

# UNITARITY AND DISCRETE SCALE INVARIANCE

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# Outline

- What is essential?
- Pionless EFT with  
S. König,  
H.W. Grießhammer  
& H.-W. Hammer
- Unitarity: light nuclei
- Unitarity: bosonic clusters  
S. Gandolfi,  
J. Carlson,  
& S.A. Vitiello
- Unitarity: matter
- Conclusion plus  
L. Contessi,  
A. Lovato,  
F. Pederiva,  
A. Roggero  
& J. Kirscher

B. Bazak  
& M. Eliyahu



# What is essential in nuclear physics?

Traditional approach

- 1) Describe NN precisely to some high energy
- 2) Append 3N forces as needed



Lost in details:

e.g., NLO, N<sup>2</sup>LO, N<sup>3</sup>LO, N<sup>4</sup>LO, ...

Here

- 1) Describe NN and 3N approximately
- 2) Treat everything else in perturbation theory  
cf. atomic systems in QED

“simplicity emerging from complexity”

Expansion around unitarity

König, Grießhammer,  
Hammer + v.K. '15 '16  
König '16  
vK '17

$$A = 2$$

$$T_2(k \ll R^{-1}) = \frac{4\pi}{m} \left( a_2^{-1} + ik - \frac{r_2}{2} k^2 + \frac{P_2}{4} k^4 + \dots \right)^{-1} + (l > 0)$$

scattering length      effective range      shape parameter  
 { }  
 unitarity limit       $a_2^{-1} \approx \sqrt{mB_2} \rightarrow 0$        $\frac{r_2}{2} \sim R, \frac{P_2}{4} \sim R^3, \dots$  typically

$$T_2(|a_2^{-1}| \ll k \ll R^{-1}) = \frac{4\pi}{m} (ik)^{-1} \left( 1 + \mathcal{O}\left(\frac{1}{ka_2}, kR\right) \right)$$

unitarity window

no parameter!

"universality"

$$\left| a_{2,I=1,I_3=0} m_\pi \right|^{-1} \simeq 0.06$$

nucleons  
 $R^{-1} \sim m_\pi$

$^1S_0$

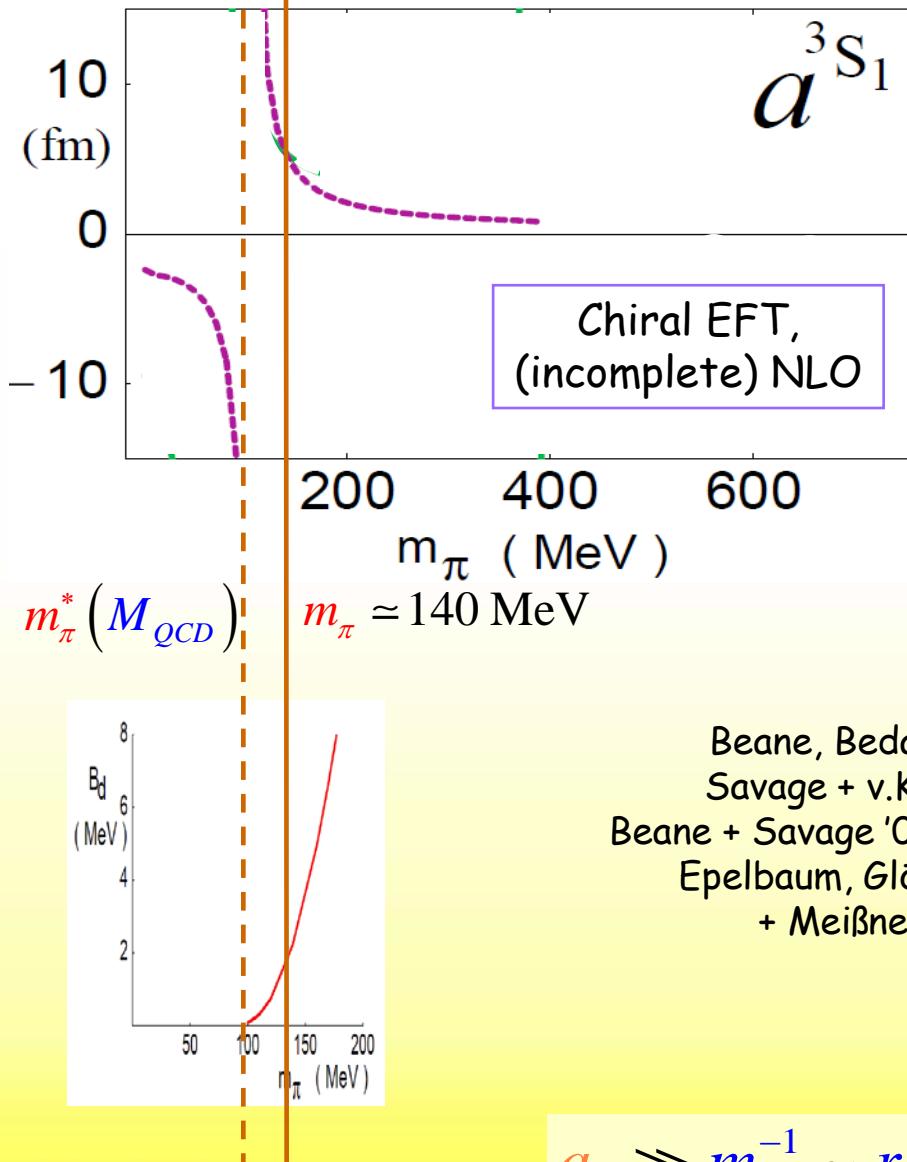
$$\left| a_{2,I=1,I_3=+1} m_\pi \right|^{-1} - \left| a_{2,I=1,I_3=0} m_\pi \right|^{-1} \simeq 0.12$$

$$\left| a_{2,I=1,I_3=-1} m_\pi \right|^{-1} - \left| a_{2,I=1,I_3=0} m_\pi \right|^{-1} \simeq 0.02$$

$^3S_1$

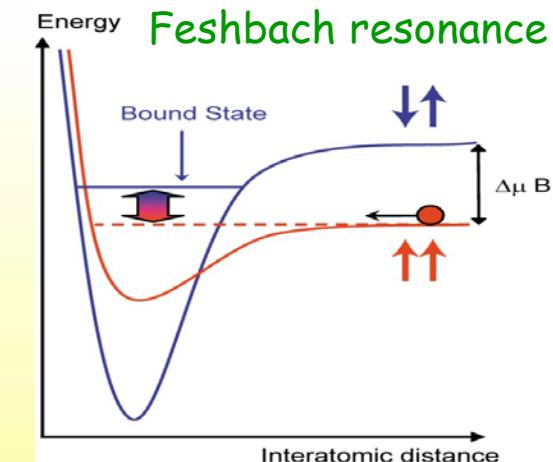
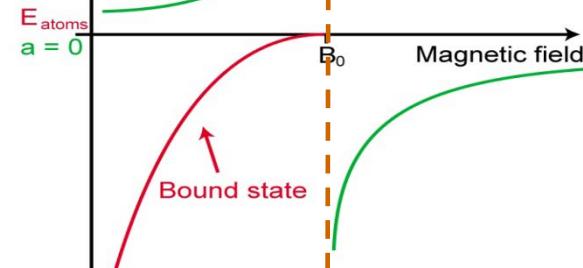
$$\left| a_{2,I=0} m_\pi \right|^{-1} \simeq 0.26$$

unitarity limit



$$a_2 \gg m_\pi^{-1} \sim r_2$$

Scattering length  $a$   
Energy



or "accidentally", e.g.  ${}^4\text{He}$  atoms

$$a_2 \gg l_{\text{vdW}} \sim r_2$$

MIT webpage

ground  
states

$$Q_A \sim \sqrt{2m_N B_A / A}$$

$A$	$Q_A / m_\pi$
2	0.3
3	0.5
4	0.8
5	0.7
6	0.7
...	...
56	0.9
...	...



