Duality in Convex Optimization: Application to Minimal Area Surfaces

String Dualities and Geometry

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- Systoles and Extremal Length
- Minimal Area Problem
- Two surfaces
- Convex optimization and Duality
- The Primal and the Dual
- The Results.

Matthew Headrick, BZ,

– Convex programs for minimal area problems (Feb. 2018)

– Minimal area metrics with crossing bands of geodesics, (Feb. 2018)

– String diagrams from minimal area metrics of non-positive curvature (Mar. 2018)

Systoles and Extremal Length

A classic problem deals with isoperimetric inequalities.

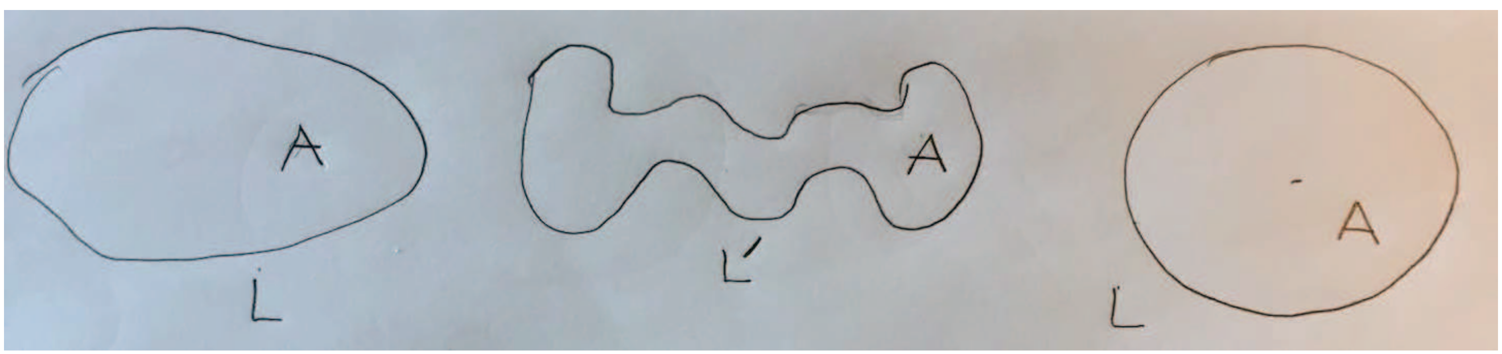
For a region $D \subset \mathbb{R}^2$ (with the flat euclidean metric) and a region D:

$$A = \text{Area}(D)\,, \quad L = \text{length}(\partial D)$$

The ratio L^2/A is bounded below:

$$\frac{L^2}{A} \geq C \quad \rightarrow \quad A \leq CL^2 \quad \rightarrow \quad A \leq \frac{1}{4\pi} L^2$$

Clearly for a given area A we can make L as big as we want.

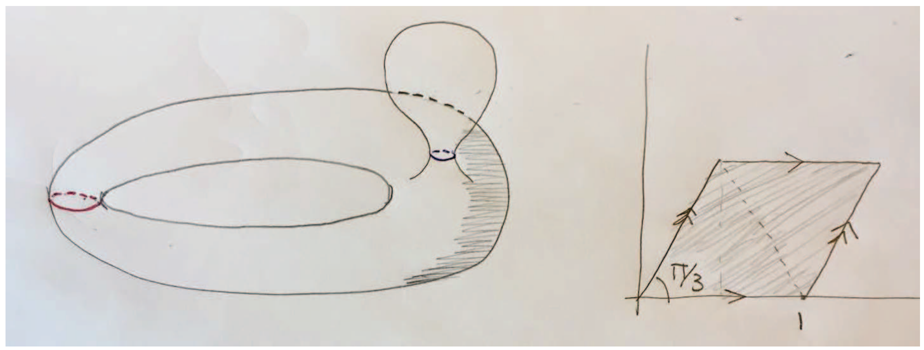


On a manifold M with a Riemannian metric one defines the systole as the length L of the shortest non-contractible curve.

For a given L, what can we tell about the area for general metric?

$$A \;\geq\; C(\text{systole})^2 = C\,L^2 \quad \rightarrow \quad \frac{L^2}{A} \leq C'$$

This is an isosystolic inequality. Can one find C'?



