Some applications of Renormalization Group Optimized perturbation (zero and finite temperatures)

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Introduction/motivation

Approach: (unconventional) resummation of perturbative expansions

General: relevant both at $T = 0$ or $T \neq 0$ (and finite density)

• $T = 0$: gives estimates of chiral symmetry breaking order parameters:
$F_\pi(m_q = 0)/\Lambda_{\overline{MS}}^{\rm QCD}$:
$F_\pi \simeq 92.2\text{MeV} \to F_\pi(m_q = 0) \to \Lambda_{\overline{MS}}^{n_f=3} \to \alpha_S^{\overline{MS}}(\mu = m_Z)$.

N^3LO: $F_\pi^{m_q=0}/\Lambda_{\overline{MS}}^{n_f=3} \simeq 0.25 \pm .01 \to \alpha_S(m_Z) \simeq 0.1174 \pm .001 \pm .001$
(JLK, A.Neveu, '13))
(compares well with latest (2016) α_S world average [PDG2016])

In this talk I illustrate $\langle\bar{q}q\rangle$ RGOPT determination at N^3LO:
$\langle\bar{q}q\rangle_{m_q=0}^{1/3}(2\,\text{GeV}) \simeq -(0.84 \pm 0.01)\Lambda_{\overline{MS}}$ (JLK, A.Neveu, '15))
Note: parameter free determination!

NB: can also address gluon 'mass' determination in principle...

More motivations (thermal context)

Complete QCD phase diagram far from being confirmed:

$T \neq 0, \mu = 0$ well-established from lattice: no sharp phase transition, continuous crossover at $T_c \simeq 154 \pm 9$ MeV

Goal: more analytical approximations, ultimately in regions not much accessible on the lattice: large density (chemical potential) due to the (in)famous "sign problem"
(i.e. complex Euclidian fermion determinant on the lattice)

•RGOPT considerably reduces the well-known problems of unstable +badly scale-dependent thermal perturbative expansions

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  (σ model: many similarities with QCD but simpler)
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2. (Variationally) Optimized Perturbation (OPT)

Trick ($T = 0$): add and subtract a mass, consider $m\,\delta$ as interaction:

$$\mathcal{L}_{QCD}(g, m) \to \mathcal{L}_{QCD}(\delta\, g, m(1-\delta)) \;\; \text{(e.g. in QCD } g \equiv 4\pi\alpha_S)$$

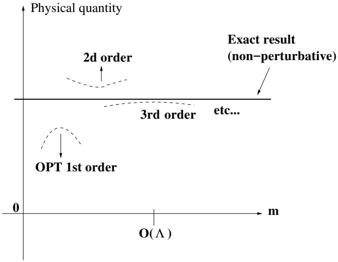
where $0 < \delta < 1$ interpolates between $\mathcal{L}_{free}$ and *massless* $\mathcal{L}_{int}$;
e.g. in QCD (quark) mass $m_q \to m$: arbitrary trial parameter

• Take any standard (renormalized) QCD pert. series, expand in δ *after:*
$m_q \to m\,(1-\delta)$; $\quad g \to \delta\, g$
then take $\delta \to 1$ (to recover original massless theory):

BUT a m-dependence remains at any finite δ^k-order:
fixed typically by stationarity prescription: optimization (OPT):
$\frac{\partial}{\partial m}$(physical quantity) $= 0$ for $m = \bar{m}_{opt}(\alpha_S) \neq 0$:
• $T = 0$: exhibits *dimensional transmutation*: $\bar{m}_{opt}(g) \sim \mu\, e^{-const./g}$

• $T \neq 0$ similar idea: "screened perturbation" (SPT), or *resummed* "hard thermal loop (HTLpt)", etc: expand around a quasi-particle mass.
But does this 'cheap trick' always work? and why?

Expected behaviour (ideally)



But not quite what happens, except in simple models:
•Convergence proof of this procedure for $D = 1$ ϕ^4 oscillator
(cancels large pert. order factorial divergences!) Guida et al '95

particular case of 'order-dependent mapping' Seznec, Zinn-Justin '79

•But QFT multi-loop calculations (specially $T \neq 0$) (very) difficult:
$\rightarrow$ what about convergence? not much apparent

•Main pb at higher order: OPT: $\partial_m(...) = 0$ has multi-solutions (some complex!), how to choose right one, if no nonperturbative "insight"??

3. RG compatible OPT ($\equiv$ RGOPT)

Our main additional ingredient to OPT (JLK, A. Neveu '10):

Consider a *physical* quantity (i.e. perturbatively RG invariant) (for example, in thermal context, the pressure $P(m, g, T)$):

in addition to OPT: $\frac{\partial}{\partial m} P^{(k)}(m, g, \delta = 1)|_{m \equiv \tilde{m}} \equiv 0$,
Require (δ-modified!) result at order δ^k to satisfy a standard (perturbative) Renormalization Group (RG) equation:

$$\mathrm{RG}\left(P^{(k)}(m, g, \delta = 1)\right) = 0$$

with standard RG operator ($g = 4\pi\alpha_S$ for QCD):

$$\mathrm{RG} \equiv \mu \frac{d}{d\,\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_m(g)\, m \frac{\partial}{\partial m}$$

$\beta(g) \equiv -b_0 g^2 - b_1 g^3 + \cdots,\ \ \gamma_m(g) \equiv \gamma_0 g + \gamma_1 g^2 + \cdots$

$\rightarrow$ Additional nontrivial constraint: contains a priori more consistent RG information than $\partial_m P(m)$ optimization.

→ If combined with OPT, RG Eq. reduces to massless form:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]P^{(k)}(m,g,\delta=1)=0$$

Then using OPT AND RG completely fix $m \equiv \bar{m}$ and $g \equiv \bar{g}$.