$$Q_2 \sim a_{2,I=0}^{-1} \equiv \aleph_1 \ll M_{lo}$$

$$Q_{A \geq 3} \sim M_{lo}$$

$^1S_0$  unitarity

$^3S_1$  unitarity

König, Grießhammer,  
Hammer + v.K. '16  
v.K. '17

König, Grießhammer,  
Hammer + v.K. '15  
König '16

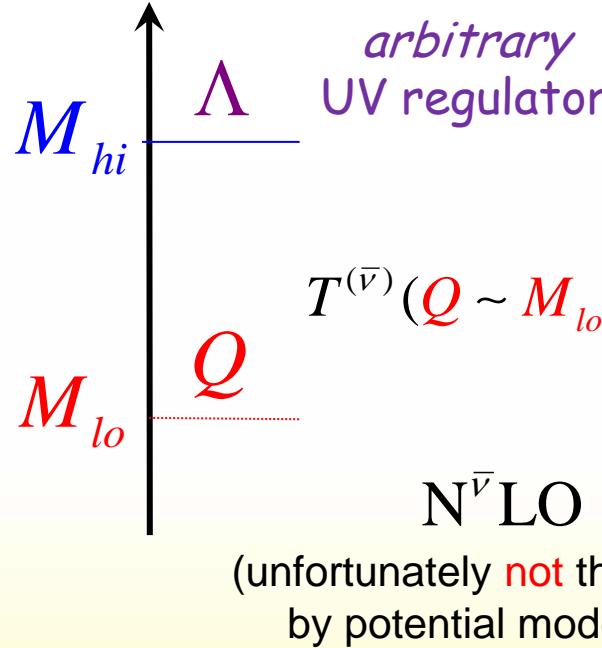
$$\left| a_{2,I=1,I_3=0} \right|^{-1} \equiv \aleph_0 \ll M_{lo}$$

$$\left| a_{2,I=1,I_3=+1} \right|^{-1} - \left| a_{2,I=1,I_3=0} \right|^{-1} \sim \alpha m_N \sim \aleph_0 \ll M_{lo}$$

$$\left| a_{2,I=1,I_3=-1} \right|^{-1} - \left| a_{2,I=1,I_3=0} \right|^{-1} \sim m_d - m_u \ll \aleph_0$$

# Effective Field Theory <sup>C</sup>

momentum scales



non-analytic functions,  
from solution of dynamical eq.  
(e.g. Lippmann-Schwinger)

$$T^{(\bar{v})}(Q \sim M_{lo} \ll M_{hi}) \propto \sum_{v=0}^{\bar{v}} \left[ \frac{Q}{M_{hi}} \right]^v F^{(v)} \left( \frac{Q}{M_{lo}}, \frac{Q}{\Lambda}; \gamma_i^{(v)} \left( \frac{\Lambda}{M_{lo}}, \frac{M_{lo}}{M_{hi}} \right) \right) + \mathcal{O} \left( \frac{Q^{\bar{v}+1}}{M_{hi}^{\bar{v}+1}}, \frac{Q^{\bar{v}+1}}{M_{hi}^{\bar{v}} \Lambda} \right)$$

controlled

"low-energy constants"

RG invariance  
(absent in "chiral potentials")

$$\frac{\Lambda}{T^{(\bar{v})}} \frac{\partial T^{(\bar{v})}}{\partial \Lambda} = \mathcal{O} \left( \frac{Q^{\bar{v}+1}}{M_{hi}^{\bar{v}} \Lambda} \right)$$

model independent

(**OTHERWISE**, NOT ERROR ESTIMATE)

to minimize cutoff errors,  $\Lambda \gtrsim M_{hi}$   
for realistic error estimate,  $\Lambda \in [M_{hi}, \infty)$

(**OTHERWISE**, SENSITIVE TO HIGH-MOM DETAILS)

# Pionless EFT

$$Q \sim M_{lo} \ll M_{hi} \sim m_\pi$$

- d.o.f.: nucleons
- symmetries: Lorentz, P, T, B, SU(3)<sub>c</sub>, U(1)<sub>em</sub>  
(trivial)

most general action

$$\begin{aligned}
 S_{EFT} = & \int \frac{dt}{2m} \int d^3r \left\{ \psi^+ \left( 2im \frac{\partial}{\partial t} + \vec{\nabla}^2 \right) \psi \right. \\
 & - 4\pi \sum_{I=0,1} C_{0I} \psi^+ \psi^+ \underbrace{P_I}_{\text{projector on isospin } I} \psi \psi \\
 & - \frac{(4\pi)^2}{3} D_0 \psi^+ \psi^+ \psi^+ \psi \psi \psi \\
 & \left. + \dots \right\} \text{ more derivatives,} \\
 & \quad \text{more bodies,} \\
 & \quad \text{isospin violation}
 \end{aligned}$$

Universality:  
first orders  
apply also to  
neutral atoms

$$m_\pi \rightarrow 1/l_{vdW} \quad \text{where} \quad V(r) = -\frac{l_{vdW}^4}{2mr^6} + \dots$$

Bedaque, Hammer  
+ v.K. '99'00  
Bedaque, Braaten  
+ Hammer '01

...

Now,  
two expansions:

$$\mathfrak{N}_1/M_{lo} \equiv Q_2/Q_3 \approx 0.4$$

around two-body unitarity

$$M_{lo}/M_{hi} \sim Q_3/m_\pi \approx 0.5$$

standard Pionless EFT

similar:

$$\mathfrak{N}_1 \sim M_{lo}^2/M_{hi}$$

starts at NLO

For simplicity, also:

$$\left. \begin{array}{l} \alpha m_N \sim \mathfrak{N}_0 \sim M_{lo}^2/M_{hi} \\ m_d - m_u \sim M_{lo}^3/M_{hi}^2 \end{array} \right\} \begin{array}{l} \text{starts at NLO} \\ \text{starts at N^2LO} \end{array}$$

v.K. '97'98

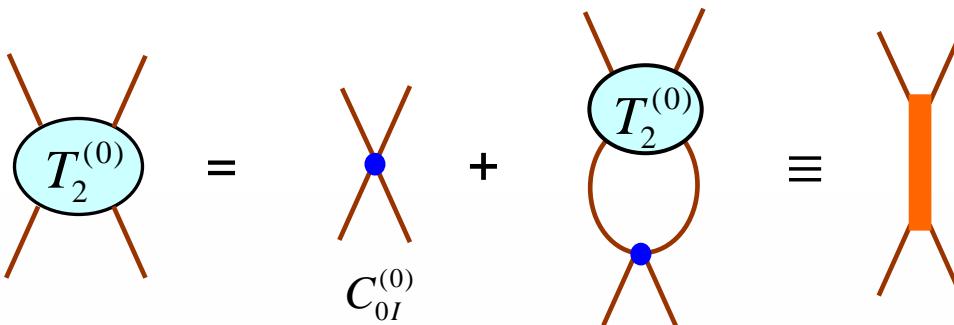
Kaplan, Savage + Wise '98

Chen, Rupak + Savage '99

$A = 2$

...

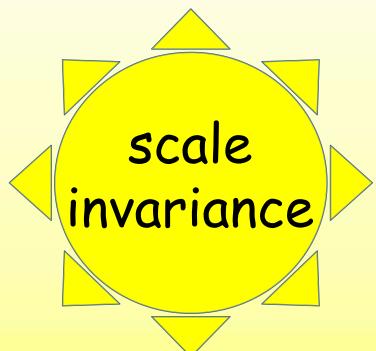
LO



$$C_{0I}^{(0)}(\Lambda) = -\frac{1}{\theta_0 \Lambda} \quad \leftrightarrow \quad m B_2^{(0)} = 0$$

regulator-dependent number

Mehen, Stewart + Wise '00  
König, Grießhammer,  
Hammer + v.K. '16  
König '16  
v.K. '17



$$\begin{aligned} x &\rightarrow \alpha x \\ \frac{t}{m} &\rightarrow \alpha^2 \frac{t}{m} \end{aligned}$$

$$\begin{aligned} \Lambda &\rightarrow \alpha^{-1} \Lambda \\ \psi &\rightarrow \alpha^{-3/2} \psi \end{aligned}$$

