

BEYOND THE STANDARD MODEL: EXERCISES

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Problem 1: Custodial symmetry

1. Show that the Lie algebras of the groups $\text{SO}(4)$ and $\text{SU}(2) \times \text{SU}(2)$ are isomorphic.

Hint: Show first that any unitary transformation $U_L V U_R^\dagger$ of the matrix $V = i v^a \sigma^a$ corresponds to a rotation of the four coefficients (v^a) . (Here U_L and U_R are any $\text{SU}(2)$ matrices, $\sigma^{1,2,3}$ are the Pauli matrices and $\sigma^4 = -i\mathbb{1}$.)

2. Being a complex doublet, the Standard Model Higgs field has four real degrees of freedom. Show that the Higgs potential

$$V(\phi) = \lambda \left(\phi^\dagger \phi - \frac{v^2}{2} \right)^2$$

is invariant under an $\text{SO}(4)$ symmetry which rotates them. Show that the action of this symmetry can also be expressed as the $\text{SU}(2)_L \times \text{SU}(2)_R$ transformation

$$\mathcal{H} \rightarrow U_L \mathcal{H} U_R^\dagger$$

where U_L and U_R are two $\text{SU}(2)$ matrices and where

$$\mathcal{H} = \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_+^* & \phi_0 \end{pmatrix}.$$

3. As you know from the Standard Model lectures, the Higgs field takes a vacuum expectation value

$$\langle \phi \rangle = \begin{pmatrix} \langle \phi_+ \rangle \\ \langle \phi_0 \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}.$$

What is the subgroup H_c of $\text{SU}(2)_L \times \text{SU}(2)_R$ which is left unbroken after spontaneous symmetry breaking? This H_c symmetry is called *custodial symmetry*.

4. We have introduced custodial symmetry as a symmetry of the Higgs potential only. Indeed it is not a symmetry of the entire Lagrangian. Can one assign a H_c representation to the would-be Goldstone bosons of electroweak symmetry breaking? What can be concluded from this for the gauge fields W_μ^i ? Which of the Standard Model couplings break H_c explicitly?
5. Recall from the Standard Model lectures that the Z boson is the linear combination

$$Z_\mu = -\sin \theta_w B_\mu + \cos \theta_w W_\mu^3.$$

We define the ρ parameter by

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_w}.$$

Argue that the Standard Model (tree-level) relation $\rho = 1$ can be understood as a consequence of custodial symmetry.

Remarks: In the literature one frequently encounters the equivalent T parameter

$$T = \frac{1}{\alpha_{\text{EM}}}(\rho - 1)$$

which is normalized to be 0 at the tree level in the Standard Model. Experimental bounds on the T parameter, as well as on similar parameters measuring deviations from Standard Model predictions for the electroweak sector, provide strong constraints on new physics; in this case, on any effects violating custodial symmetry.

Solution

1. Since $\det U_L = \det U_R = 1$ one has $\det V' = \det(U_L V U_R^\dagger) = \det V = \sum_a (i v^a)^2 \det \sigma^a = \sum_a (v^a)^2$, hence $|v|^2 = |v'|^2$ and so we have a map $\text{SU}(2) \times \text{SU}(2) \rightarrow \text{SO}(4)$. This is a group homomorphism, but not an isomorphism since its kernel $\pm \text{Id}$ is nontrivial — in fact, on the group level, one has $\text{SO}(4) \simeq (\text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2$. Here we will only show that the Lie algebras are isomorphic, by demonstrating that this map has full rank when linearized about the identity. To this end we explicitly construct the rotation matrix O^{ab} corresponding to (U_L, U_R^\dagger) :

$$\begin{aligned} i \sigma^a v'^a &= i U_L \sigma^a U_R^\dagger v^a \\ \Rightarrow \text{tr} \left(U_L \sigma^a U_R^\dagger \sigma^{\dagger b} \right) v^a &= \text{tr} \left(\sigma^a \sigma^{\dagger b} \right) v'^a = 2 v'^b \\ \Rightarrow O^{ab} &= \frac{1}{2} \text{tr} \left(\sigma^{\dagger a} U_L \sigma^b U_R^\dagger \right). \end{aligned}$$

