Discussion on Strong Field beyond Dipole Approximations

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Outline

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- My exact Volkov solution
- Dipole Approximation
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- My guess

- The exact non-dipole Volkov solution
- Gauge Invariance

The exact Volkov solution

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)=\frac{(\vec{p}-q\vec{A})^2}{2m}\Psi(\vec{r},t)$$

Use the light cone coordinate

$$\eta = t-\frac{z}{c} \qquad \zeta = t+\frac{z}{c}$$

The vector potential of this laser is

$$\vec{A}(\vec{r},t)=\vec{A}(\eta)=(A_x(\eta),A_y(\eta),0)$$

In these variables, the Schrödinger equation is

$$i\hbar\left(\frac{\partial\Psi(\vec{r},t)}{\partial\eta}+\frac{\partial\Psi(\vec{r},t)}{\partial\zeta}\right)=\frac{(\overrightarrow{p}-q\overrightarrow{A})^2}{2m}\Psi(\vec{r},t)$$

To find a solution

$$\Psi(\vec{r},t)=e^{-iEt+i\overrightarrow{p}\cdot\vec{r}}f(\eta)g(\zeta)$$

Put this solution into the Schrödinger equation leads to an equation of f(η) and g(ζ)

With the natural unit system

$$\hbar = c = 1$$

And, we set

$$m = e = 1$$

$$i(\frac{\partial f}{\partial \eta} g + f \frac{\partial g}{\partial \zeta}) = H_I(\eta) f g + i p_z (\frac{\partial f}{\partial \eta} g - f \frac{\partial g}{\partial \zeta}) - \frac{1}{2} (\frac{\partial^2 f}{\partial \eta^2} g + f \frac{\partial^2 g}{\partial \zeta^2}) + \frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \zeta}$$

Where we have used

$$H_I(\eta) = \frac{1}{2} A^2(\eta) + \overrightarrow{A}(\eta) \cdot \overrightarrow{P} \qquad E = \frac{1}{2} \overrightarrow{P}^2$$

A simpler guess

$$g(\zeta) = 1$$

Then the equation reduces to an equation of f(η)

$$\frac{1}{2}f''(\eta) + i(1-p_z)f'(\eta) - H_I(\eta)f(\eta) = 0$$

$$\alpha = (1 - \frac{p_z}{c}) = (1 - p_z)$$

Method 1: Set that

$$f(\eta) = G(\eta)f_o(\eta) \qquad f_o(\eta) = e^{-i\int^{\eta} d\xi \frac{H_I(\xi)}{\alpha}}$$

Then we will have an equation of G(η) and the solution is

$$^{++}G(\eta) = Te^{i\int^{\eta} d\xi\left[\frac{f_o^*(\xi)\frac{d^2}{d\xi^2}f_o(\xi)}{2\alpha}\right]}$$

For any general function F(ξ),

$$Te^{i\int^{\eta} d\xi F(\xi)} = 1 + i\int_{\xi_o}^{\eta} d\xi_1 F(\xi_1) + i^2\int_{\xi_o}^{\eta} d\xi_1 F(\xi_1)\left(\int_{\xi_o}^{\xi_1} d\xi_2 F(\xi_2)\right) + \ldots$$

++ : P. L. He, D. Lao, and Feng He,
Strong Field Theories beyond Dipole Approximations in Nonrelativistic Regimes,
PRL 118, 163203 (2017)

My Volkov solution

$$f(\eta) = e^{-\int^{\eta} d\xi i\alpha} F(\eta)$$

$$f'(\eta) = e^{-\int^{\eta} d\xi i\alpha} [F'(\eta) - i\alpha F(\eta)]$$

$$f''(\eta) = e^{-\int^{\eta} d\xi i\alpha} [F''(\eta) - 2i\alpha F'(\eta) - \alpha^2 F(\eta)]$$