$$S_{EFT}^{(0)} \rightarrow S_{EFT}^{(0)}$$

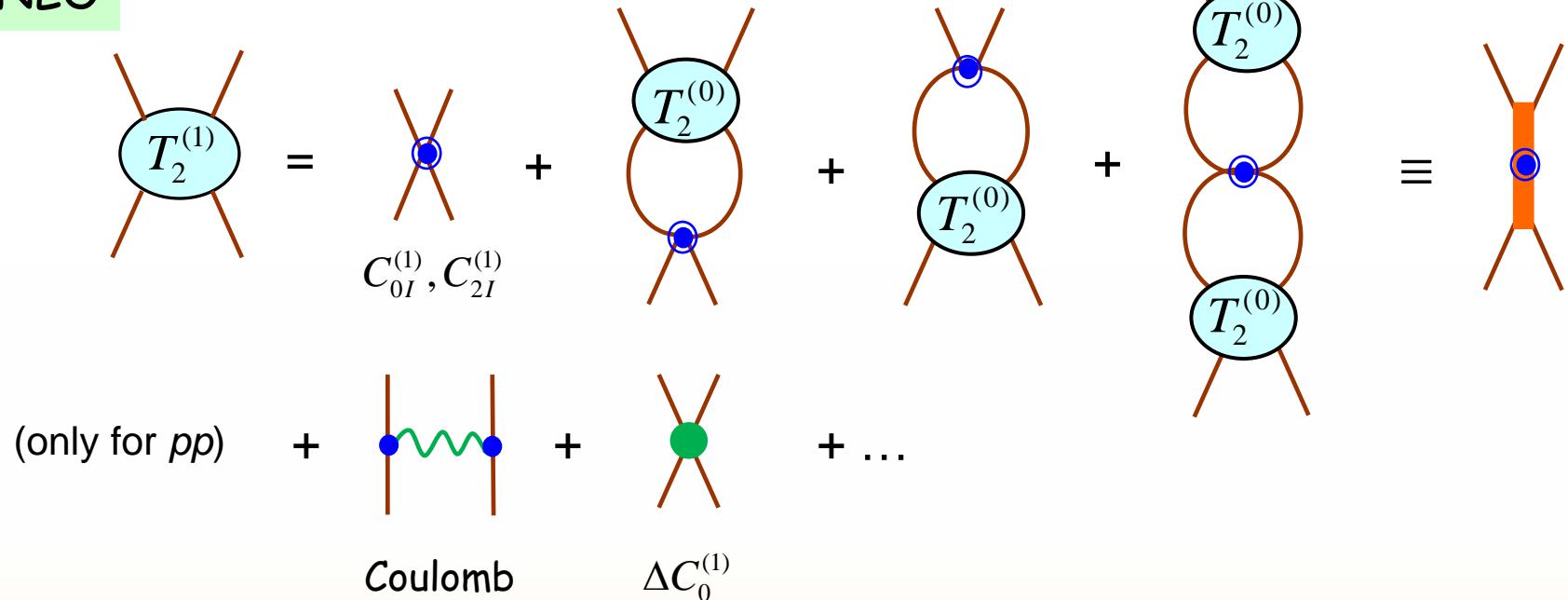
(includes Wigner spin-isospin symmetry)

Mehen, Stewart + Wise '00

...

$SU(4)_W$

# NLO



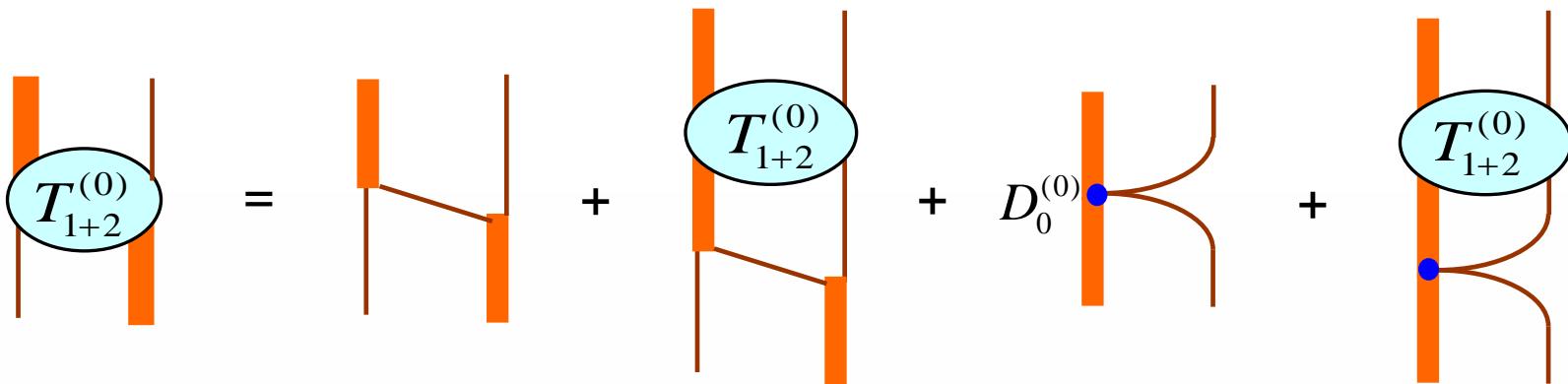
$$\begin{array}{ll}
 a_{2,I=0} = 5.42 \text{ fm} & \xrightarrow{\quad} C_{0I=0}^{(1)} \\
 r_{2,I=0} = 1.75 \text{ fm} & \xrightarrow{\quad} C_{2I=0}^{(1)} \\
 a_{2,I=1,I_3=0} = -23.71 \text{ fm} & \xrightarrow{\quad} C_{0I=1}^{(1)} \\
 r_{2,I=1,I_3=0} = 2.73 \text{ fm} & \xrightarrow{\quad} C_{2I=1}^{(1)} \\
 a_{2,I=1,I_3=+1} = -7.81 \text{ fm} & \xrightarrow{\quad} \Delta C_0^{(1)}
 \end{array}
 \left\{
 \begin{array}{ll}
 a_{2,I=1,I_3=-1}^{(1)} = a_{2,I=1,I_3=0} & -18.7 \text{ fm (? exp)} \\
 r_{2,I=1,I_3=-1}^{(1)} = r_{2,I=1,I_3=0} & \text{vs. } ? \text{ (exp)} \\
 r_{2,I=1,I_3=+1}^{(1)} = r_{2,I=1,I_3=0} & 2.79 \text{ fm (exp)}
 \end{array}
 \right.$$

etc.

scale invariance  
broken explicitly

e.g.,  $mB_2^{(2)} = \frac{1}{a_2^2}$

LO

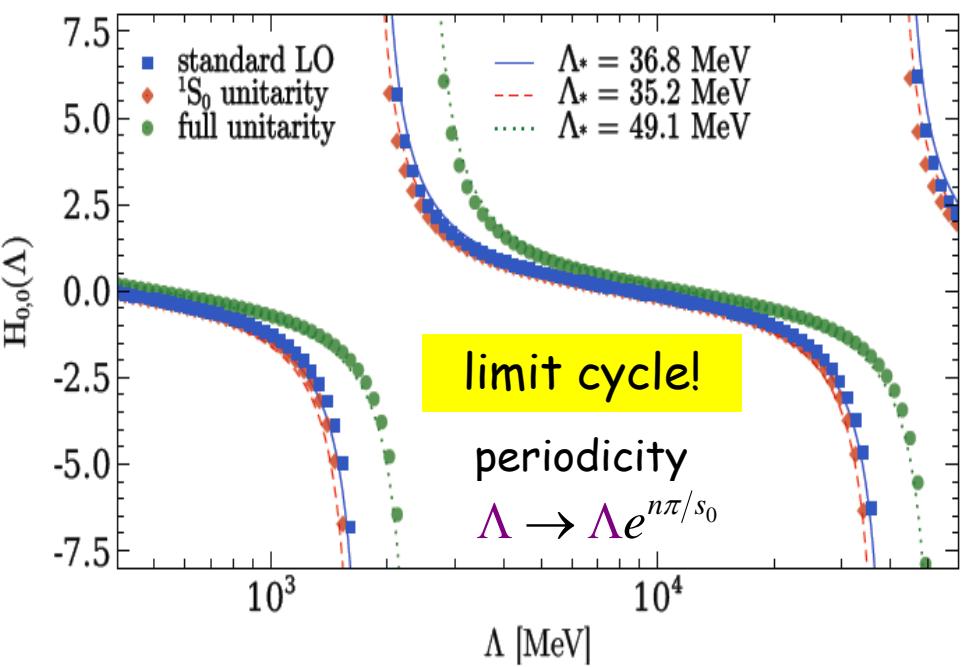
 $A = 3$ 

$$B_t = 8.48 \text{ MeV} \rightarrow D_0^{(0)}$$

3-body interaction



König, Grießhammer, Hammer + v.K.'15 '16



$$H_{0,0}(\Lambda) \equiv \frac{\Lambda^2 D_0^{(0)}(\Lambda)}{m C_0^{(0)2}(\Lambda)} \quad s_0 = 1.00624... \\ \simeq \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) + \arctan s_0^{-1})}{\sin(s_0 \ln(\Lambda/\Lambda_*) - \arctan s_0^{-1})}$$

$$D_0^{(0)}(\Lambda) \propto \frac{1}{\Lambda^4} \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) + \arctan s_0^{-1})}{\sin(s_0 \ln(\Lambda/\Lambda_*) - \arctan s_0^{-1})}$$

dimensionful parameter  
(dimensional transmutation)



$$\begin{array}{ll} x \rightarrow \alpha_n x & \Lambda \rightarrow \alpha_n^{-1} \Lambda \\ \frac{t}{m} \rightarrow \alpha_n^2 \frac{t}{m} & \psi \rightarrow \alpha_n^{-3/2} \psi \end{array} \quad \left. \right\} S_{EFT}^{(0)} \rightarrow S_{EFT}^{(0)}$$

$$\alpha_n = \exp(n\pi/s_0) = (22.7)^n \quad n = \dots, 0, 1, 2, \dots$$



ground state

$$mB_{3n}^{(0)} = k_*^2 \exp(-2n\pi/s_0) \lesssim M_{hi}^2$$

## Efimov states

Efimov '71

...

$$\ln\left(\frac{k_*}{\Lambda_*}\right) = \ln \beta, \mod \pi/s_0 \quad \beta \simeq 0.383$$