But $\Lambda_{\overline{MS}}(g)$ satisfies by def.:
$[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}]\,\Lambda_{\overline{MS}} \equiv 0$ consistently at a given pert. order for $\beta(g)$.

Thus equivalent to:

$$\frac{\partial}{\partial\, m}\left(\frac{P^k(m,g,\delta=1)}{\Lambda_{\overline{MS}}(g)}\right)=0\ ;\quad \frac{\partial}{\partial\, g}\left(\frac{P^k(m,g,\delta=1)}{\Lambda_{\overline{MS}}(g)}\right)=0 \text{ for } \bar{m},\bar{g}$$

Optimal $\bar{m}, \bar{g} = 4\pi\bar{\alpha}_S$ unphysical: final (physical) result from $P(\bar{m},\bar{g},T)$

At $T=0$ reproduces at first order exact nonperturbative results in simpler models [e.g. Gross-Neveu model]

OPT + RG = RGOPT main new features

•Standard OPT: embarrassing freedom (a priori) in interpolating form:
e.g. why not $m \to m\,(1-\delta)^a$?
Most previous works ($T=0$ or SPT, HTLpt,...($T \neq 0$):
linear interpolation "add and subtract" ($a=1$) without deep justification
but generally (we have shown) $a=1$ spoils RG invariance!

•OPT,RG Eqs: many solutions at increasing δ^k-orders

$\to$ Our approach restores RG, +requires optimal solution to match perturbation (i.e. Asymptotic Freedom for QCD ($T=0$)):
$\alpha_S \to 0$ ($\mu \to \infty$): $\bar{g}(\mu) \sim \frac{1}{2b_0 \ln \frac{\mu}{\bar{m}}} + \cdots$, $\bar{m} \sim \Lambda_{QCD}$

$\to$ At successive orders AF-compatible optimal solution (often unique) *only* appears for universal critical a:
$m \to m\,(1-\delta)^{\frac{\gamma_0}{b_0}}$ (in general $\frac{\gamma_0}{b_0} \neq 1$)

$\to$ Goes beyond simple "add and subtract" trick

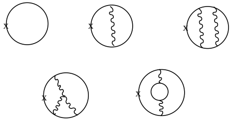
+ It removes any spurious solutions incompatible with AF

− But does not always avoid complex solutions
(if occur, possibly cured by renormalization scheme change)

4. Application: $T = 0$ QCD quark condensate (JLK, A. Neveu '15)

$\langle \bar{q} q \rangle$: chiral symmetry breaking order parameter:
$SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_{L+R}$, n_f massless quarks. ($n_f = 2, 3$)

• Start from perturbative result, known to 3 loops (Chetyrkin et al '94; Chetyrkin +Maier '10)



$$m \langle \bar{q} q \rangle (m, g)_{\overline{MS}} = 3 \frac{m^4}{2\pi^2} \left[\frac{1}{2} - L_m + \frac{g}{\pi^2} (L_m^2 - \frac{5}{6} L_m + \frac{5}{12}) \right.$$
$$\left. + (\frac{g}{16\pi^2})^2 [f_{30}(n_f) L_m^3 + f_{31}(n_f) L_m^2 + f_{32}(n_f) L_m + f_{33}(n_f)] \right], \quad (L_m \equiv \ln \frac{m}{\mu})$$

Important rk: this finite part (after mass + coupling renormalization) is not separately RG-invariant:
$\mu \frac{d}{d\mu} m \langle \bar{q} q \rangle = \mathcal{O}(m^4) \equiv$ vacuum energy anomalous dimension

•Known generic RG feature (but often missed in literature): needs appropriate 'subtraction' terms (consistently RG-determined) to recover perturbative RG invariance

First attempt: naive (direct) optimization of $m\langle\bar{q}q\rangle$?

NB works pretty well for F_π/Λ_{QCD}:
$i\langle 0|TA^i_\mu(p)A^j_\nu(0)|0\rangle \equiv \delta^{ij} g_{\mu\nu} F^2_\pi(m=0)$; $A^i_\mu \equiv \bar{q}\gamma_\mu\gamma_5\frac{\tau_i}{2}\,q$
(empirical convergence of RGOPT observed already at 2-loops)

•For $m\langle\bar{q}q\rangle$: One-loop order: no nontrivial OPT +RG solution...

•Higher orders (2- and 3-loops): gives right order of magnitude, but ambiguous results: plagued by large, unphysical, imaginary parts
$\rightarrow$ no conclusive stability/convergence trend (appears slow at best)

• Instabilities traced to strong sensitivity to regularization choice:

with naive cutoff the (dominant) one-loop quadratic divergence gives correct (negative) sign of $\langle\bar{q}q\rangle$ (a pillar of the success of Nambu-Jona-Lasinio model!)
but (one-loop) sign flips in ($D = 4 - \epsilon$) dimensional regularization ($\overline{\rm MS}$)

Yet important to keep benefits of $\overline{\rm MS}$: high order QCD perturbative results available: crucial for stability/convergence check.

$\rightarrow$ Like any other variational methods it is sensible to start from a suitable quantity to optimize: here the spectral density of the Dirac operator, intimately related to $\langle\bar{q}q\rangle$.