Loewner (1949) found that for a torus

$$\frac{L^2}{A} \leq \frac{2}{\sqrt{3}}$$

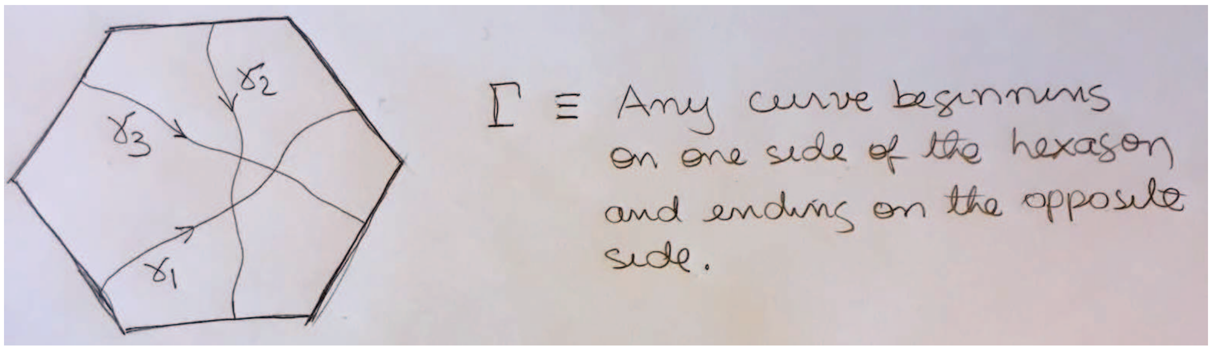
and the equality occurs for flat torus with $\tau = \exp(i\pi/3)$.

The constant is not known for genus two nor is the metric nor the conformal structure known.

In conformal geometry a metric is a function $\rho(z)$ with $z = x + iy$ and with a length element and area element

$$ds = \rho|dz|\,, \quad dA = \rho^2\, dx \wedge dy\,.$$

Define Γ as a collection of curves of interest on some surface M.



Pick a metric ρ on the surface.

Find the systole in Γ: $L(\Gamma, \rho)$

Find the area $A(\rho)$

Compute the ratio and **maximize** over ρ:

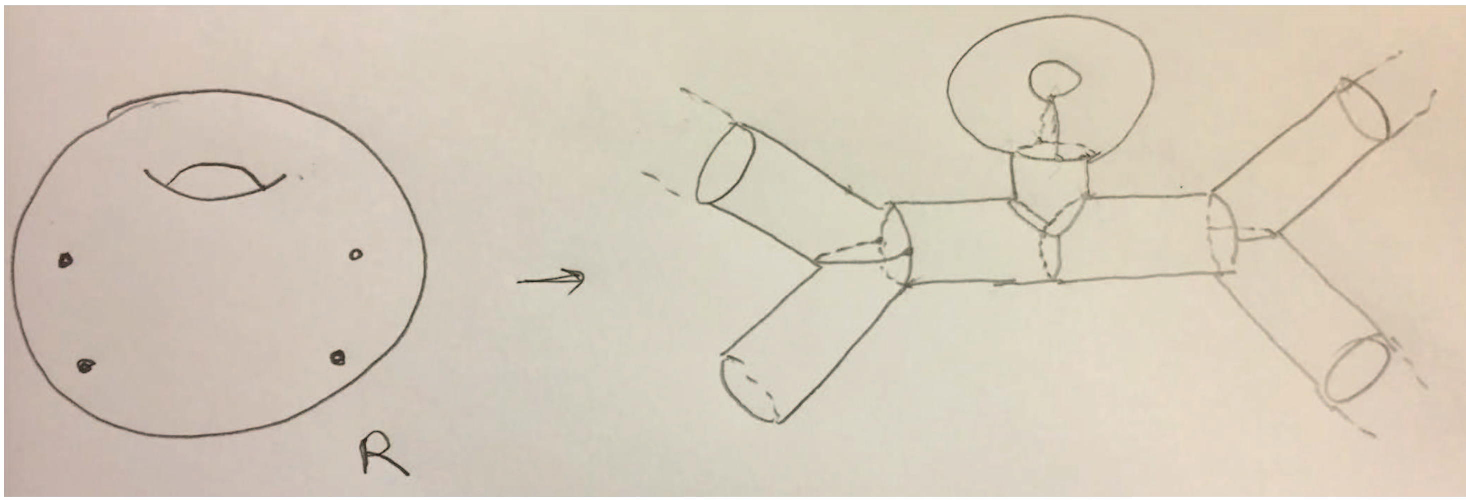
$$\text{Extremal length } = \lambda \equiv \text{Max}_\rho \left(\frac{L^2(\Gamma,\rho)}{A(\rho)}\right)$$

