of the coupling constant** for $N_f$ flavors properly regularised in the infrared.

$$g_q^2(\mu) = \frac{g_0^2}{1 + \beta_0 g_0^2 \ln \left( \frac{\mu^2 + x^2 m_0^2}{x^2 m_0^2} \right)}$$

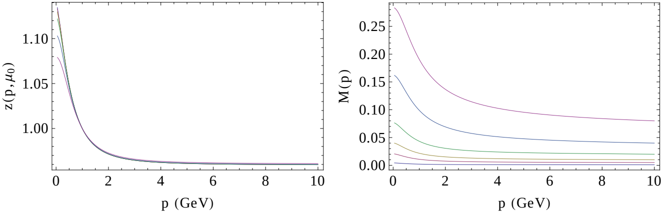
# Finding the solution:

- We set $m_0 = 0{,}4$ GeV.
- The initial contions for the flow are:
  - The value of $M(10GeV)$
  - $g_0$ and $x$
- Starting condition for the iteration:

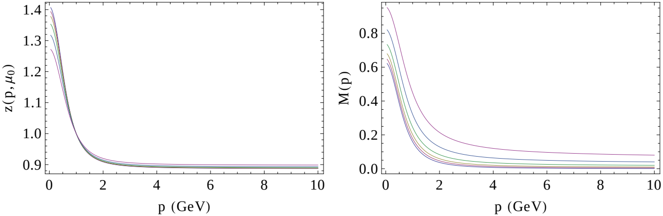
$$Z_\psi(q) = 1$$

$$M(q) = b_1 \left( \ln \frac{q^2 + m_0^2}{m_0^2} \right)^{\alpha} + \frac{b_1}{q^2} \left( \ln \frac{q^2 + m_0^2}{m_0^2} \right)^{-(\alpha+1)}$$

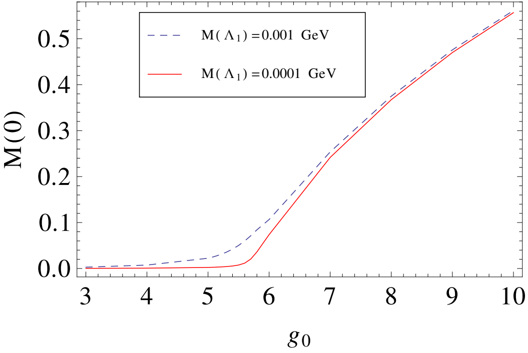
with $b_1$ so $M(10GeV)$ has the appropiate value



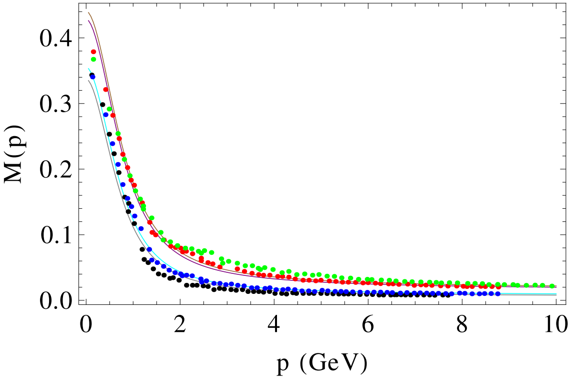

$Z_\psi(p)$ and $M(p)$ for different values of $M(10GeV) = 0{,}001, 0{,}005, 0{,}01, 0{,}02, 0{,}04, 0{,}08$. Parameters: $N_f = 2$, $N_c = 3$, $m_0 = 0{,}4$ GeV, $g_0 = 4$ and $x = 5$.




$Z_\psi(p)$ and $M(p)$ for different values of $M(10GeV) = 0{,}001, 0{,}005, 0{,}01, 0{,}02, 0{,}04, 0{,}08$. Parameters: $N_f = 2$, $N_c = 3$, $m_0 = 0{,}4$ GeV, $g_0 = 11$ and $x = 5$.



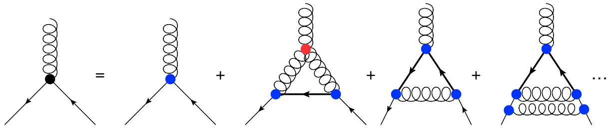

Constituent quark mass $M(p=0)$ as a function of the coupling parameter $g_0$ for two values of the ultraviolet mass $M(\Lambda_1)$. The variation of $g_0$ is done by keeping $\Lambda_{QCD}$ fixed.




Comparision with lattice data from [*Oliveira et al. arxiv:1605.09632*] for $M(p)$ for $M(10GeV) = 0{,}008, 0{,}01, 0{,}02, 0{,}022$. Parametres: $N_f = 2$, $N_c = 3$, $m_0 = 0{,}4$ GeV, $g_0 = 7$ and $x = 5$.