Linearizing $U_L = \mathbb{1} + \frac{i}{2} \sigma^i \alpha^i + \dots$, $U_R^\dagger = \mathbb{1} - \frac{i}{2} \sigma^j \beta^j + \dots$ (where (α^i) and (β^j) contain the six real $\text{SU}(2) \times \text{SU}(2)$ parameters for $i, j = 1, 2, 3$) gives

$$O^{ab} = \delta^{ab} + \frac{i}{4} \text{tr} \left(\sigma^{\dagger a} \sigma^i \sigma^b \right) \alpha^i - \frac{i}{4} \text{tr} \left(\sigma^{\dagger a} \sigma^b \sigma^j \right) \beta^j + \dots$$

To explicitly find the $\text{SO}(4)$ generators one can use the following identities (which are easily checked when taking into account that the Pauli matrices satisfy the $\text{SU}(2)$ and Clifford algebras):

$$\begin{aligned} \text{tr} \left(\sigma^4 \sigma^i \sigma^j \right) &= -2i \delta^{ij} \\ \text{tr} \left(\sigma^{\dagger 4} \sigma^i \sigma^j \right) &= 2i \delta^{ij} \\ \text{tr} \left(\sigma^{\dagger 4} \sigma^i \sigma^4 \right) &= \text{tr} \left(\sigma^{\dagger 4} \sigma^4 \sigma^i \right) = \text{tr} \sigma^i = 0 \\ \text{tr} \left(\sigma^i \sigma^j \sigma^k \right) &= 2i \epsilon^{ijk}. \end{aligned}$$

This gives e.g. for $\vec{\alpha} = \vec{\beta} = (1, 0, 0)$

$$\frac{1}{4} \text{tr} \left(\sigma^{\dagger a} \sigma^i \sigma^b \right) \alpha^i - \frac{1}{4} \text{tr} \left(\sigma^{\dagger a} \sigma^b \sigma^j \right) \beta^j = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

or for $\vec{\alpha} = -\vec{\beta} = (1, 0, 0)$

$$\frac{1}{4} \text{tr} \left(\sigma^{\dagger a} \sigma^i \sigma^b \right) \alpha^i - \frac{1}{4} \text{tr} \left(\sigma^{\dagger a} \sigma^b \sigma^j \right) \beta^j = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is readily verified that other combinations of basis vectors for $\vec{\alpha}$ and $\vec{\beta}$ lead to the four remaining SO(4) generators.

2. With $\phi = (\phi_+, \phi_0) = (\phi^1 + i\phi^2, -\phi^3 + i\phi^4)$, SO(4) and SU(2) × SU(2) invariance become manifest in writing

$$\phi^\dagger \phi = \sum_a (\phi^a)^2 = \det \mathcal{H}$$

and one checks that $\mathcal{H} = i\sigma^a \phi^a$ as in part 1.

3. One has

$$\langle \mathcal{H} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$$

which is left invariant under $\langle \mathcal{H} \rangle \rightarrow U_L \langle \mathcal{H} \rangle U_R^\dagger$ iff $U_L = U_R$. Hence the unbroken subgroup is the diagonal subgroup $H_c = \text{SU}(2)_{\text{diag}}$.

4. The would-be Goldstone bosons $\phi_{1,2,4}$ form a triplet under H_c , and the W_μ^i also form a triplet. Custodial symmetry is explicitly broken by the hypercharge gauge coupling and Yukawa couplings.
5. The fact that (W_μ^a) forms a triplet implies that the mass terms for all its components are the same,

$$\mathcal{L} = -\frac{1}{2} m_W^2 ((W_\mu^1)^2 + (W_\mu^2)^2 + (W_\mu^3)^2) + \dots$$

Given the Z boson as a linear combination of W_μ^3 and B_μ (the orthogonal combination forming the massless photon) it follows that $m_Z^2 \cos^2 \theta_w = m_W^2$.

Problem 2: The SMEFT

1. Show that the only dimension-5 operator in the Standard Model EFT is the Weinberg operator

$$\mathcal{L}_5 = \frac{\kappa_{ij}}{\Lambda} \phi \phi \ell_i \ell_j + \text{h.c.}$$

Here κ_{ij} are some dimensionless coefficients, Λ is a scale, ϕ is the Standard Model Higgs doublet and ℓ_i ($i = 1, 2, 3$) are the lepton doublets.

