Introduction to QCD and Loop Calculations

24th Vietnam School of Physics, Quy Nhon, August 2018

Exercise 1: Gauge transformations

Consider the Lagrangian for the quark fields

$$\mathcal{L}_q(q_f,\, m_f) = \sum_{j,k=1}^{N_c} \bar{q}_f^{\,j}(x)\; (i\,\gamma_\mu \boldsymbol{D}^\mu[\boldsymbol{A}] - m_f)_{jk}\; q_f^k(x)\;, \tag{1}$$

where the covariant derivative $\boldsymbol{D}^\mu$ is given by

$$\boldsymbol{D}^\mu[A] = \partial^\mu + i\, g_s\, \boldsymbol{A}_\mu\;, \tag{2}$$

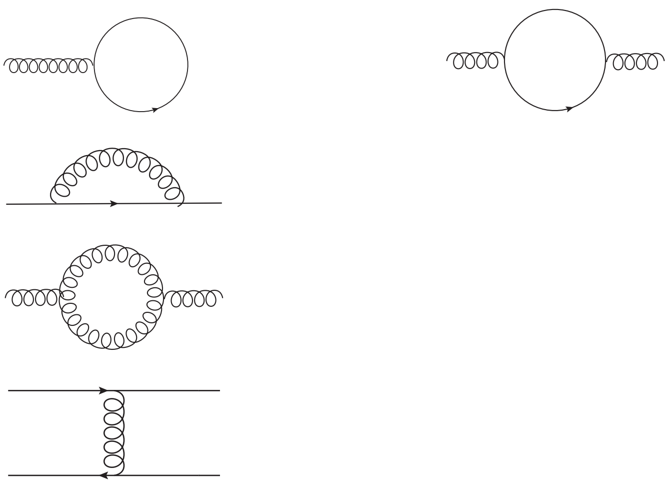
with $\boldsymbol{A}^\mu = t^a\, A_a^\mu\,,\;\; a = 1\,\ldots\, N_c^2-1$.
The quark and gluon fields transform under local gauge transformations $U(x)$ as

$$\begin{aligned} q'_j(x) &= U_{jk}\, q^k(x) \\ \boldsymbol{A}'_\mu(x) &= U(x)\boldsymbol{A}_\mu(x)U^{-1}(x) + \frac{i}{g_s}\big(\partial_\mu U(x)\big)U^{-1}(x)\;. \end{aligned} \tag{3}$$

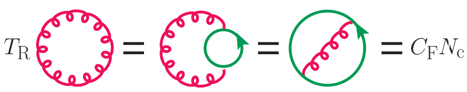
Show explicitly that $\boldsymbol{D}^\mu[\boldsymbol{A}']\, q'(x) = U(x)\Big(\boldsymbol{D}^\mu[\boldsymbol{A}]\, q(x)\Big)$ and therefore the Lagrangian $\mathcal{L}_q$ in Eq. (1) is invariant under local gauge transformations.

Exercise 2: Colour factors

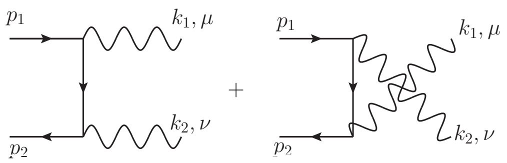
(a) Calculate the colour factors for the following Feynman diagrams:



(b) Derive the explicit value for C_F by considering pictorial identities like

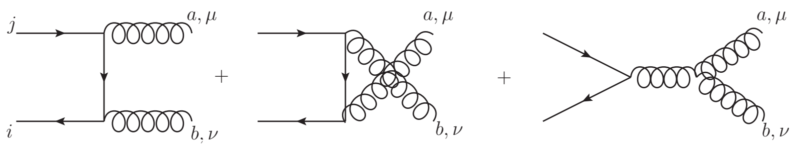


Exercise 3a: QED Ward Identity



Calculate the tree level amplitude $\mathcal{M}_{\mu\nu}$ for $e^+e^- \to \gamma\gamma$ (assuming massless electrons) and show explicitly that $k_1^\mu M_{\mu\nu} = 0, k_2^\mu M_{\mu\nu} = 0$.

Exercise 3b: Gauge invariance of QCD amplitudes



Calculate the tree level amplitude $\mathcal{M}_{\mu\nu}^{\mathrm{QCD}}$ for $q\,\bar{q} \to g\,g$ (with massless quarks) and show explicitly that $k_1^\mu M_{\mu\nu}^{\mathrm{QCD}}$ only vanishes if k_2 is the momentum of a physical gluon.

Exercise 4: One-loop tensor Integrals

In the representation of tensor integrals, for tensor ranks $r \geq 2$, higher dimensional integrals I_N^{D+2m} arise as coefficients of metric tensors $(g^{\mu\nu})^{\otimes m}$. To see how these integrals arise, start from the representation in terms of Feynman parameters and quadratic forms in the loop momentum, but this time with loop momenta in the numerator:

$$L_N^{\mu_1\mu_2} \quad = \quad \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty \frac{d^D l}{i\pi^{\frac{D}{2}}}\, l^{\mu_1} l^{\mu_2} \left[l^2 - R^2 + i\delta\right]^{-N} .$$

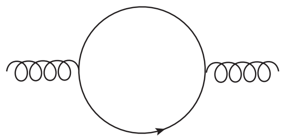
Then make the ansatz

$$L_N^{\mu_1\mu_2} = K\, g^{\mu_1\mu_2}$$

and determine K in terms of I_N^{D+2}. This can be done using the general D-dimensional formula for scalar integrals:

$$\begin{aligned} I_N^D &= \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty d\bar{l}\, \left[l^2 - R^2\right]^{-N} \\ &= (-1)^N \Gamma(N - \frac{D}{2}) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l)\ \left[R^2\right]^{D/2-N} . \end{aligned} \tag{4}$$

Exercise 5: One-loop renormalisation



Calculate the fermion loop contribution to the gluon selfenergy (shown above) and then extract the fermionic contribution to the renormalisation constant Z_3, $Z_3^{(F)}$.

Useful formulas for Traces, 1-point and 2-point integrals:

$$\begin{aligned}
\mathrm{Trace}[\gamma_\mu \not{p}\, \gamma_\nu \not{k}\,] &= 4\,(k_\mu p_\nu + k_\nu p_\mu - g_{\mu\nu} k\cdot p)\ ,\ \mathrm{Trace}[\gamma_\mu\gamma_\nu\,] = 4\, g_{\mu\nu} \\
I_1^D(m) &= \frac{1}{\epsilon}\, m^2 + \mathcal{O}(\epsilon^0) \\
I_2^D(p^2, m, m) &= \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \\
I_2^{D,\mu}(p^2, m, m) &= -\frac{1}{2}\, p^\mu\, I_2^D(p^2, m, m) = -\frac{1}{2}\, p^\mu\, \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \\
I_2^{D,\mu\nu}(p^2, m, m) &= g^{\mu\nu}\, \frac{1}{\epsilon}\left(-\frac{1}{12}\, p^2 + \frac{1}{2}\, m^2\right) + p^\mu p^\nu\, \frac{1}{\epsilon}\, \frac{1}{3} + \mathcal{O}(\epsilon^0)\ . \qquad (5)
\end{aligned}$$