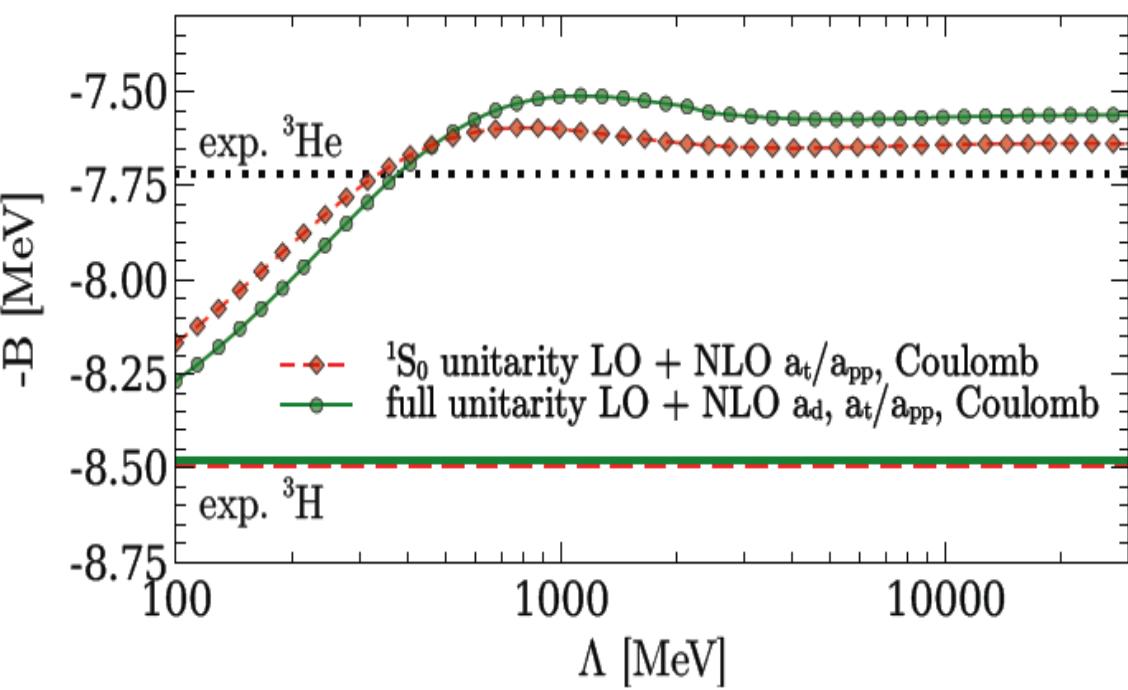
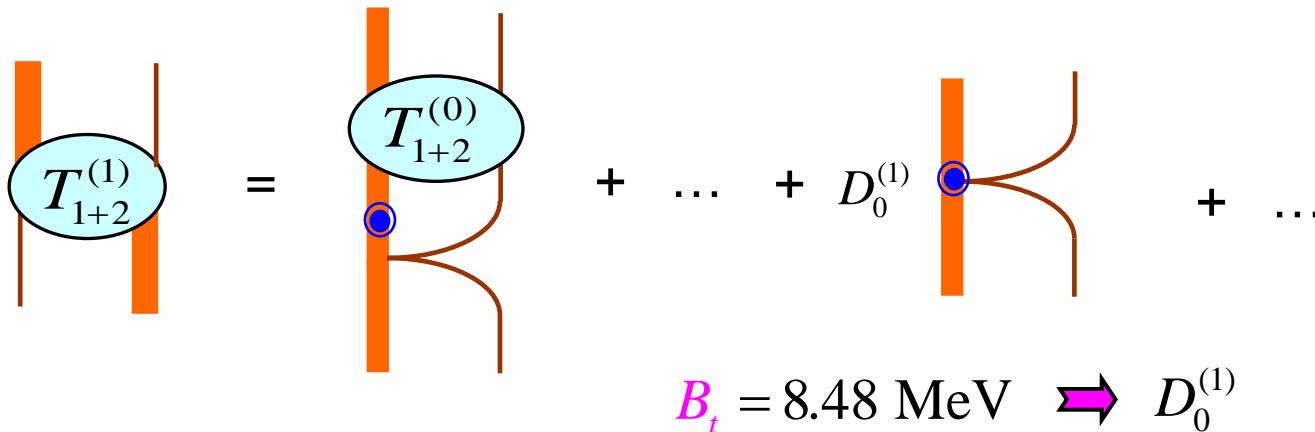
Braaten + Hammer '06

(includes Wigner spin-isospin symmetry)

Bedaque, Hammer + v.K. '00

$SU(4)_W$

NLO

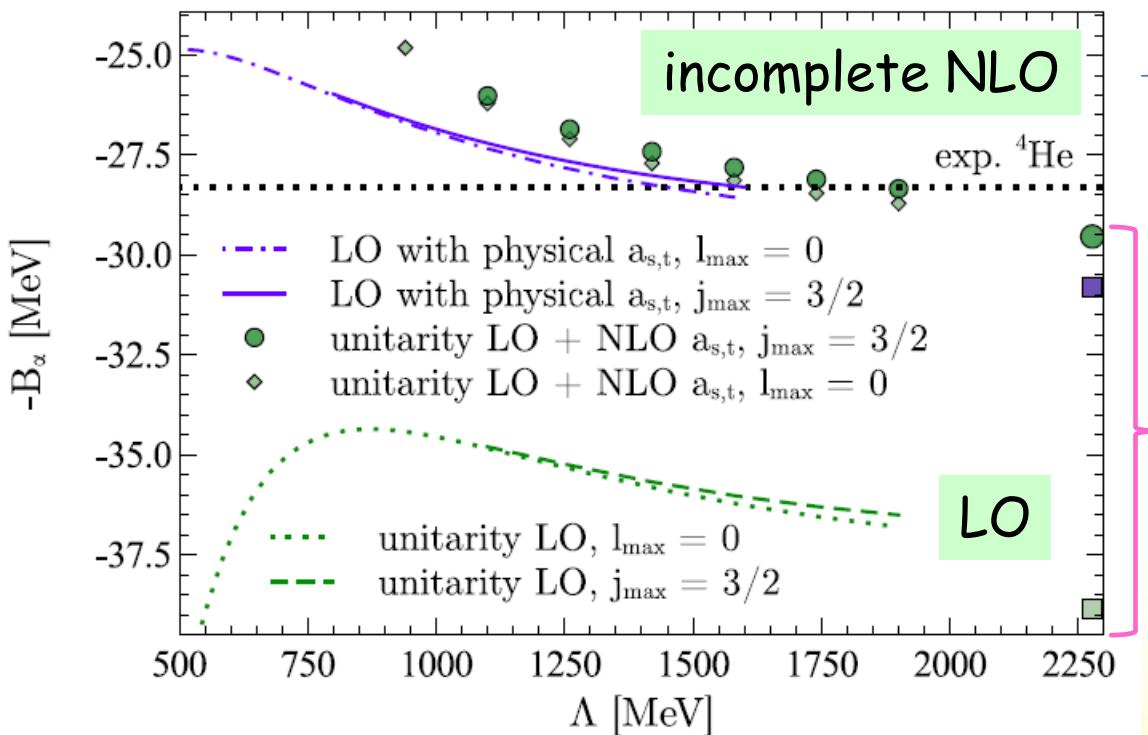


$$B_h^{(1)} - B_t \approx -(0.92 \pm 0.18) \text{ MeV}$$

vs.

$$-0.764 \text{ MeV} (\text{exp})$$

# (Fadeev-Yakubovski)



König, Grießhammer,  
Hammer + v.K. '16

$A = 4$

$$\begin{cases} r_{2,I} = 0 \\ \alpha = 0 \end{cases}$$

extrapolated values

$$B_\alpha(\Lambda) = \underbrace{B_\alpha(\infty)}_{\text{finite}} \left[ 1 + \beta_\alpha \left( \frac{Q_\alpha}{\Lambda} \right)^2 + \dots \right]$$