Put these back into the equation

$$\frac{1}{2} f''(\eta) + i\alpha f'(\eta) - H_I(\eta) f(\eta) = 0$$

$$F''(\eta) + \left[\alpha^2 - 2H_I(\eta)\right] F(\eta) = 0$$

With

$$B = 2\alpha \qquad \xi(\eta) = -2H_I(\eta)$$

The equation becomes

$$F''(\eta) + \left[\frac{1}{4}B^2 + \xi(\eta)\right] F(\eta) = 0$$

$$F = F_o + F_1 + F_2 + \ldots$$

Define differential operator

$$L = \frac{\partial^2}{\partial \eta^2} + \frac{1}{4}B^2$$

The 0th order equation

$$F_o''(\eta)+\frac{1}{4}B^2F_o(\eta)=0 \quad \Rightarrow \quad LF_o(\eta)=0$$

The 1st order equation

$$F_1''(\eta)+\frac{1}{4}B^2F_1(\eta)=-\xi(\eta)F_o(\eta) \quad \Rightarrow \quad LF_1(\eta)=-\xi(\eta)F_o(\eta)$$

The 2nd order equation

$$F_2''(\eta)+\frac{1}{4}B^2F_2(\eta)=-\xi(\eta)F_1(\eta) \quad \Rightarrow \quad LF_2(\eta)=-\xi(\eta)F_1(\eta)$$

The nth order equation

$$LF_n(\eta)=-\xi(\eta)F_{n-1}(\eta)$$

$$\begin{aligned}F_n(\eta)&=-L^{-1}\xi(\eta)F_{n-1}(\eta)\\&=-L^{-1}\xi(\eta)[-L^{-1}\xi(\eta)F_{n-2}(\eta)]\end{aligned}$$

$$F_n(\eta)=(-1)^n[L^{-1}\xi(\eta)]^nF_o$$

$$F(\eta)=\sum_n^{\infty}f_n=\sum_n^{\infty}(-1)^n[L^{-1}\xi(\eta)]^nF_o$$

Green’ s function of operator L

The Green’ s of operator L satisfies

$$\left(\frac{d^2}{dt^2}+\omega_o^2\right)G(t-t')=\delta(t-t')$$

$$\delta(t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{i\omega(t-t')}$$

$$G(t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\tilde{G}(\omega)e^{i\omega(t-t')}$$

$$\tilde{G}(\omega)=-\frac{1}{\omega^2-\omega_o^2}$$

Then we can find the G by taking inverse Fourier transform

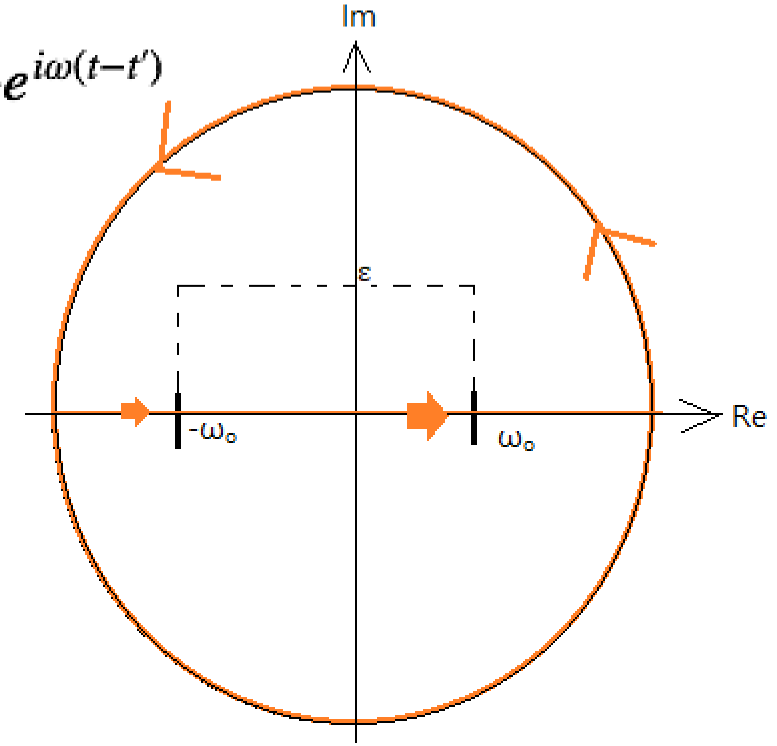
$$G(t-t') = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2-\omega_o^2}e^{i\omega(t-t')}$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(\omega+\omega_o)(\omega-\omega_o)}e^{i\omega(t-t')}$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{\epsilon\to 0}\frac{1}{(\omega+\omega_o-i\epsilon)(\omega-\omega_o-i\epsilon)}e^{i\omega(t-t')}$$