2. Show that the dimension-6 operator

$$\mathcal{L}_6 \supset \frac{\zeta}{\Lambda^2} |\phi^\dagger D_\mu \phi|^2$$

violates the custodial symmetry of Exercise 1. Calculate the correction to the ρ parameter induced by $\zeta \neq 0$.

Solution

1. The building blocks for any candidate operator are the SM fermion fields, the Higgs, the covariant derivative, the field strengths and their duals. By dimensional analysis and Lorentz invariance, any candidate operator can contain either no fermions or two fermions.

- No fermions: All objects constructed out of field strengths, dual field strengths and D_μ s with all Lorentz indices contracted have even dimensions. Therefore all candidate dimension-5 operator must contain an odd total power of ϕ and ϕ^\dagger , and thus cannot be gauge invariant (seen most easily by noting that the hypercharges cannot add up to 0).
- Two fermions
 - and a field strength, a dual field strength or two covariant derivatives: by Lorentz invariance this would need to involve a bilinear of either left-handed or right-handed spinors. None of these is neutral under hypercharge, which would again be required to form a gauge-invariant object.
 - and a covariant derivative and a Higgs: note that by Lorentz invariance this needs to involve a left-handed and a right-handed fermion whose hypercharges need to sum up to $\pm\frac{1}{2}$. No such bilinear exists in the Standard Model.
 - and two Higgs fields: we need two fermions of the same chirality whose hypercharges should sum up to ± 1 or 0. The only possibility is the Weinberg operator, which is also invariant under $SU(3)_c \times SU(2)_L$.

2. Replacing $\phi \rightarrow \langle \phi \rangle$ gives

$$\begin{aligned}
 \mathcal{L}_6 &\supset \frac{\zeta}{\Lambda^2} \left| \begin{pmatrix} 0 & \frac{v}{\sqrt{2}} \end{pmatrix} \left(\frac{1}{2} g' B_\mu + \frac{1}{2} g \sum_i \sigma^i W_\mu^i \right) \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \right|^2 \\
 &= \frac{\zeta}{8\Lambda^2} v^4 |g' B_\mu - g W_\mu^3|^2 \\
 &= \frac{\zeta}{8\Lambda^2} v^4 (g'^2 + g^2) Z_\mu^2 \\
 &= \frac{1}{2} \frac{\zeta v^2}{\Lambda^2} (m_Z^{(\text{SM})})^2 Z_\mu^2
 \end{aligned}$$

hence this operator gives a contribution to the Z mass without affecting the W mass, which shows that it breaks the custodial symmetry. The resulting contribution to the ρ parameter is

$$\Delta\rho = -\frac{\zeta v^2}{\Lambda^2}$$

Problem 3: Supersymmetry

1. Deduce from the supersymmetry algebra

$$\{Q_\alpha, Q_{\dot{\beta}}^\dagger\} = 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu, \quad [P_\mu, Q_\alpha] = [P_\mu, Q_{\dot{\beta}}^\dagger] = \{Q_\alpha, Q_\beta\} = \{Q_{\dot{\alpha}}^\dagger, Q_{\dot{\beta}}^\dagger\} = 0$$

that all the masses of the states forming a supermultiplet are the same, up to supersymmetry-breaking terms.

2. Supersymmetry allows for Yukawa terms (scalar-fermion-fermion interactions) of the type

$$\mathcal{L}_{\text{Yuk}} = \varphi \psi \psi' + \text{h.c.}$$

where ψ and ψ' are left-handed spinors and φ is any complex scalar field forming a chiral supermultiplet together with a left-handed spinor. It does *not* allow for Yukawa terms of the type $\varphi^*\psi\psi' + \text{h.c.}$ with φ^* the superpartner of a right-handed spinor.

Find all Yukawa couplings between the MSSM Higgs and matter fields and their superpartners which are allowed by gauge invariance and supersymmetry. Give an argument why a supersymmetric extension of the Standard Model needs at least two Higgs doublets.