→ no 4BF at

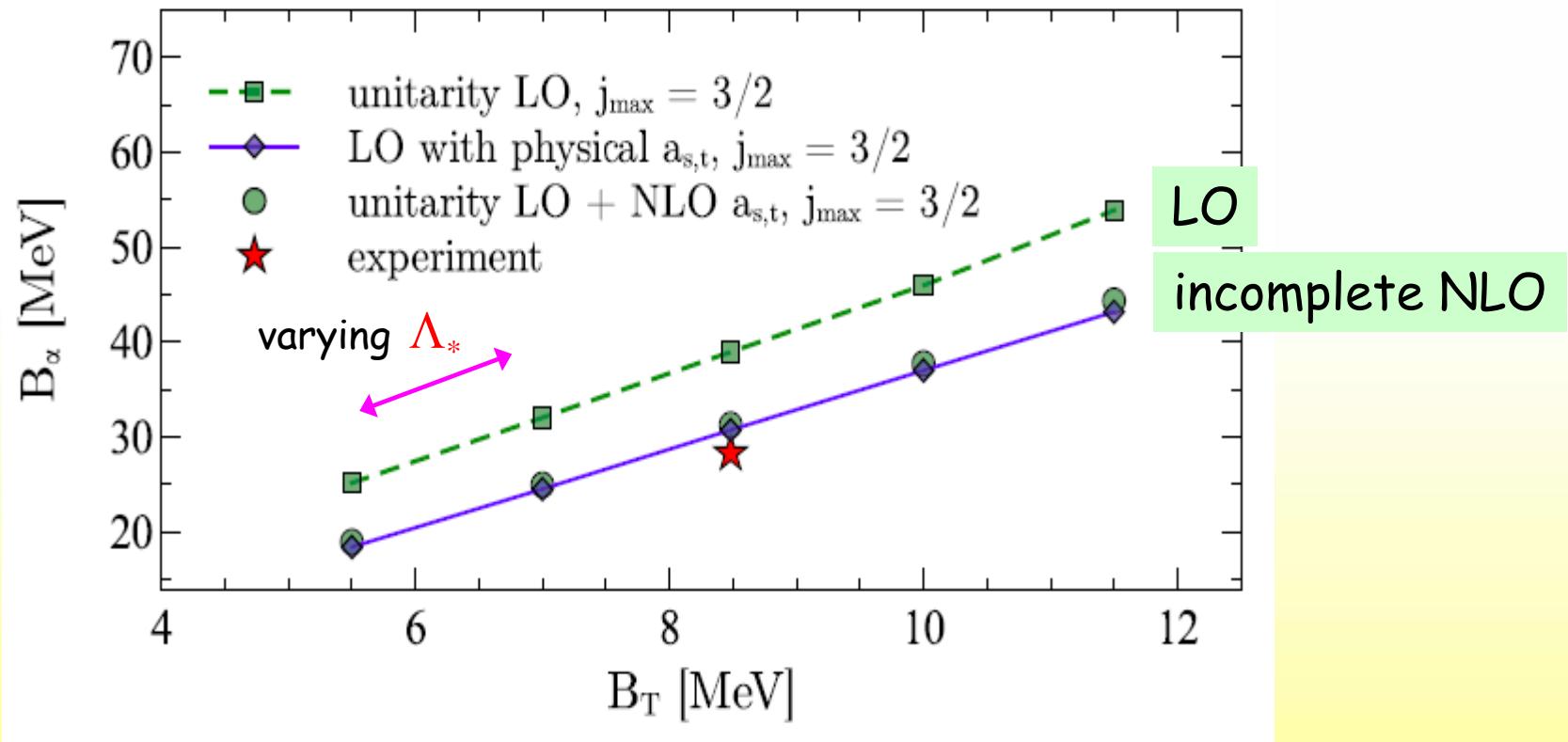
LO  
incomplete NLO

Hammer, Meiñner + Platter '04 '05  
Hammer + Platter '07  
Stetcu, Barrett + v.K. '07  
...

$$\begin{cases} B_\alpha^{(1,\text{inc})} \approx (29.5 \pm 8.7) \text{ MeV} \\ B_{\alpha^*}^{(0)} - B_t \approx 0 \end{cases}$$

vs.  
28.4 MeV (exp)  
-0.3 MeV (exp)

