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Birkhoff theorem in theories with torsion

Francisco José Maldonado Torralba
University of Cape Town
Collaborators: A. de la Cruz Dombriz, A. Mazumdar

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Why torsion theories?

Sciama (1962) and Kibble (1961)

- Poincaré gauge invariance $\longrightarrow$ Naturally defined spin fields
- More degrees of freedom $\longrightarrow$ New possibilities to explain open problems in cosmology

Introduction
Why torsion theories?

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- More degrees of freedom $\longrightarrow$ New possibilities to explain open problems in cosmology

General connections

An arbitrary connection has D^3 degrees of freedom in D dimensions:

- $\frac{D^2(D-1)}{2}$ in the antisymmetric part (*torsion*):

$$T^{\rho}_{\mu\nu} \equiv \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} .$$

- $\frac{D^2(D+1)}{2}$ in the *non-metricity* tensor:

$$Q_{\rho\mu\nu} = \nabla_{\rho} g_{\mu\nu} .$$

- Remark: A different connection does not always lead to different phenomenology (e.g. Teleparallel Gravity).

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Properties of the theories with torsion

- Gauge theory of the Poincaré Group.
- Metricity

$$\nabla_\rho g_{\mu\nu} = 0$$

- Non symmetric connection

$$T^\rho_{\mu\nu} \equiv \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \neq 0$$

- Relation between the connection and the Levi-Civita one

$$\mathring{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + K^\rho_{\mu\nu}$$

where

$$K^\rho_{\mu\nu} = \frac{1}{2}\left(T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T^\rho_{\mu\nu}\right)$$

Decomposition of the torsion tensor

$$T_{\mu\nu\rho} = \frac{1}{3}\left(T_\nu g_{\mu\rho} - T_\mu g_{\mu\nu}\right) - \frac{1}{6}\varepsilon_{\mu\nu\rho\sigma} S^\sigma + q_{\mu\nu\rho},$$

where

$$\underbrace{T_\mu = T^\nu{}_{\mu\nu}}_{\text{Trace}},\ \underbrace{S^\mu = \varepsilon^{\rho\sigma\nu\mu} T_{\rho\sigma\nu}}_{\text{Axial vector}},\ \underbrace{q^\mu_{\nu\rho}\ s.t.\ q^\nu_{\mu\nu} = 0\ and\ \varepsilon^{\rho\sigma\nu\mu} q_{\rho\sigma\nu} = 0}_{\text{Tensor}}$$

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Theories that fulfill the Birkhoff theorem

R. Rauch, H. T. Nieh, Physical Review D, 24(8), 2029 (1981)

Theorem

In the absence of matter the Schwarzschild metric with vanishing torsion is the unique SO(3) spherically symmetric solution to the vacuum field equations

$$\mathscr{L}_1 = -\lambda R + \alpha R^2$$

$$\begin{aligned}\mathscr{L}_2 &= -\lambda R + \frac{1}{12}(4a+b+3\lambda)\, T_{\alpha\beta\gamma}T^{\alpha\beta\gamma} \\ &+ \frac{1}{6}(-2a+b-3\lambda)\, T_{\alpha\beta\gamma}T^{\beta\gamma\alpha} + \frac{1}{3}(-a+2c-3\lambda)\, T^{\beta}_{\ \beta\alpha}T_{\gamma}^{\ \gamma\alpha}\end{aligned}$$

Most general stable Lagangian

J. A. R. Cembranos et. al., The European Physical Journal C, 77(11), 755 (2017)

$$\begin{aligned}\mathscr{L}_S &= -\lambda R + \frac{1}{12}(4a+b+3\lambda)\, T_{\alpha\beta\gamma}T^{\alpha\beta\gamma} \\ &+ \frac{1}{6}(-2a+b-3\lambda)\, T_{\alpha\beta\gamma}T^{\beta\gamma\alpha} + \frac{1}{3}(-a+2c-3\lambda)\, T^{\beta}_{\ \alpha\beta}T_{\gamma}^{\ \alpha\gamma} \\ &+ 2\tau R_{[\alpha\beta]}R^{[\alpha\beta]}\end{aligned}$$

where λ, a, b, c, τ are constants, such that $b+3\lambda>0$, $c+3\lambda>0$, and $\tau<0$. The antisymmetric part of the Ricci tensor will introduce derivatives of the torsion tensor (i.e. propagating torsion)

$$R_{[\mu\nu]} = \nabla_\sigma\left(T^{\sigma}_{\ \mu\nu} + \delta^{\sigma}_{\mu}T_\nu - \delta^{\sigma}_{\nu}T_\mu\right) - 2T_\sigma T^{\sigma}_{\ \mu\nu}$$