4. $\langle \bar{q}q \rangle$ and Spectral density $\rho(\lambda)$

Euclidean Dirac operator:
$i\,\not{D}\,u_n(x) = \lambda_n\,u_n(x)$; $\not{D} \equiv \not{\partial} + g\,\not{A}$;
NB $i\,\not{D}\,(\gamma_5 u_n(x)) = -\lambda_n\,(\gamma_5 u_n(x))$

On a lattice: $\rho(\lambda) \equiv \frac{1}{V}\langle \sum_n \delta(\lambda - \lambda_n^{[A]})\rangle_A$

$V \to \infty$: spectrum becomes dense, and
$\langle \bar{q}q \rangle \equiv \frac{1}{V}\,\mathrm{Tr}\frac{1}{m+\not{D}} \to \langle \bar{q}q \rangle_{V\to\infty}(m) \equiv -2m\int_0^\infty d\lambda\frac{\rho(\lambda)}{\lambda^2+m^2}$

$\rho(\lambda)$: spectral density of the (euclidean) Dirac operator.

Banks-Casher relation ('80): $\langle \bar{q}q \rangle(m \to 0) \equiv -\pi\rho(0)$
(using e.g. $\lim_{m\to 0}\,\frac{1}{m-i\lambda} = i\,PV(\frac{1}{\lambda}) + \pi\delta(\lambda)$)

'Washes out' large λ problems (e.g. quadratic UV divergences)

Conversely: $-\rho(\lambda) = \frac{1}{2\pi}\left(\langle \bar{q}q \rangle(i\lambda+\epsilon) - \langle \bar{q}q \rangle(i\lambda-\epsilon)\right)|_{\epsilon\to 0}$
i.e. $\rho(\lambda)$ determined by discontinuities of $\langle \bar{q}q \rangle(m)$ across imaginary axis.

Perturbative expansion: $\to \ln(m \to i\lambda)$ discontinuities
$\to$ no contributions from divergence and non-log terms (like anom. dim.)

Adapting OPT and RG Eqs. to spectral density

- Perturbative logarithmic discontinuities simply from

$$\ln^n\left(\frac{m}{\mu}\right) \to \frac{1}{2i\pi}\left[\left(\ln\frac{|\lambda|}{\mu}+i\frac{\pi}{2}\right)^n - \left(\ln\frac{|\lambda|}{\mu}-i\frac{\pi}{2}\right)^n\right] \qquad (1)$$

i.e. $\ln\left(\frac{m}{\mu}\right) \to 1/2$; $\ln^2\left(\frac{m}{\mu}\right) \to \ln\frac{|\lambda|}{\mu}$; $\ln^3\left(\frac{m}{\mu}\right) \to \frac{3}{2}\ln^2\frac{|\lambda|}{\mu} - \frac{\pi^2}{8}$; $\cdots$

- Modified perturbation: intuitively λ plays the role of m, so:

$$\rho_{pert}(\lambda, g) \to \rho_{opt}(\lambda(1-\delta)^{\frac{4}{3}\frac{\gamma_0}{2b_0}}, \delta g)\,; \text{ expand in } \delta\,; \quad \delta \to 1 \qquad (2)$$

- OPT Eq.: $$\frac{\partial}{\partial\lambda}\rho_{opt}(g,\lambda) = 0 \text{ for } \lambda = \bar{\lambda}_{opt}(g) \neq 0 \qquad (3)$$

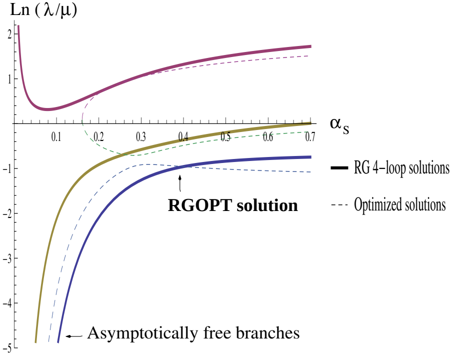
- Using $\frac{\partial}{\partial m}\frac{m}{\lambda^2+m^2} = -\frac{\partial}{\partial\lambda}\frac{\lambda}{\lambda^2+m^2}$, one finds $\rho(\lambda)$ obeys RG eq.:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_m(g)\,\lambda\frac{\partial}{\partial\lambda} - \gamma_m(g)\right]\rho(g,\lambda) = 0 \qquad (4)$$

$\to$ RGOPT recipe: -After (1), (2), solve (3), (4) for optimal $\bar{\lambda}$, $\bar{g}$;
then $\rho(\bar{\lambda},\bar{g}) \simeq \rho(0) \equiv -\langle\bar{q}q\rangle(m=0)/\pi$.

Example: RG and OPT $\rho(\lambda)$ solutions: up to 4-loops

NB $\langle \bar{q}q \rangle_{pert}$ exactly known at present to 3-loop α_S^2 order.
But 1) RG properties determine next (4-loop) $\alpha_S^3 \ln^p(m/\mu)$ coefficients,
2) non-logarithmic terms do not contribute to $\rho_{pert}(\lambda)$:
$\rightarrow$ we get $\rho_{pert}(\lambda)$ exact to 4-loop!



RGOPT 2,3,4-loop results for $\langle \bar{q} q \rangle$ ($n_f = 2$)

δ^k, RG order	$\ln \frac{\bar{\lambda}}{\mu}$	$\bar{\alpha}_S$	$\frac{-\langle \bar{q} q \rangle^{1/3}}{\bar{\Lambda}_2}(\bar{\mu})$	$\frac{\bar{\mu}}{\bar{\Lambda}_2}$	$\frac{-\langle \bar{q} q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_2}$
δ, RG 2-loop	-0.45	0.480	0.822	2.8	0.821
δ^2, RG 3-loop	-0.703	0.430	0.794	3.104	0.783
δ^3, RG 4-loop	-0.820	0.391	0.796	3.446	0.773

(and similarly behaving results for $n_f = 3$).