## Tjon line



# Unitary bosons

$N = 3$

Similar 3BF

$N = 4$

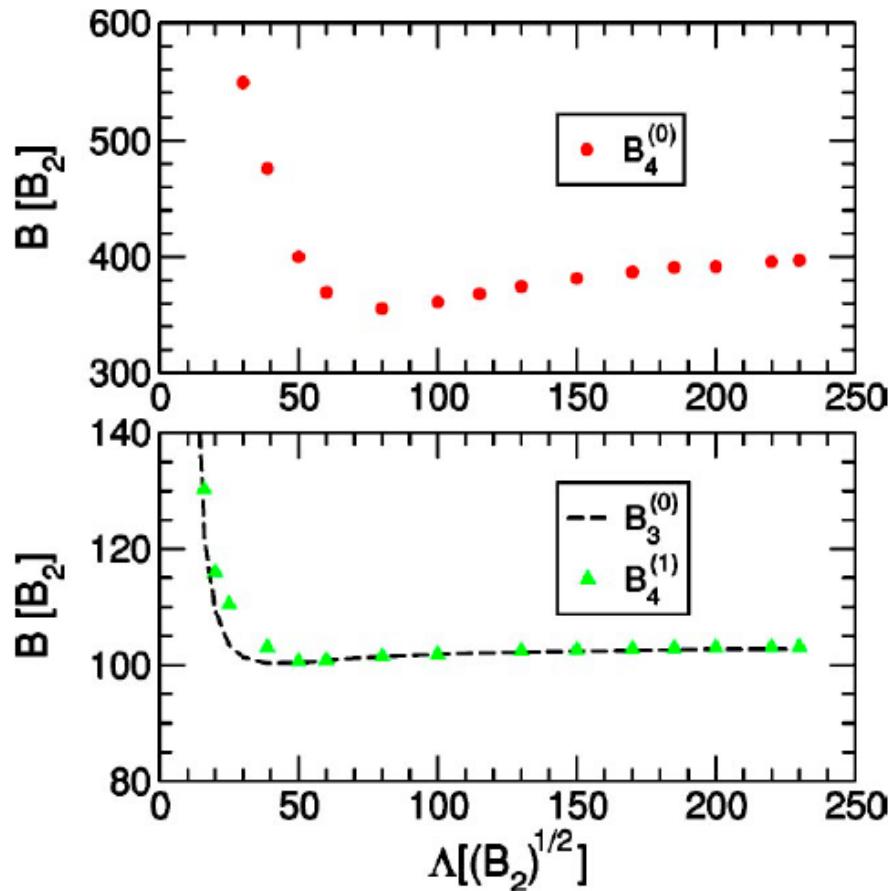
No 4BF

LO

Bedaque, Hammer + v.K. '99 '00

Hammer + Platter '07

Efimov  
descendants



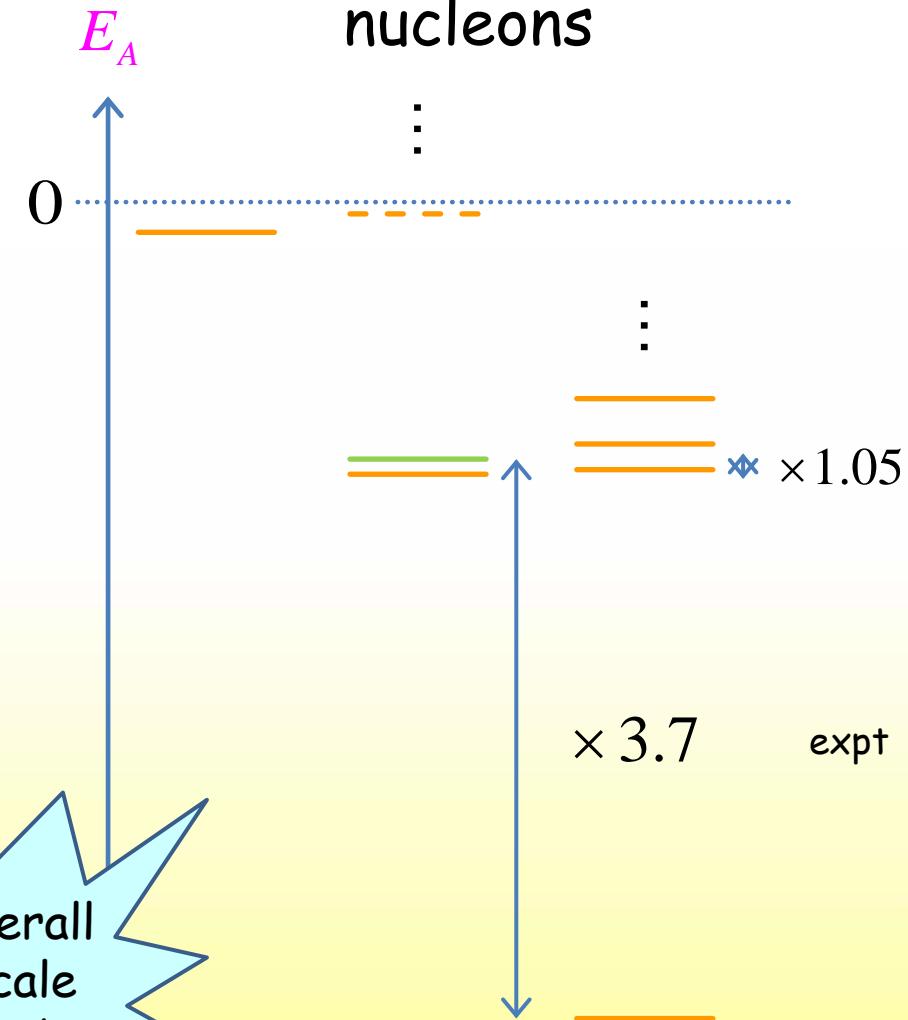
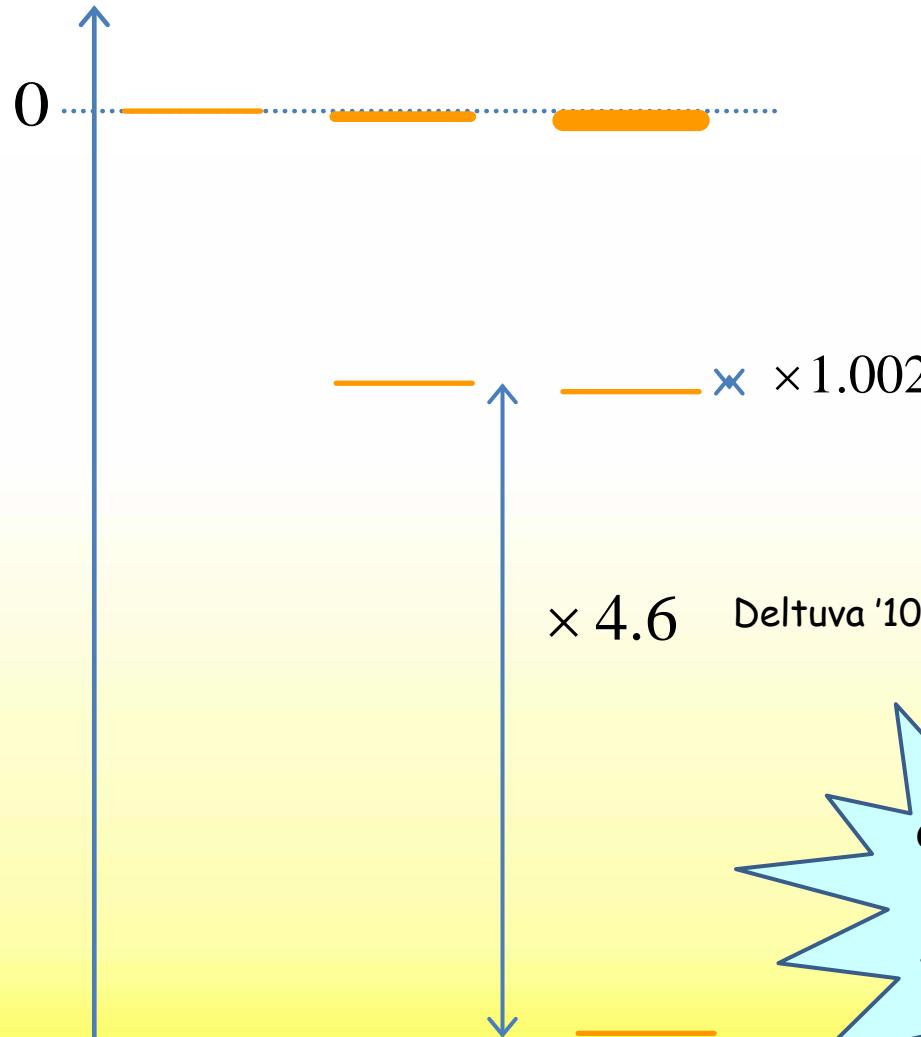
ground state

first  
excited state

# Schematically

unitary bosons

nucleons



overall  
scale  
set by

$\Lambda_*$

$N = 2$

$N = 3$

$N = 4$

$A = 2$

$A = 3$

$A = 4$

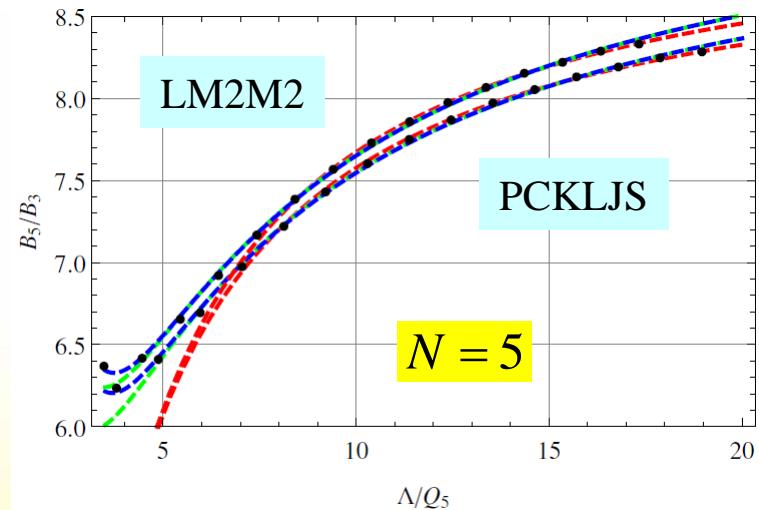
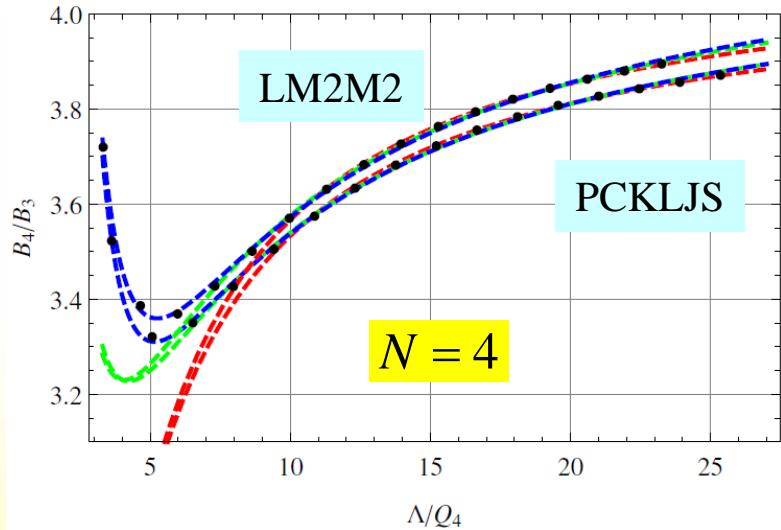
# Bosons

$N \geq 4$

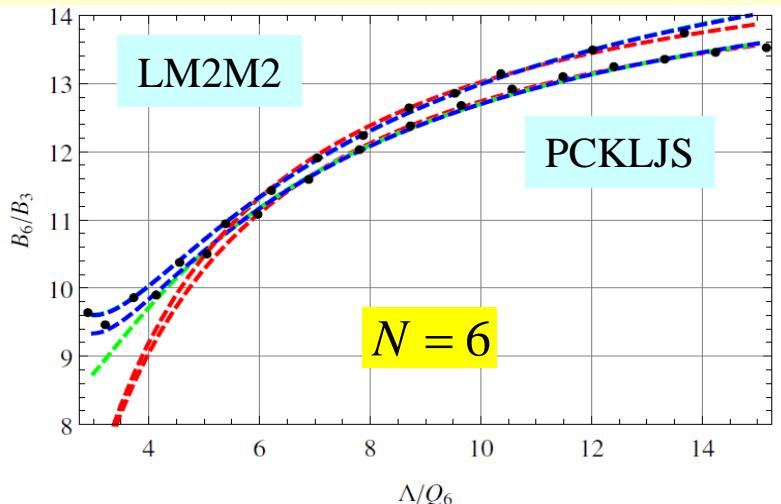
No more-body forces

LO

Bazak, Eliyahu + v.K. '16



(Stochastic Variational)



$$B_N^{(0)}(\Lambda) = \underbrace{B_N^{(0)}(\infty)}_{\text{within } \sim 10\% \text{ of model results}} \left[ 1 + \alpha_N \frac{Q_N}{\Lambda} + \beta_N \left( \frac{Q_N}{\Lambda} \right)^2 + \dots \right]$$

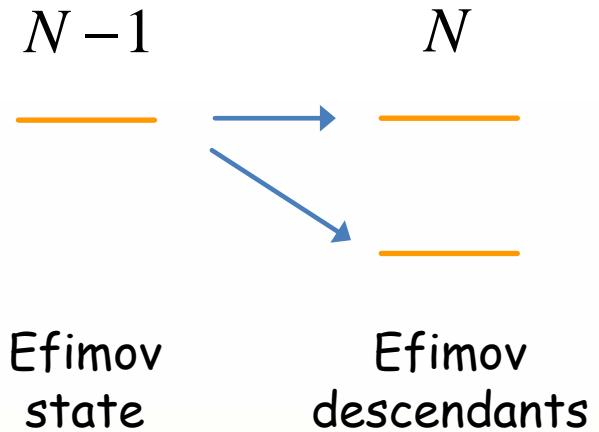
within  $\sim 10\%$   
of model results

$\alpha_N, \beta_N, \dots = \mathcal{O}(1)$

# Unitary bosons



doubling  
Hammer + Platter '06  
von Stecher '10'11  
Gattobigio, Kievsky + Viviani '11'12