$$= \frac{\sin[\omega_o(t-t')]}{\omega_o}$$

$$= \frac{1}{\omega_o}\sin[\omega_o(t-t')]$$



$$G(t-t') = \Theta(t-t')\frac{1}{\omega_o}\sin[\omega_o(t-t')]$$

Then we can use this Green's function of L to calculate F_n, for example,

$$F_1 = -L^{-1}\xi F_o$$

$$= \int_{-\infty}^{\eta} d\eta_1 \frac{1}{\frac{1}{2}B} \sin\left[\frac{1}{2}B(\eta - \eta_1)\right](-2c^2 H_I)F_o$$

The equivalence between 2 solutions?

Interaction Picture

The time dependent Schrödinger equation is

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle = (H_o + H_I(t))|\Psi(t)\rangle$$

Introduce

$$|\Psi(t)\rangle = e^{\frac{-iH_0t}{\hbar}}|\tilde{\Psi}(t)\rangle$$

Then

$$i\hbar\frac{d}{dt}\left(e^{\frac{-iH_0t}{\hbar}}|\tilde{\Psi}(t)\rangle\right) = (H_o + H_I)\left(e^{\frac{-iH_0t}{\hbar}}|\tilde{\Psi}(t)\rangle\right)$$

$$i\hbar\frac{d}{dt}|\tilde{\Psi}(t)\rangle = V(t)|\tilde{\Psi}(t)\rangle$$

Where

$$V(t) = e^{\frac{-iH_0t}{\hbar}}H_I(t)e^{\frac{-iH_0t}{\hbar}}$$

Using the perturbation expansion,

$$|\tilde{\Psi}(t)\rangle = |\tilde{\Psi}(t)\rangle^{(0)} + |\tilde{\Psi}(t)\rangle^{(1)} + |\tilde{\Psi}(t)\rangle^{(2)} + \ldots$$

$$i\hbar\frac{d}{dt}|\tilde{\Psi}(t)\rangle^{(0)} = 0$$

$$|\tilde{\Psi}(t)\rangle^{(0)} = |\tilde{\Psi}(0)\rangle^{(0)} = |\tilde{\Psi}(0)\rangle$$

$$i\hbar\frac{d}{dt}|\tilde{\Psi}(t)\rangle^{(1)} = V(t)|\tilde{\Psi}(0)\rangle^{(0)}$$

$$|\tilde{\Psi}(t)\rangle^{(1)} = \frac{1}{i\hbar} = \int_0^{t'} dt' V(t')|\tilde{\Psi}(0)\rangle^{(0)}$$

Dipole Approximation

For the electromagnetic wave-hydrogen interacting Hamiltonian,

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} - \frac{e^2}{4\pi\epsilon_o r}$$
$$= \left[\frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_o r}\right] + \left[-q\frac{\vec{A}\cdot\vec{p}}{m} + q^2\frac{A^2}{2m}\right] = H_o + H_I(t)$$

For calculating the transition amplitude between state $|\tilde{\Psi}(t)>$ and any eigenstate of hydrogen atom |n> in interaction picture

$$\langle n|\Psi(t)\rangle = \langle n|e^{\frac{iH_o t}{\hbar}}|\tilde{\Psi}(t)\rangle = e^{\frac{iE_n t}{\hbar}}\left[\langle n|\tilde{\Psi}(0)\rangle^{(0)} + \langle n|\tilde{\Psi}(0)\rangle^{(1)}\right]$$