4 Since L^2/A is scale invariant, given a ρ one can scale it until the systole is equal to one. This becomes a minimal area problem. Extremal Length is a conformal invariant.

Minimal Area Problem:

> *Given a genus g Riemann surface with $n \geq 0$ marked points ($n \geq 2$ for $g = 0$) find the metric of minimal (reduced) area under the condition that the length of any non-contractible closed curve be greater than or equal to 2π.*

The extremal metric on the surface R tells us how the surface is built with vertices and propagators.



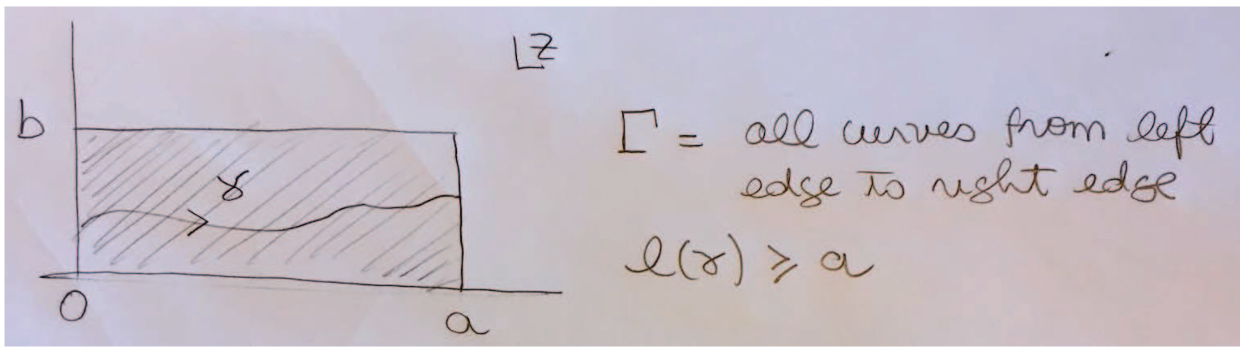
Expect that the surface is covered by bands of systolic geodesics!

The minimal area metrics are known for genus zero and any number of punctures (classical closed string field theory).

Some metrics are known for any $\mathcal{M}_{g,n}$ but some are not. Metrics are not known when bands of systolic geodesics cross.

Find simple case where systolic geodesics cross.

The basic problem in conformal geometry ask for the least area metric on a rectangle!



If we identify the left and right edges, the surface is an annulus, and the curves must begin and end with the same value of y.

The metric $\rho = 1$ is admissible: all length constraints work.

In fact, $\rho = 1$ is the minimal area metric!

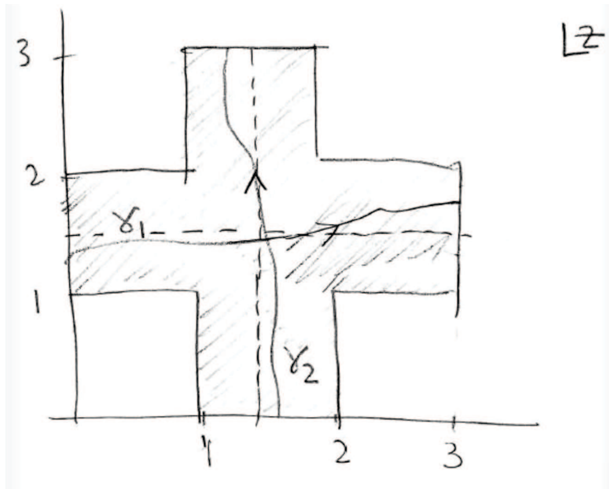
Both for the rectangle and for the annulus!