Ground states

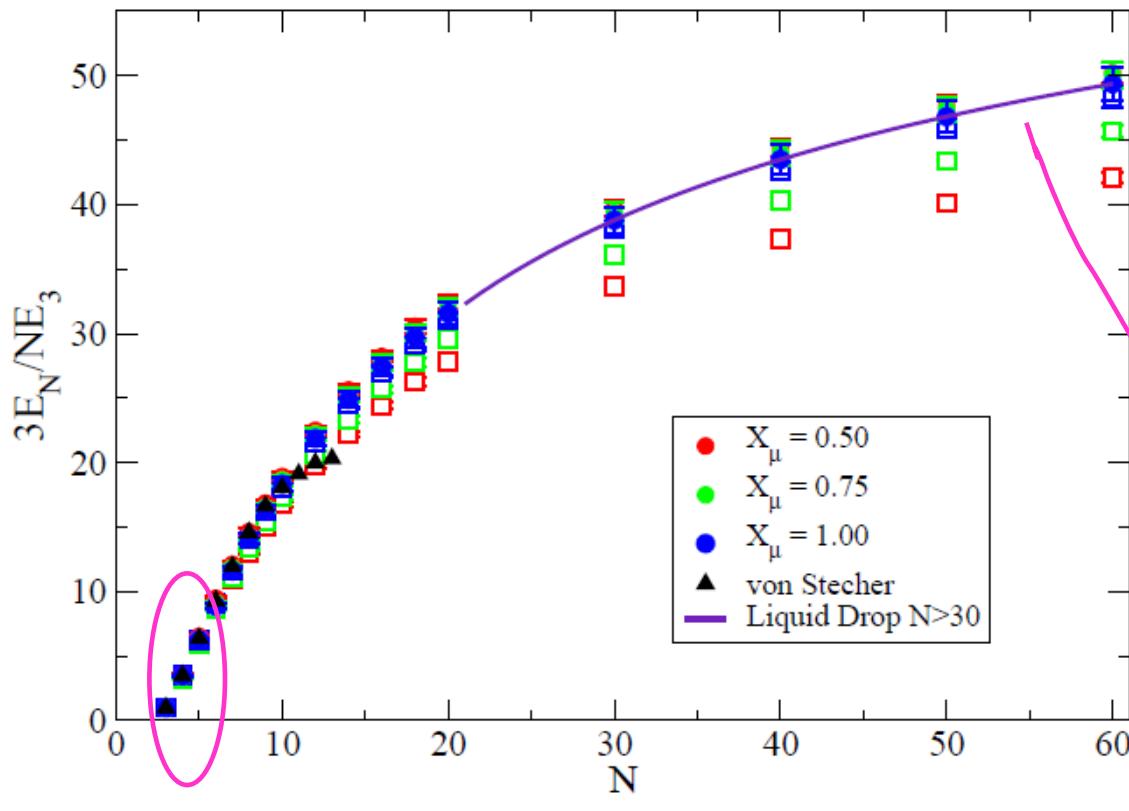
single scale  $\rightarrow$  
$$\frac{B_N^{(0)}(\Lambda_*)}{N} = \kappa_N \frac{B_3(\Lambda_*)}{3}$$

universal numbers

# (Variational and Diffusion Monte Carlo)

Gandolfi, Carlson,  
Vitiello + vK '17

LO



$$\begin{aligned}\Lambda_2/Q_3 &\simeq 26 - 38 \\ \Lambda_3/\Lambda_2 &= 0.5 - 1\end{aligned}$$

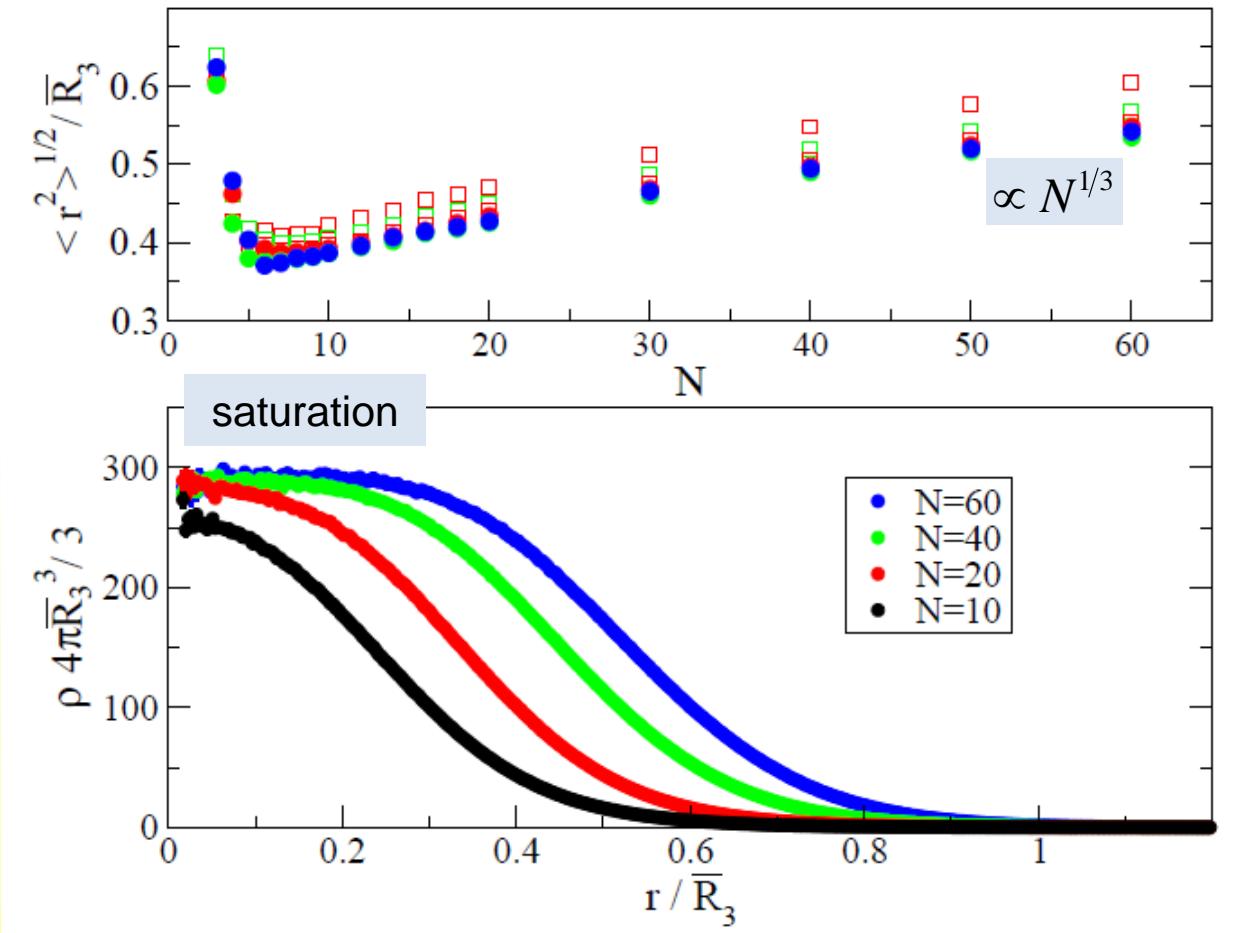
$$\kappa_N \approx \frac{3}{N} (N-2)^2$$

$$\kappa_N = \kappa_\infty \left[ 1 + \eta N^{-1/3} + \mathcal{O}(N^{-2/3}) \right]$$