3. Among the MSSM Yukawa terms which you found in the previous exercise, identify the subset containing an even number of superpartners. These Yukawa terms are therefore allowed by a discrete R parity symmetry which acts as (SM field) \rightarrow (SM field) and (superpartner) \rightarrow ($-$ superpartner). Show that the terms allowed by R parity are precisely those which respect baryon and lepton number conservation.

Solution

1. Given an eigenstate of P^2 with eigenvalue m^2 , any state obtained by acting with Q or Q^\dagger is also an eigenstate with the same eigenvalue. This follows immediately from the fact that P_μ commutes with Q and Q^\dagger .
2. The only gauge invariant combinations are terms of the form

$$\begin{aligned}\mathcal{L}_{\text{Yuk}} = & \phi_u q \bar{u} + \phi_d q \bar{d} + \phi_d \ell \bar{e} + \text{h.c.} \\ & + \tilde{\ell} \bar{\ell} + \tilde{\ell} q \bar{d} + \tilde{u} \bar{d} + \text{h.c.}\end{aligned}$$

(suppressing indices and couplings; note that the last term has the color indices contracted with an ϵ symbol) as well as terms trivially related to these by replacing the scalar field with its fermionic superpartner and one of the fermions with its scalar superpartner. One needs at least two Higgs doublets since with only ϕ_u or ϕ_d one cannot give masses to both up-type and down-type quarks.

3. All terms in the first line (as well as the terms related to them as described above) are even under R -parity. One may assign lepton number $L = +1$ to ℓ and its superpartner, and $L = -1$ to \bar{e} and its superpartner, and similarly baryon number $B = \pm \frac{1}{3}$ to (anti)quark superfields. Then demanding that each term be neutral under B and L is equivalent to demanding each term to be R -parity even.

Problem 4: The abelian pseudo-Goldstone Higgs

The subject of this exercise is a toy model which is at least qualitatively similar to the minimal composite Higgs model but less cumbersome. We take $G/H = \text{SO}(3)/\text{SO}(2)$ rather than $\text{SO}(5)/\text{SO}(4)$; this means of course that the resulting pseudo-Goldstone Higgs field is at best associated with a $\text{U}(1)$ gauge group rather than with the full $\text{SU}(2)_L \times \text{U}(1)_Y$ of the Standard Model. We also use a perturbative sigma model potential rather than strong gauge dynamics to model spontaneous symmetry breaking. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^A \partial^\mu \Phi^A - \frac{\kappa^2}{8} (\Phi^A \Phi^A - f^2)^2$$

where Φ^A ($A = 1, 2, 3$) are scalar fields.

1. Give explicit matrix expressions for a basis $\{T^A\}$ of hermitian $\text{SO}(3)$ generators, chosen such that the generators $T^{1,2,3}$ generate rotations around the (x^1, x^2, x^3) axes respectively, and normalized such as to satisfy $\text{tr } T^A T^B = 2\delta^{AB}$.

The potential is evidently minimized for any Φ^A satisfying $\Phi^A \Phi^A = f^2$. Without loss of generality we can choose

$$\langle \vec{\Phi} \rangle = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}$$

in the above basis. The fluctuations around this vacuum are conveniently parameterized by the field redefinition

$$\vec{\Phi}(x) = U(\Pi_1(x), \Pi_2(x)) \begin{pmatrix} 0 \\ 0 \\ f + \sigma(x) \end{pmatrix}, \quad U(\Pi_1(x), \Pi_2(x)) = \exp \left(i \frac{\Pi_a(x) T^a}{f} \right).$$

Here $a = 1, 2$ and Π_1 and Π_2 are the Goldstone bosons corresponding to fluctuations along the $\text{SO}(3)/\text{SO}(2) \simeq S^2$ manifold, while σ is the orthogonal mode.

2. Show that

$$\vec{\Phi} = (f + \sigma) \begin{pmatrix} \sin \left(\frac{\Pi}{f} \right) \frac{\Pi_2}{\Pi} \\ \sin \left(\frac{\Pi}{f} \right) \frac{\Pi_1}{\Pi} \\ \cos \left(\frac{\Pi}{f} \right) \end{pmatrix}.$$

where $\Pi(x) \equiv \sqrt{\Pi_a(x) \Pi_a(x)}$.