Outline

Spherically symmetric spacetime

The most general spherically symmetric metric is

$$ds^2 = -\psi(t,r)\,dt^2 + \phi(t,r)\,dr^2 + \rho^2(t,r)\left(d\theta^2 + sin^2\theta\, d\varphi^2\right)$$

Torsion components (S. Sur and A. S. Bhatia (2013))

$T^{t}{}_{tr} = a(r)$	$T^{\theta_j}{}_{t\theta_i} = e^{a\theta_j} e^{b}_{\theta_i} \varepsilon_{ab} f(r)$
$T^{r}{}_{tr} = b(r)$	$T^{\theta_j}{}_{r\theta_i} = e^{a\theta_j} e^{b}_{\theta_i} \varepsilon_{ab} g(r)$
$T^{\theta_j}{}_{t\theta_i} = \delta^{\theta_j}_{\theta_i} c(r)$	$T^{t}{}_{\theta_i\theta_j} = \varepsilon_{ij} \sin(\theta) h(r)$
$T^{\theta_j}{}_{r\theta_i} = \delta^{\theta_j}_{\theta_i} d(r)$	$T^{r}{}_{\theta_i\theta_j} = \varepsilon_{ij} \sin(\theta) l(r)$

- *Trace*, $T_\mu \propto a(r), b(r), c(r), d(r)$
- *Axial vector* $S^\mu \propto f(r), g(r), h(r), l(r)$
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Palatini formalism

- Variations with respect to the metric $\Longrightarrow$ Einstein Equations
- Variations with respect to the connection $\Longrightarrow$ Cartan Equations

By a first look at these ones we find that in a spherically symmetric spacetime

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   - General aspects of the theories with torsion
2. Birkhoff theorem
   - Previous literature
   - Spherically symmetric system
3. Study through torsion decomposition
   - General considerations
   - Non-null axial part
   - Non-null trace part

Axial part

- *Trace*, $T_{\mu} \propto \cancel{a(r), b(r), c(r), d(r)}$
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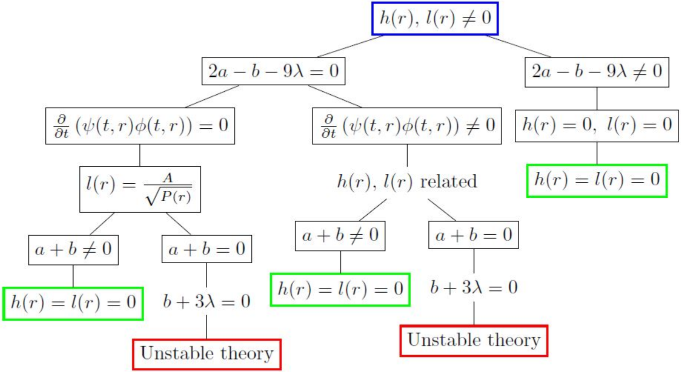
- *Tensor* $q^{\mu}_{\nu\rho} \propto \cancel{a(r), b(r), c(r), d(r)}, f(r), g(r), h(r), l(r)$

Motivation:

- This is the part of the torsion that couples with the spin tensor, so it is the most physically relevant invariant. (FJMT, J.A.R. Cembranos, J. Gigante Valcarcel, arXiv: 1805.09577 (2018))

Tree of decision

A. Cruz-Dombriz, FJMT, A. Mazumdar, in preparation, 2018.
Green⟶Birkhoff theorem holds, Red⟶Tachyon instability



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Motivations:

- Involves the rest of the torsion components.
- It can help solving the general proof.

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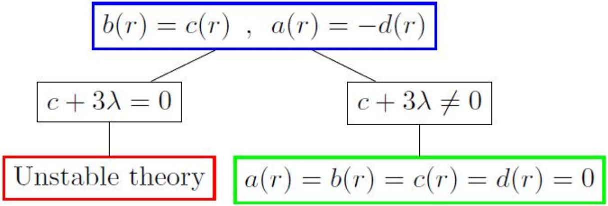

Birkhoff with torsion

A. Cruz-Dombriz, FJMT, A. Mazumdar, in preparation, 2018

Theorem

In the absence of matter the Schwarzschild metric with vanishing torsion is the unique SO(3) spherically symmetric solution to the vacuum field equations, given that one of these two conditions meet:

- *Only the functions that contribute to the total antisymmetric torsion are non-null.*
- *The functions that contribute to the trace torsion are non-null.*

Conclusions and prospects

- Find new theories where the Birkhoff theorem applies.
- Show the advantages of using the torsion decomposition to solve complicated problems.

- Prospects
  - Find the general proof of the Birkhoff theorem.
  - Modelling of neutron stars in torsion theories with stable solutions.