NB: $\langle \bar{q} q \rangle_{RGI} = \langle \bar{q} q \rangle(\mu)\,(2b_0\, g)^{\frac{\gamma_0}{2b_0}} \left(1 + (\frac{\gamma_1}{2b_0} - \frac{\gamma_0\, b_1}{2b_0^2})\, g + \cdots\right)$

- stability/convergence exhibited;
- already realistic value at 2-loop order

RG Evolution to reference scale $\mu = 2$ GeV:

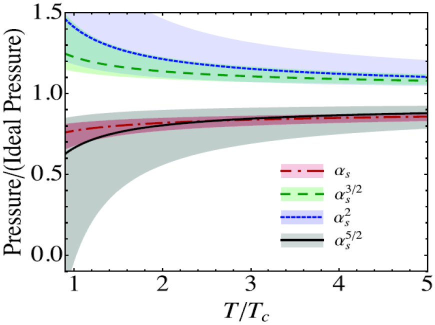
$$-\langle \bar{q} q \rangle^{1/3}_{n_f=2}(2\text{GeV}) = (0.833_{(4-loop)} - 0.845_{(3-loop)})\bar{\Lambda}_2$$

$\rightarrow$ Using most precise $\bar{\Lambda}_2$ lattice result: $\bar{\Lambda}_2 = 331 \pm 21$
(quark potential, Karbstein et al '14) finally gives:
$-\langle \bar{q} q \rangle^{1/3}_{n_f=2}(\text{rgopt}, 2\,GeV) \simeq 278 \pm 2(rgopt) \pm 18(\bar{\Lambda}_2)\; MeV$

•compares with latest lattice result (Engel et al '14):
$-\langle \bar{q} q \rangle^{1/3}_{n_f=2}(\mu = 2\text{GeV}) = 261 \pm 6 \pm 8$ MeV

Thermal perturbative expansion (QCD, $g\phi^4$, ...)

known problem: poorly convergent and very scale-dependent (ordinary) perturbative expansions:



QCD (pure glue) pressure at successive (standard) perturbation orders
shaded regions: scale-dependence for $\pi T < \mu < 4\pi T$
(illustration from Andersen, Strickland, Su '10)

Problems of thermal perturbation (QCD but generic)

Main culprit: mix up of *hard* $p \sim T$ and *soft* $p \sim \alpha_S T$ modes.

Thermal 'Debye' screening mass $m_D^2 \sim \alpha_S T^2$ gives IR cutoff,
BUT $\Rightarrow$ perturbative expansion in $\sqrt{\alpha_S}$ in QCD
$\rightarrow$ often advocated reason for slower convergence

Yet many interesting QGP physics features happen at not that large coupling $\alpha_S(\sim 2\pi T_c) \sim .5$, ($\alpha_S(\sim 2\pi T_c) \sim 0.3$ for pure glue)

Many efforts to improve this (review e.g. Blaizot, Iancu, Rebhan '03):

Screened PT (SPT) (Karsch et al '97), $\sim$ Hard Thermal Loop (HTL) resummation (Andersen, Braaten, Strickland '99); Functional RG, 2-particle irreducible (2PI) formalism (Blaizot, Iancu, Rebhan '01; Berges, Borsanyi, Reinosa, J. Serreau '05)

RGOPT($T \neq 0$): essentially treats thermal mass 'RG consistently':
$\rightarrow$ UV divergences induce its anomalous dimension.

(NB some qualitative connections with recently advocated "massive scheme" approach (Blaizot, Wschebor '14)

5. RGOPT(thermal ϕ^4) (JLK, M.B Pinto '15, '16)

•Start from 2-loop vacuum energy $m \neq 0$, $T \neq 0$ ($\overline{\text{MS}}$ scheme):

$$(4\pi)^2 \mathcal{F}_0 = \mathcal{E}_0 - \frac{m^4}{8}(3 - 4\ln\frac{m}{\mu}) - \frac{T^4}{2} J_0(\frac{m}{T})$$

$$+\frac{1}{8}(\frac{g}{16\pi^2})\left[(1 - 2\ln\frac{m}{\mu})m^2 - T^2\, J_1(\frac{m}{T})\right]^2$$

$$J_0(\frac{m}{T}) \sim \int_0^\infty dp \frac{1}{\sqrt{p^2+m^2}} \frac{1}{e^{\sqrt{p^2+m^2}}-1} \;;\quad \partial_x J_0(x) \equiv -2x\, J_1(x)$$

First step: $\mathcal{E}_0$: necessary *finite* vacuum energy subtraction:
$\mathcal{E}_0(g,m) = -m^4\left(\frac{s_0}{g} + s_1 + s_2 g + \cdots\right)$
(T-independent, determined by requiring RG invariance:)

$$s_0 = \frac{1}{2(b_0 - 4\gamma_0)} = 8\pi^2;\quad s_1 = \frac{(b_1 - 4\gamma_1)}{8\gamma_0\,(b_0 - 4\gamma_0)} = -1,\ \ldots;$$

missed by SPT, or HTLpt (hot QCD): explains the important scale dependence observed in those approaches (more below)

Next: expand in δ, $\delta \to 1$ after $m^2 \to m^2(1-\delta)^a$; $g \to \delta g$
RG only consistent for $a = 2\gamma_0/b_0$ ($= 1/3$ for ϕ^4: while $a = 1$ in SPT)

Practical bonus: non-trivial mass gap $\bar{m}(g,T)$ already at one-loop.
NB $1/g$ in $\mathcal{E}_0$ automatically cancels in (optimized) energy.