3. Obtain the Lagrangian in terms of the fields σ and Π_a . Rewrite it in terms of σ and the complex field $\phi = \frac{\Pi_1 - i\Pi_2}{\sqrt{2}}$.
4. The Lagrangian which you have obtained should make the unbroken $\text{U}(1) \simeq \text{SO}(2)$ symmetry manifest. Gauge this symmetry by replacing derivatives of ϕ by covariant derivatives. This constitutes an explicit breaking of G , which will induce a potential for ϕ through loop corrections and eventually contribute to ϕ taking a vacuum expectation value $\langle \phi \rangle = \frac{v}{\sqrt{2}}$. Taking v as given, calculate the relation between v , the $\text{U}(1)$ gauge coupling e , and the mass of the $\text{U}(1)$ gauge boson. Moreover, calculate the couplings between the physical Higgs boson $h(x)$ (defined by $\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))$ in unitary gauge) and the gauge field. Compare your results with an ordinary abelian Higgs model.

Solution

1. The generators in this basis are

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. By explicit calculation one shows that

$$\begin{aligned}
i\Pi_a T^a &= \begin{pmatrix} 0 & 0 & \Pi_2 \\ 0 & 0 & \Pi_1 \\ -\Pi_2 & -\Pi_1 & 0 \end{pmatrix} \\
(i\Pi_a T^a)^2 &= - \begin{pmatrix} \Pi_2^2 & \Pi_1 \Pi_2 & 0 \\ \Pi_1 \Pi_2 & \Pi_1^2 & 0 \\ 0 & 0 & \Pi^2 \end{pmatrix} \\
(i\Pi_a T^a)^{2n} &= (-)^n \Pi^{2n} \begin{pmatrix} \frac{\Pi_2^2}{\Pi^2} & \frac{\Pi_1 \Pi_2}{\Pi^2} & 0 \\ \frac{\Pi_1 \Pi_2}{\Pi^2} & \frac{\Pi_1^2}{\Pi^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{by induction}) \\
(i\Pi_a T^a)^{2n+1} &= (-)^n \Pi^{2n+1} \begin{pmatrix} 0 & 0 & \frac{\Pi_2}{\Pi} \\ 0 & 0 & \frac{\Pi_1}{\Pi} \\ -\frac{\Pi_2}{\Pi} & -\frac{\Pi_1}{\Pi} & 0 \end{pmatrix}.
\end{aligned}$$

By splitting the exponential series in the definition of U into its even and odd parts

$$U(\vec{\Pi}) = \exp\left(i \frac{\Pi_a T^a}{f}\right) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(i \frac{\Pi_a T^a}{f}\right)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(i \frac{\Pi_a T^a}{f}\right)^{2n+1}$$

and inserting the results for $(i\Pi_a T^a)^{2n}$ and $(i\Pi_a T^a)^{2n+1}$ one obtains the desired result.

3. Writing $\Phi^A(x) = (f + \sigma(x)) X^A(x)$ one has

$$\vec{X}^2 = \sin^2 \frac{\Pi}{f} \underbrace{\left(\frac{\Pi_2^2}{\Pi^2} + \frac{\Pi_1^2}{\Pi^2} \right)}_{=1} + \cos^2 \frac{\Pi}{f} = 1$$

and in particular

$$0 = \frac{1}{2} \partial_\mu (\vec{X}^2) = \vec{X} \cdot \partial_\mu \vec{X}.$$

Noting that $\vec{\Phi}^2 = (f + \sigma)^2 \vec{X}^2 = (f + \sigma)^2$ the potential becomes

$$\frac{\kappa^2}{8} (\vec{\Phi}^2 - f^2) = (f + \sigma)^2 - f^2 = \frac{\kappa^2}{8} (\sigma^2 + 2\sigma f).$$

The kinetic term is

$$\begin{aligned}
\frac{1}{2} \partial_\mu \Phi^A \partial^\mu \Phi^A &= \frac{1}{2} \left((\partial_\mu \sigma) \vec{X} + (f + \sigma) \partial_\mu \vec{X} \right)^2 \\
&= \frac{1}{2} \left((\partial_\mu \sigma)^2 + (f + \sigma)^2 (\partial_\mu \vec{X})^2 \right).
\end{aligned}$$