Consider the last term

$$\langle n|\tilde{\Psi}(0)\rangle^{(1)} = \frac{1}{i\hbar}\int_0^{t'} dt'\langle n|V(t')|\tilde{\Psi}(0)\rangle^{(0)}$$
$$= \frac{1}{i\hbar}\int_0^{t'} dt'\langle n|e^{\frac{iH_o t}{\hbar}}H_I(t)e^{\frac{-iH_o t}{\hbar}}|\tilde{\Psi}(0)\rangle^{(0)}$$

For the vector potential A is in the form of plane wave,

$$\vec{A} = \hat{\epsilon} A e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

For the wavelength of E.M. wave is much longer than the size of atom,

$$\frac{a_o}{\lambda} \ll 1 \qquad\qquad k \to 0$$

$$e^{i(\vec{k}\cdot\vec{r}-\omega t)} \to e^{-i\omega t}$$

One may show that

$$\hbar\omega = E_n - E_m$$

Consider the commutator

$$[x, H_o] = \left[x, \frac{p_x^2 + p_y^2 + p_x^2}{2m} - -\frac{e^2}{4\pi\epsilon_0 r}\right]$$

$$= \frac{1}{2m}[x, p_x^2] = \frac{1}{2m}([x, p_x]p_x + p_x[x, p_x])$$

$$= \frac{i\hbar}{m} p_x$$

Hence,

$$\overrightarrow{p} = \frac{m}{i\hbar}[\vec{r}, H_o]$$

Then,

$$\langle n | \overrightarrow{p} | m \rangle = \frac{m}{i\hbar}\langle n | [\vec{r}, H_o] | m \rangle$$

$$= m\frac{i(E_n - E_m)}{\hbar}\langle n | \vec{r} | m \rangle$$

And the matrix element of interacting hamiltonian is

$$\langle n|e\frac{\vec{A}\cdot\vec{p}}{m}|m\rangle = \frac{i(E_n - E_m)}{\hbar}eA\vec{\epsilon}\cdot\langle n|\vec{r}|m\rangle$$

$$= e\frac{i\omega}{\hbar}A\vec{\epsilon}\cdot\langle n|\vec{r}|m\rangle$$

$$= e\frac{\partial\vec{A}}{\partial t}\cdot\langle n|\vec{r}|m\rangle$$

$$= -\langle n|e\vec{r}\cdot\vec{E}|m\rangle$$

So, the approximation

$$e^{i(\vec{k}\cdot\vec{r}-\omega t)} \rightarrow e^{-i\omega t}$$

is called the dipole approximation.

Velocity gauge and Length gauge

Electric field and magnetic field are invariance under the gauge transformation.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}' \qquad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi = -\frac{\partial \vec{A}'}{\partial t} - \vec{\nabla}\phi'$$

Within dipole approximation, the Hamiltonian

$$H = \frac{(\vec{p} - q\vec{A}(t))^2}{2m} + V(\vec{r})$$

is called the Velocity gauge

$$\overrightarrow{A}' = \overrightarrow{A} + \overrightarrow{\nabla}\chi \qquad\qquad \phi' = \phi - \frac{\partial \mathscr{X}}{\partial t}$$

The length gauge is achieved by setting

$$\chi(\vec{r}, t) = -\vec{r} \cdot \overrightarrow{A}(t)$$

Then

$$\overrightarrow{A}' = \overrightarrow{A} + \overrightarrow{\nabla}\chi = \overrightarrow{A} - \overrightarrow{A} = 0$$

$$\phi' = \phi - \frac{\partial \mathscr{X}}{\partial t} = 0 + \frac{\partial}{\partial t}[\vec{r} \cdot \overrightarrow{A}(t)] = \vec{r} \cdot \frac{\partial \overrightarrow{A}}{\partial t}$$

Notice that the electric field

$$\overrightarrow{E} = -\frac{\partial \overrightarrow{A}}{\partial t}$$

$$\phi' = -\vec{r} \cdot \overrightarrow{E}$$

$$\phi' = -e\vec{r} \cdot \overrightarrow{E}$$

What's wrong with the gauge invariance?

Thanks for listening