One-loop RGOPT ($\mathcal{O}(\delta^0)$)

Exact OPT "thermal mass gap" $\bar{m}$ self-consistent solution of

$$\bar{m}^2 = \frac{g}{2}\left[b_0\bar{m}^2(\ln\frac{\bar{m}^2}{\mu^2}-1)+T^2 J_1(\frac{\bar{m}}{T})\right] \quad (\bar{m}^2(T=0)=\mu^2 e^{1+\frac{2}{b_0\,g}})$$

or from simple quadratic Eq. for $m/T \lesssim 1$ (sufficient for most purpose):

$$\left(\frac{1}{b_0\,g}+\gamma_E+\ln\frac{\mu}{4\pi T}\right)(\frac{m}{T})^2+2\pi\,\frac{m}{T}-2\frac{\pi^2}{3}=0$$

$$\frac{\bar{m}}{T}=\pi\frac{\sqrt{1+\frac{2}{3}(\frac{1}{b_0\,g}+L_T)}-1}{\frac{1}{b_0\,g}+L_T} \sim \pi(\frac{2}{3}\sqrt{b_0 g}-b_0 g+\frac{1}{2\sqrt{6}}(3-2L_T)(b_0 g)^{3/2}+\cdots)$$

$$\frac{P}{P_0}(G)=1-\frac{5}{4}G-\frac{15}{2}G^2(G+1)+\frac{5}{3}\sqrt{6}\left[G(1+\frac{3}{2}G)\right]^{3/2}+\cdots$$

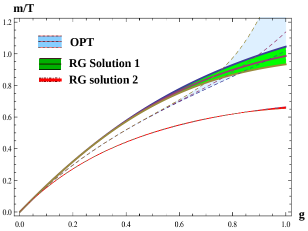
$L_T \equiv \gamma_E+\ln\frac{\mu}{4\pi T}$, $1/G \equiv 1/(b_0 g(\mu))+L_T$; $P_0=\pi^2 T^4/90$

$\Rightarrow$ $\bar{m}$**, and thus** $P(\bar{m})$ **are explicitly exactly scale-invariant**

+ reproduces much simply exact (all orders) known large N results

(Drummond et al '98)

RGOPT mass and Pressure: two-loop order



$(g(\mu)/24)^{1/2} \equiv g(\mu)$ with scale-dependence $\pi T < \mu < 4\pi T$

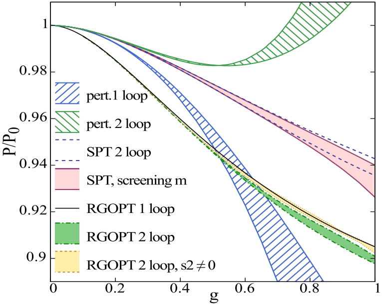
At two-loops a remnant scale-dependence reappears from unperfectly matched running coupling $\leftrightarrow$ mass and pressure:
but moderate in $P(g)$, shown to be of higher (4-loop) order $\mathcal{O}(g^3 \ln \mu)$.

Generic: RGOPT at $\mathcal{O}(g^k) \to \ln \mu$ appears at $\mathcal{O}(g^{k+1})$ for *any* $\bar{m}$, but $\bar{m}^2 \sim g T^2$: so $P \simeq \bar{m}^4/g + \cdots$ has leading μ-dependence at $\mathcal{O}(g^{k+2})$.

•Note intersection of RG and OPT solutions: can fix a 'nonperturbative' $g(\mu \sim \alpha\pi T)$ (alternative to running coupling)

•The other RG solution with tiny scale-dependence (red) is driven by (spurious perturbative) UV fixed point: to be discarded.

Two-loop RGOPT compared with standard PT and SPT



[JLK, M.B Pinto, PRL 116 (2016); PRD 92 (2015)]

Definite scale-dependence improvement (a factor ~ 3) w.r.t. SPT [J.O. Andersen et al '01]

-Improvement should be more drastic at 3-loops, where SPT scale dependence increases rather strongly.

NB RGOPT pressure has a different slope at $g \sim 0$: not a problem: g not fixed at a physical scale (similar to e.g. large N different slope)

[Alternatively: solving OPT Eq. as $\bar{g}(m)$: $P \equiv P(m)$; where m is arbitrary: now using *physical* (screening) mass $P(m \to m_D(g))$:

$\to$ recover standard perturbative terms of $P(g)$]

Closer to QCD: $O(N)$ nonlinear σ model (NLSM)

[G. Ferreri, JLK, M.B. Pinto, R.0 Ramos, arXiv:1709...]

(1+1)D NLSM shares many properties with QCD: asymptotic freedom, mass gap. Also $T \neq 0$ pressure, trace anomaly have QCD-similar shape
Other nonperturbative $T \neq 0$ results available for comparison
(lattice [Giacosa et al '12], $1/N$ expansion [Andersen et al '04], others)

$$\mathcal{L}_0 = \frac{1}{2}(\partial\pi_i)^2 + \frac{g(\pi_i\partial\pi_i)^2}{2(1-g\pi_i^2)} - \frac{m^2}{g}\left(1-g\pi_i^2\right)^{1/2}$$

two-loop pressure from:

•Advantage w.r.t. QCD: exact T-dependence at 2-loops:

$$P_{\rm pert.2loop} = -\frac{(N-1)}{2}\left[I_0(m,T) + \frac{(N-3)}{4}m^2 g I_1(m,T)^2\right] + \mathcal{E}_0,$$

$$I_0(m,T) = \frac{1}{2\pi}\left(m^2(1-\ln\frac{m}{\mu}) + 4T^2 K_0(\frac{m}{T})\right)$$

$$K_0(x) = \int_0^\infty dz \ln\left(1-e^{-\sqrt{z^2+x^2}}\right), \quad I_1(m,T) = \partial I_0(m,T)/\partial m^2$$

NLSM peculiarity: standard perturbative RG invariance

$\mathcal{E}_0$ in P_{2-loop}: $\equiv \frac{m^2}{g}$ already there in Lagrangian!