To simplify this further, note that

$$\partial_\mu \vec{X} = \begin{pmatrix} c_\theta \frac{\partial_\mu \Pi}{f} \frac{\Pi_2}{\Pi} + s_\theta \frac{\partial_\mu \Pi_2}{\Pi} - s_\theta \frac{\Pi_2 \partial_\mu \Pi}{\Pi^2} \\ c_\theta \frac{\partial_\mu \Pi}{f} \frac{\Pi_1}{\Pi} + s_\theta \frac{\partial_\mu \Pi_1}{\Pi} - s_\theta \frac{\Pi_1 \partial_\mu \Pi}{\Pi^2} \\ s_\theta \frac{\partial_\mu \Pi}{f} \end{pmatrix}$$

where $s_\theta \equiv \sin \theta$ and $c_\theta \equiv \cos \theta$ with $\theta \equiv \frac{\Pi}{f}$. By substitution one obtains

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma \\ & + \frac{1}{2} (f + \sigma)^2 \left(\sin^2 \left(\frac{\Pi}{f} \right) \frac{(\partial_\mu \Pi_1)^2 + (\partial_\mu \Pi_2)^2}{\Pi^2} + \frac{1}{4 \Pi^4} \left(\frac{\Pi^2}{f^2} - \sin^2 \frac{\Pi}{f} \right) (\partial_\mu \Pi^2)^2 \right) \\ & - \frac{\kappa^2}{8} (\sigma^2 + 2\sigma f)^2.\end{aligned}$$

We finally substitute $\phi = \frac{\Pi_1 + i\Pi_2}{\sqrt{2}}$ which yields

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma \\ & + \frac{1}{2} (f + \sigma)^2 \left(\sin^2 \left(\frac{\sqrt{2}|\phi|}{f} \right) \frac{|\partial_\mu \phi|^2}{|\phi|^2} + \frac{1}{4|\phi|^4} \left(\frac{2|\phi|^2}{f^2} - \sin^2 \frac{\sqrt{2}|\phi|}{f} \right) (\partial_\mu |\phi|^2)^2 \right) \\ & - \frac{\kappa^2}{8} (\sigma^2 + 2\sigma f)^2.\end{aligned}$$

Note that, despite its appearance at first sight, the Lagrangian is not singular at $|\phi| = 0$. As expected the potential does not depend on the Goldstone modes, whose interactions are instead governed by an infinite number of derivative terms.

4. We replace $\partial_\mu \phi \rightarrow (\partial_\mu - ieA_\mu)\phi$ and $\phi = \frac{1}{\sqrt{2}}(v + h(x))$. The photon mass term is $\frac{1}{2}m_A^2 A_\mu A^\mu$ where

$$m_A^2 = e^2 f^2 \sin^2 \frac{v}{f} = e^2 v^2 \left(1 - \frac{1}{3} \left(\frac{v}{f} \right)^2 + \dots \right)$$

and the Higgs-photon-photon and Higgs-Higgs-photon-photon vertices are $g_{hAA} h A_\mu A^\mu + \frac{1}{2} g_{hhAA} h h A_\mu A^\mu$ where

$$g_{hAA} = e^2 v \left(1 - \frac{2}{3} \left(\frac{v^2}{f} \right) + \dots \right), \quad g_{hhAA} = e^2 \left(1 - 2 \left(\frac{v}{f} \right)^2 + \dots \right)$$

The respective leading terms in these expansions correspond to the abelian Higgs model. The corrections are controlled by the parameter v/f . The situation is similar in the minimal composite Higgs model, where the same parameter controls the deviations of the Higgs couplings from the Standard Model.

Problem 5: Minimal non-supersymmetric SU(5) grand unification

Let Σ be a scalar field transforming in the 24-dimensional adjoint representation of SU(5), written as a traceless Hermitian 5×5 matrix. Assuming a parity symmetry for simplicity, one may write down the following potential for Σ :

$$\mathcal{V} = -m^2 \text{tr} \Sigma \Sigma + \frac{\alpha}{4} (\text{tr} \Sigma \Sigma)^2 + \frac{\beta}{4} \text{tr} \Sigma \Sigma \Sigma \Sigma.$$

1. Verify that this potential is invariant under SU(5) gauge transformations acting as $\Sigma \rightarrow U \Sigma U^\dagger$.