# Running of the quark-gluon coupling

- Corrections for the quark-gluon vertex:



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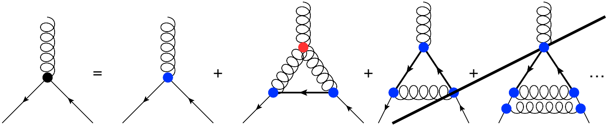



- One-loop-diagram with three-gluon vertex must be included.
- The quark propagator appears always in its full form.
- We define the quark-gluon coupling through $\lambda_1' = -\frac{1}{4g_B(d-2)} Im \sum \mathrm{Tr}(\gamma_\epsilon \Gamma_\mu P^\perp_{\mu\nu}(k) P^\perp_{\nu\rho}(r) P^\perp_{\rho\epsilon}(p))$ in the kinematic configuration corresponding to two equal and ortogonal quark-antiquark momenta (OTE).

- The $\beta$-function for $g_q$ takes the form:

$$\boxed{\begin{gathered}\beta_{g_q} = \mu \frac{dg_q}{d\mu}|_{g_\Lambda} = g_q(\gamma_\psi + \frac{1}{2}\gamma_A) + g_q \mu \frac{d\lambda_1^\Lambda(ren)}{d\mu} \\ \beta_{g_g} = g_g(\gamma_C + \frac{1}{2}\gamma_A)\end{gathered}}$$

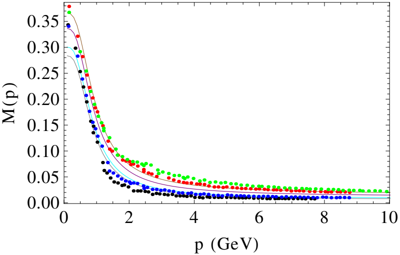
- $\gamma_C$ is computed in its one loop form.
- In order to compute $\gamma_A$ we include one loop diagrams in gluon propagator, considering full quark propagators in the diagram with a quark loop.
- $\lambda_1$, $\gamma_A$ and $\gamma_\psi$ are also coupled with $M(p)$ and $z_\psi(p)$.

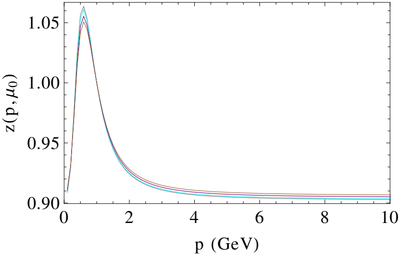
# Initial conditions for the flow

- The relation between both coupling constants at $10\,GeV$ (the starting point of the flow) can be done perturbately.
- $$g_q(\mu) = g_g(\mu)\left(1 + g_g^2(\delta Z_\psi - \delta Z_c + \delta\lambda_1^\Lambda)\right)$$
- In the OTE configuration, for large $\mu$:

$$g_q(\mu) = g_g(\mu)\left(1 + \frac{g_g^2 N}{64\pi^2}(5 - 3\log(2))\right)$$

- In this case the flow of the coupling constant is determined only by one parameter, $g_g(10\,GeV)$. However, this is not a completely free parameter and has to be compared with the known value of the coupling constant.




M(p), with initial condition M(10GeV)=0.008, 0.01, 0.015, 0.02, y $g_g(10GeV) = 1{,}85$

# Conclusions

## To sumarize:

- Based on the fact that, at low energies, **the coupling** $g_g$ **differs significantly from the coupling** $g_q$ **in the matter sector** we treat both constant on different footing.
- **Two and three point correlation functions in Yang-Mill sector** have shown to be **well reproduced by a one loop analysis** using a massive Lagrangian.
- We propose a **systematic expansion** scheme for QCD at low energy based on a **double expansion** in powers of the coupling strength $\mathbf{g_g}$ in the Yang-Mills sector of the theory and in powers of $\mathbf{1/N_c}$.

# Conclusions and perspectives

## Conclusions

- At leading order, this scheme reproduces the well-known **rainbow approximation**.
- It allows for a systematic study of higher order corrections.
- We are able to implement a **consistent renormalization group** improvement of the rainbow equations that yields a better control of large logarithms.
- We **solve the rainbow approximation** using the one loop ultraviolet running for the coupling and also using the quark-gluon vertex. In both cases we reproduce numerically the chiral symmetry breaking.

# Conclusions and perspectives

## Perspectives

We are begining to use the present scheme to calculate mesonic properties such as the mass spectrum or decay rates.

Thanks