If it would be ignored, reconstructed consistently by RG 'subtraction':

$\mathcal{E}_0(g,m) = m^2 \left(\frac{s_0}{g} + s_1 + s_2 g + \cdots \right)$

$s_0 = \frac{(N-1)}{4\pi(b_0-2\gamma_0)} = 1\ ,\ \ s_1 = (b_1 - 2\gamma_1)\frac{s_0}{2\gamma_0} = 0$

A property not expected in any other models: consequence of NLSM renormalization with only two counterterms (coupling Z_g and field Z_π)):
$Z_m = Z_g Z_\pi^{-1/2}$

•RG consistency: $m \to m(1-\delta)^{\frac{\gamma_0}{b_0}}$, $\frac{\gamma_0}{b_0} = (N-3)/(N-2)/2$ for NLSM

Other properties similar to ϕ^4:

•Exact one-loop scale invariance
•Very moderate remnant scale dependence at 2-loops
•Also reproduces exact (all orders) known large N (LN) results
(Andersen et al '04)

One-loop RGOPT for NLSM pressure

Exact T-dependent mass gap $\bar{m}(g,T)$ from $\partial_m P(m) = 0$:

$$\ln\frac{\bar{m}}{\mu} = -\frac{1}{b_0\, g(\mu)} - 2K_1\left(\frac{\bar{m}}{T}\right), \qquad \left(b_0^{\rm nlsm} = \frac{N-2}{2\pi}\right)$$

more explicitly for $T=0$: $\bar{m} = \mu e^{-\frac{1}{b_0\, g(\mu)}} = \Lambda^{\rm 1-loop}_{\overline{\rm MS}}$

and for $T \gg m$:

$$\frac{\bar{m}}{T} = \frac{\pi\, b_0\, g}{1 - b_0\, g\, L_T}, \quad \left(L_T \equiv \ln\frac{\mu\, e^{\gamma_E}}{4\pi T}\right)$$

$$P^{\rm RGOPT}_{1L,exact} = -\frac{(N-1)}{\pi} T^2 \left[K_0(\bar{x}) + \frac{\bar{x}^2}{8}\,(1+4K_1(\bar{x}))\right], \quad (\bar{x} \equiv \bar{m}/T)$$

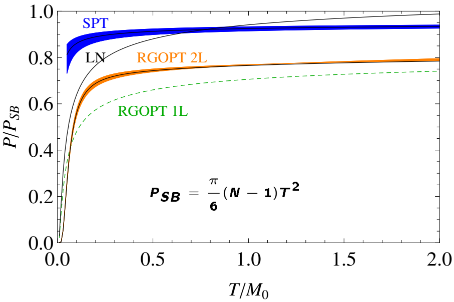
$$P^{\rm RGOPT}_{1L}(T \gg m) \simeq 1 - \frac{3}{2} b_0 g\left(\frac{4\pi T}{e^{\gamma_E}}\right)$$

with one-loop running $g^{-1}(\mu) = g^{-1}(M_0) + b_0 \ln\frac{\mu}{M_0})$

$\rightarrow$ Remarkable property of RGOPT: running with T consistently included!
(for standard perturbation, and SPT, HTLpt, $\mu \sim 2\pi T$ put 'by hand'
e.g. to get correct Stefan-Boltzmann limit)

RGOPT NLSM mass and pressure: two-loops

$P/P_{SB}(N=4, g(M_0)=1)$ vs standard perturbation (PT), large N (LN), and SPT $\equiv$ ignoring RG-induced subtraction; $m^2 \to m^2(1-\delta)$:



(shaded range: scale-dependence $\pi T < \mu \equiv M < 4\pi T$)

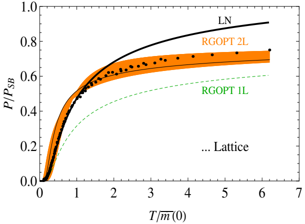
$\to$ A moderate scale-dependence reappears, from unperfectly matched 2-loop standard running coupling.

RGOPT(NLSM) vs lattice results

•NLSM $T \neq 0$ lattice simulations: only done for $N = 3$
[E. Seel, D. Smith, S. Lottini, F. Giacosa '12]

•Comparison for large coupling: $g(M_0) = 2\pi$

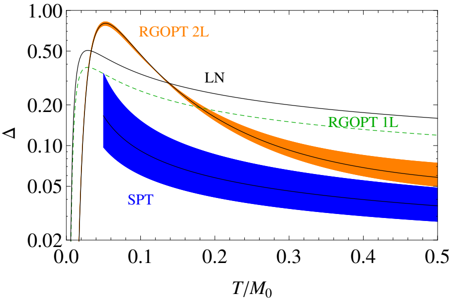
•Drawback: for such large coupling 2-loop RGOPT remnant scale dependence much more sizable.



shaded regions: scale-dependence $\pi T < \mu = M < 4\pi T$
However 2-loop good agreement is rather accidental for large $g(M_0)$ (not genuine RGOPT prediction)

NLSM interaction measure (trace anomaly)

(normalized) $T^2\,\Delta_{\text{2D NLSM}} \equiv \mathcal{E} - P = S\,T - 2P \equiv T^3 \partial_T (\frac{P}{T^2})$



$N = 4, g(M_0) = 1$ (shaded regions: scale-dependence $\pi T < \mu = M < 4\pi T$)

•2-loop Δ_{SPT}: small, monotonic behaviour + sizeable scale dependence.

•RGOPT shape 'qualitatively' comparable to QCD, showing a peak
(but no spontaneous sym breaking/phase transition in 2D NLSM
(Mermin-Wagner-Coleman theorem): reflects broken conformal
invariance (mass gap).