2. Find the extrema of the potential assuming that $\alpha \geq 0$ and $\beta \geq 0$. Find a condition on α and β such that the ground state of the theory breaks $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$ rather than $SU(5) \rightarrow SU(4) \times U(1)$.
Hint: It is convenient to assume that Σ is diagonal (which can always be achieved by a gauge transformation).
3. Assuming that the SM Higgs doublet is embedded in a 5-dimensional fundamental representation Φ , write down the terms involving Σ and Φ which can contribute to the SM Higgs mass term after GUT symmetry breaking.

Solution

1. This follows immediately from the cyclic invariance of the trace and $U^\dagger U = \mathbb{1}$.
2. We write $\Sigma = v \text{diag}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$ where the σ_i are real numbers satisfying the constraint $\sum_i \sigma_i = 0$. We therefore need to solve

$$\frac{d}{d\sigma_i} \hat{\mathcal{V}} = \frac{d}{d\lambda} \hat{\mathcal{V}} = 0$$

where $\hat{\mathcal{V}}$ is the potential with a Lagrange multiplier term added,

$$\hat{\mathcal{V}} = -\frac{m^2}{2} \sum_i \sigma_i^2 + \frac{\alpha}{4} \left(\sum_i \sigma_i^2 \right)^2 + \frac{\beta}{4} \sum_i \sigma_i^4 + \lambda \sum_i \sigma_i.$$

This implies that

$$-m^2 \sigma_i + \alpha \sigma_i \sum_j \sigma_j^2 + \beta \sigma_i^3 + \lambda = 0.$$

Taking the trace of this equation and using $\sum_i \sigma_i = 0$ yields that

$$\lambda = -\frac{\beta}{5} \sum_i \sigma_i^3,$$

hence

$$-m^2 \sigma_i + \alpha \sigma_i \sum_j \sigma_j^2 + \beta \sigma_i^3 - \frac{\beta}{5} \sum_j \sigma_j^3 = 0.$$

For each σ_i this is a cubic equation with at most three real roots. The only solution where all of the σ_i are given by the same root is that of unbroken $SU(5)$ symmetry, $\sigma_i = 0 \quad \forall i$. We first discuss the $SU(5) \rightarrow SU(4) \times U(1)$ and $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$ solutions in more detail, where only two distinct roots appear.

- (a) Three of the σ_i can take one value and the other two another. Tracelessness then implies that we can set

$$\langle \Sigma \rangle = v_3 \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

and therefore

$$\sum_i \sigma_i^2 = 30 v_3^2, \quad \sum_i \sigma_i^3 = -30 v_3^3$$

Replacing either $\sigma_i \rightarrow 2 v_3$ or $\sigma_i \rightarrow -3 v_3$ yields

$$v_3^2 = \frac{m^2}{30\alpha + 7\beta}.$$

The potential at this point is

$$\mathcal{V} = -\frac{15 m^4}{60\alpha + 14\beta}.$$

- (b) Four of the σ_i can take one value and the remaining σ_i another. By tracelessness

$$\langle \Sigma \rangle = v_4 \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix}$$

which implies

$$\sum_i \sigma_i^2 = 20 v_4^2, \quad \sum_i \sigma_i^3 = -60 v_4^3$$

Both $\sigma_i = v_4$ and $\sigma_i = -4 v_4$ then lead to

$$v_4^2 = \frac{m^2}{20\alpha + 13\beta}.$$

The potential at this point is

$$\mathcal{V} = -\frac{15 m^4}{60\alpha + 39\beta}.$$

In conclusion, if β is strictly positive then the $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$ breaking minimum will be energetically preferred, independently of the value of α . If $\beta = 0$ then both minima are degenerate.

Other critical points can have three different σ_i . They can be analyzed similarly but are more cumbersome to deal with.

3. The terms contributing to the Higgs mass at the tree level are

$$\mathcal{L} \supset -\frac{\gamma}{2} \Phi^\dagger \Sigma^2 \Phi - \frac{1}{2} m^2 \Phi^\dagger \Phi.$$

As in the supersymmetric theory discussed in the lectures, a severe fine-tuning of parameters is needed to make sure that the doublet mass is of the order of the electroweak scale (another manifestation of the hierarchy problem).