Bazak, Eliyahu  
+ v.K. '16

$$\kappa_\infty = 90 \pm 10 \quad \eta = -1.7 \pm 0.3$$

LO



$$\bar{R}_3 \equiv (2mB_3)^{-1/2}$$

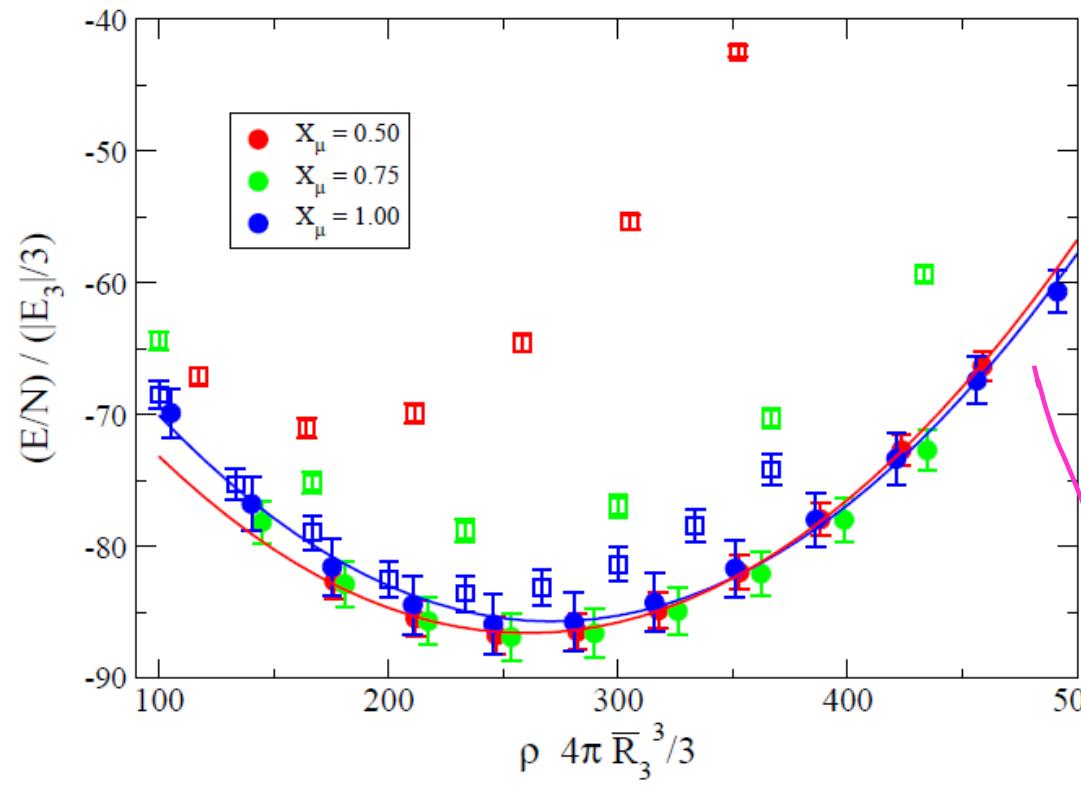
$N = 20, 40, 60$

## Equation of State

periodic boundary conditions

Gandolfi, Carlson,  
Vitiello + vK '17

LO



$$\frac{4\pi \bar{R}_3^3}{3} \rho_0 = 275 \pm 20$$

$$\kappa_\infty(\rho) = \kappa_\infty(\rho_0) \left[ 1 - \gamma \left( \frac{\rho - \rho_0}{\rho_0} \right)^2 + \mathcal{O} \left( \left( \frac{\rho - \rho_0}{\rho_0} \right)^3 \right) \right]$$

$$\kappa_\infty(\rho_0) = 87 \pm 5 \quad \gamma = 0.42 \pm 0.05$$

	unitary bosons, LO	${}^4\text{He}$ atoms
$Q_2/Q_3$	0	0.12
$Q_3/M_{hi}$	0.05	0.4
$Q_4/Q_3$	1.9	1.8
$Q_6/Q_3$	3.0	4.0
$Q_{40}/Q_3$	6.6	8.0
$Q_\infty/Q_3$	9.5	13.4
$k_0/Q_\infty$	2.0	0.66
$\eta$	-1.7	-2.7
$\gamma$	0.42	1.9

sensitivity to  
short-range  
physics!?

$$k_0 \equiv \left(6\pi^2 \rho_0/g\right)^{1/3}$$

( $g$ : degeneracy)

Gandolfi,  
Carlson, Vitiello  
+ v.K. '17

Kalos *et al.* '81  
Pandharipande  
*et al.* '83

$A \geq 4$

## Nucleons

LO

No more-body forces  
+ saturation

$A = 16$

$B_{^{16}\text{O}}^{(0)}$  (MeV)

Contessi, Lovato, Pederiva,  
Roggero, Kirscher + v.K. '17

$\Lambda$	$m_\pi = 140$ MeV	$m_\pi = 510$ MeV	$m_\pi = 805$ MeV
$2 \text{ fm}^{-1}$	$-97.19 \pm 0.06$	$-116.59 \pm 0.08$	$-350.69 \pm 0.05$
$4 \text{ fm}^{-1}$	$-92.23 \pm 0.14$	$-137.15 \pm 0.15$	$-362.92 \pm 0.07$
$6 \text{ fm}^{-1}$	$-97.51 \pm 0.14$	$-143.84 \pm 0.17$	$-382.17 \pm 0.25$
$8 \text{ fm}^{-1}$	$-100.97 \pm 0.20$	$-146.37 \pm 0.27$	$-402.24 \pm 0.39$
$\rightarrow \infty$	$-115_{\pm 8 \text{ (stat)}}^{\pm 1 \text{ (sys)}}$	$-151_{\pm 10 \text{ (stat)}}^{\pm 2 \text{ (sys)}}$	$-504_{\pm 12 \text{ (stat)}}^{\pm 20 \text{ (sys)}}$
Exp.	$-127.62$	—	—

$A \geq 4$

## Nucleons around unitarity



doubling?

Ground states

single scale

LO



$$\frac{B_A^{(0)}(\Lambda_*)}{A} = \kappa_A \frac{B_3(\Lambda_*)}{3}$$

$$\begin{cases} \kappa_4 \simeq 3.5 & \text{same as for bosons} \\ \kappa_{A \geq 5} \simeq ? & \text{should grow slower than bosons} \end{cases}$$



nuclear matter  
within Pionless EFT?

	unitary bosons, LO	$^4\text{He}$ atoms	unitary nucleons, LO	nucleons
$Q_2/Q_3$	0	0.12	0	0.4
$Q_3/M_{hi}$	0.05	0.4	0.5	0.5
$Q_4/Q_3$	1.9	1.8	1.9	1.6
$Q_6/Q_3$	3.0	4.0	?	1.4
$Q_{40}/Q_3$	6.6	8.0	?	1.7
$Q_\infty/Q_3$	9.5	13.4	?	2.4
<hr/>				
$k_0/Q_\infty$	2.9	0.66	?	1.5
$\eta$	-1.7	-2.7	?	-1.1
$\gamma$	0.42	1.9	?	1.7

$$k_0 \equiv \left(6\pi^2 \rho_0/g\right)^{1/3}$$

( $g$ : degeneracy)

Gandolfi,  
Carlson, Vitiello  
+ v.K. '17

Kalos *et al.* '81  
Pandharipande  
*et al.* '83

König,  
Grießhammer,  
Hammer + v.K. '16

experiment,  
semi-empirical  
mass formula



# Equation of State

$$k_\rho \equiv (6\pi^2 \rho / g)^{1/3} \quad (g: \text{degeneracy})$$

Bertsch '99  
Carlson *et al.* '03

unitary fermions

scale invariance

$$\lim_{N \rightarrow \infty} \frac{E_N}{N} = \frac{3k_\rho^2}{10m} \left( \xi - \frac{\zeta}{a_2 k_\rho} + \eta r_2 k_\rho + \dots \right)$$

$\xi \approx 0.4 \quad \zeta \approx 1 \quad \eta \approx 0.1$

unitary bosons and nucleons

discrete scale invariance

$$\lim_{N \rightarrow \infty} \frac{E_N}{N} = \frac{k_\rho^2}{m} \mathfrak{G} \left[ s_0 \ln \left( k_\rho / k_* \right) \right] + \dots$$

v.K. '17

$$\mathfrak{G}[x + n\pi] = \mathfrak{G}[x]$$

$$\rightarrow \left( \lim_{N \rightarrow \infty} \frac{E_N}{N} \right)_{\text{equil}} \propto \left( \frac{k_\rho^2}{m} \right)_{\text{equil}}$$

as  $\Lambda_*$  varies

Coester line

$$\mathfrak{G}[x] \equiv \frac{3}{10} \gamma_0 - \gamma_1 \sin(2x + \phi_1) + \sum_{l=2}^{\infty} \gamma_l \sin(2lx + \phi_l)$$

simple model:  $\gamma_0 \approx 0.50$     $\gamma_1 \approx 0.43$     $\phi_1 = 0$     $\gamma_{l \geq 2} = 0$

qualitative fit:  $K_\infty \equiv \left( k_\rho^2 \frac{d^2}{dk_\rho^2} \lim_{A \rightarrow \infty} \frac{E_A}{A} \right)_{\text{equil}} \approx 170 \text{ MeV}$

vs.

$\approx 240 \text{ MeV} (\text{emp})$

# Conclusion

- ◆ Systems near unitarity can be described model-independently by Pionless EFT
    - properties given by essentially **one** parameter  $\Lambda_*$
    - details obtained in perturbation theory
  - ◆ How far can we go for nuclei?
    - more nucleons *Gezerlis et al., in progress*
    - higher orders *Bazak et al., in progress*