6. Thermal (pure glue) QCD: hard thermal loop (HTLpt)

QCD generalization of OPT = HTLpt [Andersen, Braaten, Strickland '99]: same "OPT" trick operates on a gluon "mass" term [Braaten-Pisarski '90]:

$$\mathcal{L}_{QCD}(\text{gauge}) - \frac{m^2}{2} Tr \left[G_{\mu\alpha} \langle \frac{y^\alpha y^\beta}{(y.D)^2} \rangle_y G^\mu_\beta \right], \quad D^\mu = \partial^\mu - ig\, A^\mu, \quad y^\mu = (1, \hat{\mathbf{y}})$$

(effective, gauge-invariant, but nonlocal Lagrangian):

describes screening mass $m^2 \sim \alpha_S T^2$, but also many more 'hard thermal loop' contributions [modifies vertices and gluon propagators]

$$P^{HTL,exact}_{1\text{-loop}} = (N_c^2 - 1) \times$$

$$\left\{ \frac{m^4}{64\pi^2} (C_g - \ln \frac{m}{\mu}) + \int_0^\infty \frac{d\omega}{(2\pi^3} \frac{1}{e^{\frac{\omega}{T}} - 1} \int_\omega^\infty dk\, k^2 (2\phi_T - \phi_L) \right.$$

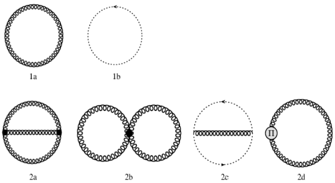
$$\left. - \frac{T}{2\pi^2} \int_0^\infty dk\, k^2 \left[2 \ln(1 - e^{-\frac{\omega_T}{T}}) + \ln(1 - e^{-\frac{\omega_L}{T}}) \right] - \frac{\pi^2 T^4}{90} \right\}$$

where $k^2 + m^2 \left[1 - \frac{\omega_L}{2k} \ln(\frac{\omega_L + k}{\omega_L - k}) \right] = 0$; $f(\omega_T) = 0$; ϕ_L, ϕ_T: complicated.

- Exact 2-loop? daunting task...

→ Clearly more complicated than gluon mass in Curci-Ferrari model [e.g. RSTW approach discussed here '15 '16]

More standard HTLpt results



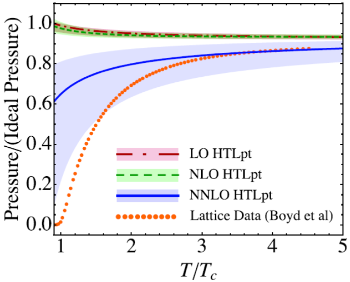
HTLpt advantage: calculated up to 3-loops α_S^2 (NNLO) [Andersen et al '99-'15] BUT only as m/T expansions
Drawback: HTLpt $\equiv$ high-T approximation (by definition)

$$P^{HTLpt}_{\text{1-loop},\overline{MS}} = P_{\text{ideal}} \times$$
$$\left[1 - \frac{15}{2}\hat{m}^2 + 30\hat{m}^3 + \frac{45}{4}\hat{m}^4 (\ln \frac{\mu}{4\pi T} + \gamma_E - \frac{7}{2} + \frac{\pi^2}{3})\right]$$

$\hat{m} \equiv \frac{m}{2\pi T}$ $\qquad$ $P_{\text{ideal}} = (N_c^2 - 1)\pi^2 \frac{T^4}{45}$

More standard HTLpt results

Pure glue up to NNLO 3-loops [Andersen, Strickland, Su '10]:



Reasonable agreement with lattice simulations (Boyd et al '96) at NNLO (3-loop), down to $T \sim 2 - 3\,T_c$, for low scale $\mu \sim \pi T - 2\pi T$.

Main issue of HTLpt however: odd increasing scale dependence at higher (NNLO) order

Moreover HTLpt (frequent) mass prescription $\bar{m} \to m_D^{pert}(\alpha_S)$ [to avoid complex optimized solutions]: may miss more "nonperturbative" information.

RGOPT adaptation of HTLpt =RGOHTL

Main RGOPT changes:

• Crucial RG invariance-restoring subtractions in Free energy (pressure): $P_{HTLpt} \to P_{HTLpt} - m^4 \frac{s_0}{\alpha_S}$: reflects its anomalous dimension.

• interpolate with $m^2(1-\delta)^{\frac{\gamma_0}{b_0}}$, where gluon 'mass' anomalous dimension defined from its (available) counterterm.

RGOPT scale dependence should improve at higher orders, from RG invariance maintained at all stages:
from subtraction terms (prior to interpolation), and from interpolation restoring RG invariance.

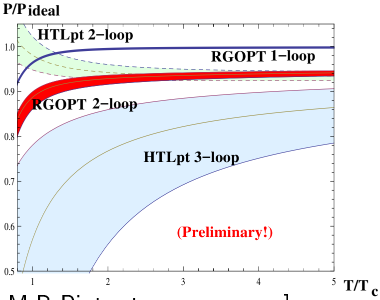
• SPT,HTLpt,... do not include these subtractions:
yet scale dependence moderate up to 2-loops,
because the (leading order) RG-unmatched term, of $\mathcal{O}(m^4 \ln\mu)$, is perturbatively formally like a (3-loop order) α_S^2 term: $m^2 \sim \alpha_S T^2$.

→ Explains why HTLpt scale dependence dramatically resurfaces at 3-loops!

Preliminary RGO(HTL) results (1- and 2-loop, pure glue)

One-loop: exactly scale-invariant pressure (like for ϕ^4 and NLSM):

2-loops: a moderate scale-dependence reappears, similar to ϕ^4, NLSM case: a factor ~ 2 improvement w.r.t. HTLpt 2-loops:



[JLK, M.B Pinto, to appear soon]

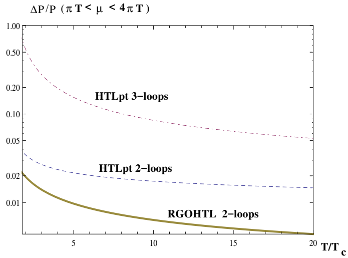
NB scale dependence should improve much at 3-loops, generically:

RGOPT at $\mathcal{O}(\alpha_S^k) \to \bar{m}(\mu)$ appears at $\mathcal{O}(\alpha_S^{k+1})$ for *any* $\bar{m}$, but $\bar{m}^2 \sim \alpha_S T^2 \to P \simeq \bar{m}_G^4/\alpha_S + \cdots$: leading μ-dependence at $\mathcal{O}(\alpha_S^{k+2})$.

• However low $T \sim T_c$ genuine pressure shape needs determining higher order subtraction terms of $\mathcal{O}(m^4 \alpha_S^2 \ln \mu)$: new calculations of 3-loop HTL integrals (neglected in standard HTLpt as formally $\mathcal{O}(\alpha_S^4)$)

(Very) preliminary RGO(HTL) approximate 3-loop results

3-loops: exact missing $m^4\alpha_S^2$ terms need extra nontrivial calculations, but $P^{3l}_{RGOHTL} \sim$
$RGOPT(P^{2l}_{HTLpt}) + m^4\alpha_S^2(C_{30}\ln^3\frac{\mu}{2\pi T} + C_{31}\ln^2\frac{\mu}{2\pi T} + C_{32}\ln\frac{\mu}{2\pi T} + C_{33})$:
leading logarithms (LL) and next-to-leading (NLL) C_{30}, C_{31} fully determined from lower orders from RG invariance
Within this LL, NLL approximation and in the $T/T_c \gtrsim 2$ range where more trustable:



We assume/expect unknown terms will not spoil this improved scale dependence.

Summary and Outlook

•OPT gives a simple procedure to resum perturbative expansions, using only perturbative information.

•Our RGOPT version includes 2 major differences w.r.t. previous OPT/SPT/HTLpt... approaches:

1) OPT +/or RG optimizations fix optimal $\bar{m}$ *and possibly* $\bar{g} = 4\pi\bar{\alpha}_S$

2) Maintaining RG invariance uniquely fixes the basic interpolation $m \to m(1-\delta)^{\gamma_0/b_0}$: discards spurious solutions and accelerates convergence.

• Applied to $T \neq 0$: exhibits improved stability + much improved scale dependence (with respect to standard PT, but also wrt SPT $\sim$ HTLpt)

•Paves the way to extend such RG-compatible methods to full QCD thermodynamics, (work in progress, starting with $T \neq 0$ pure gluodynamics) specially for exploring also finite density

Backup: RGOPT in Gross-Neveu model

•$D = 2$ $O(2N)$ GN model shares many properties with QCD (asymptotic freedom, (discrete) chiral sym., mass gap,..)

$$\mathcal{L}_{GN} = \bar{\Psi} i \not\partial \Psi + \frac{g_0}{2N} (\sum_1^N \bar{\Psi}\Psi)^2 \ (massless)$$

Standard mass-gap (massless, large N approx.):

work out $V_{eff}(\sigma \sim \langle \bar{\Psi}\Psi \rangle) \sim \frac{\sigma^2}{2g} + Tr \ln(i \not\partial - \sigma)$;

$\frac{\partial V_{eff}}{\partial \sigma} = 0$: $\quad \rightarrow \sigma \equiv M = \mu e^{-\frac{2\pi}{g}} \equiv \Lambda_{\overline{MS}}$

•Mass gap also known exactly for any N:

$$\frac{M_{exact}(N)}{\Lambda_{\overline{MS}}} = \frac{(4e)^{\frac{1}{2N-2}}}{\Gamma[1 - \frac{1}{2N-2}]}$$

(From $D = 2$ integrability: Bethe Ansatz) Forgacs et al '91

Massive (large N) GN model

$M(m,g) \equiv m(1+g\ln\frac{M}{\mu})^{-1}$: Resummed mass ($g/(2\pi) \to g$)
$= m(1 - g\ln\frac{m}{\mu} + g^2(\ln\frac{m}{\mu} + \ln^2\frac{m}{\mu}) + \cdots)$ (pert. re-expanded)

- Only fully resummed $M(m,g)$ gives right result, upon:

-identifying $\Lambda \equiv \mu e^{-1/g}$; $\to M(m,g) = \frac{m}{g\ln\frac{M}{\Lambda}} \equiv \frac{\hat{m}}{\ln\frac{M}{\Lambda}}$;
-taking *reciprocal*: $\hat{m} = M\ \ln\frac{M}{\Lambda} \to M(\hat{m} \to 0) \sim \frac{\hat{m}}{\hat{m}/\Lambda + \mathcal{O}(\hat{m}^2)} = \Lambda$

never seen in standard perturbation: $M_{pert}(m \to 0) \to 0$!

•Now (RG)OPT gives $M = \Lambda$ at *first* (and any) δ-order!
(at any order, OPT sol.: $\ln\frac{\bar{m}}{\mu} = -\frac{1}{\bar{g}}$, RG sol.: $\bar{g} = 1$)

•At δ^2-order (2-loop), RGOPT $\sim 1-2\%$ from $M_{exact}(\text{any}N)$

•Not specific to GN model: generalize to any model:
RG, OPT solutions at first (and all) orders:
$\ln\frac{\bar{m}}{\mu} = -\frac{\gamma_0}{2b_0}$; $\bar{g} = \frac{1}{\gamma_0}$ correctly resums pure RG LL, NLL,... (as far as b_0, γ_0 dependence concerned).