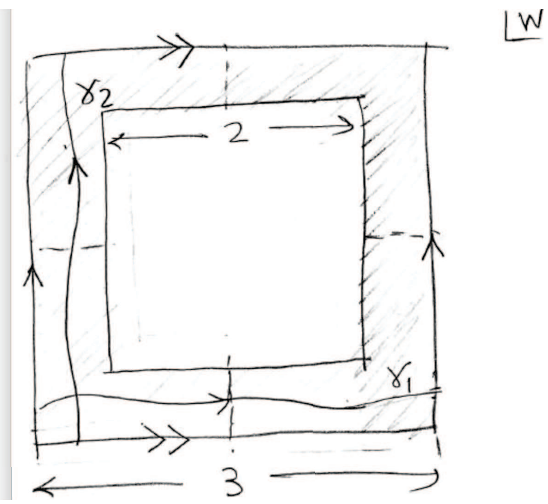
The extremal length is

$$\lambda = \frac{L^2}{A} = \frac{a^2}{ab} = \frac{a}{b} = \text{ modulus of a quadrilateral}$$

Two Surfaces

Swiss cross problem:





Torus with a boundary!

They ARE the SAME problem

∞ With $\ell_\gamma \geq 3$, the metric $\rho = 1$ is admissible.

Convex Optimization and Duality

Book by S. Boyd and L. Vandenberghe available online.

Let $C \subseteq \mathbb{R}^n$.

C is affine if the full line going through any two distinct points in C lies in C.

C is convex if the line *segment* joining any two distinct points in C lies in C.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if its domain is a convex set and for all x, y in that domain

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \forall t \in [0,1]\,.$$

A function $h : \mathbb{R}^n \to \mathbb{R}$ is affine if it is the sum of a linear function and a constant function.

It is natural to search for the minimum of a convex function and for the maximum of a concave function.

The Primal

Consider the following optimization problem presented in standard form

$$\begin{array}{rl} \text{Minimize} & f_0(x) \quad \text{over } x \in \mathcal{D} \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \ldots, m\,, \\ & h_i(x) = 0 \quad i = 1, \ldots, p\,, \end{array}$$

$x \in \mathbb{R}^n$ is the variable we minimize over.

$f_0(x)$ is the *objective* function

f_i's with $i \geq 1$ define the inequality constraints

h_i's define the equality constraints.

$\mathcal{D}$ is the common domain of all the functions.

The *feasible* set $\mathcal{F}$ is the subset of $\mathcal{D}$ where the constraints hold.

The *optimal* value $p^* = \min_{x \in \mathcal{F}} \; f_0(x)$.

x^* is said to be an *optimal* point if $x^* \in \mathcal{F}$ and $f_0(x^*) = p^*$.

The above problem is a convex program if:

– $f_0(x)$ and the $f_i(x)$ are convex functions over a convex $\mathcal{D}$

– the h_i are affine functions.

For a convex program any local minimum is in fact a global minimum.

Build the Lagrangian $L(x, \lambda, \nu)$:

$$L(x, \lambda, \nu) \equiv f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x) \,.$$

Minimizing over x yields a concave dual Lagrangian $\tilde{L}(\lambda, \nu)$:

$$\tilde{L}(\lambda, \nu) \equiv \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \,.$$

A key result:

$$\tilde{L}\,(\lambda \geq 0, \nu) \leq p^* \,.$$

With $\tilde{L}(\lambda, \nu)$ concave we define the **dual program**:

$$\begin{array}{lll} \text{Maximize} & \tilde{L}(\lambda, \nu) & \text{over} \ \ \lambda, \nu\,, \\ \text{subject to} & \lambda \geq 0\ . & \end{array}$$

This is a convex program. We have a *dual optimum* d^* for (λ^*, ν^*) if

$$d^* \ \equiv \ \sup_{\lambda \geq 0, \nu} \tilde{L}\,(\lambda, \nu) \ = \ \tilde{L}(\lambda^*, \nu^*)\,.$$

Moreover, $d^* \leq p^*$

Strong duality: $d^* = p^*$.

Strong duality is guaranteed when the primal is convex and there exists a feasible point x in the interior of $\mathcal{D}$ where the inequality constraints are strictly satisfied.

If strong duality holds with $x^* \in \mathcal{F}$ and (λ^*, ν^*), the inequality constraints and their Lagrange multipliers satisfy **complementary slackness**:

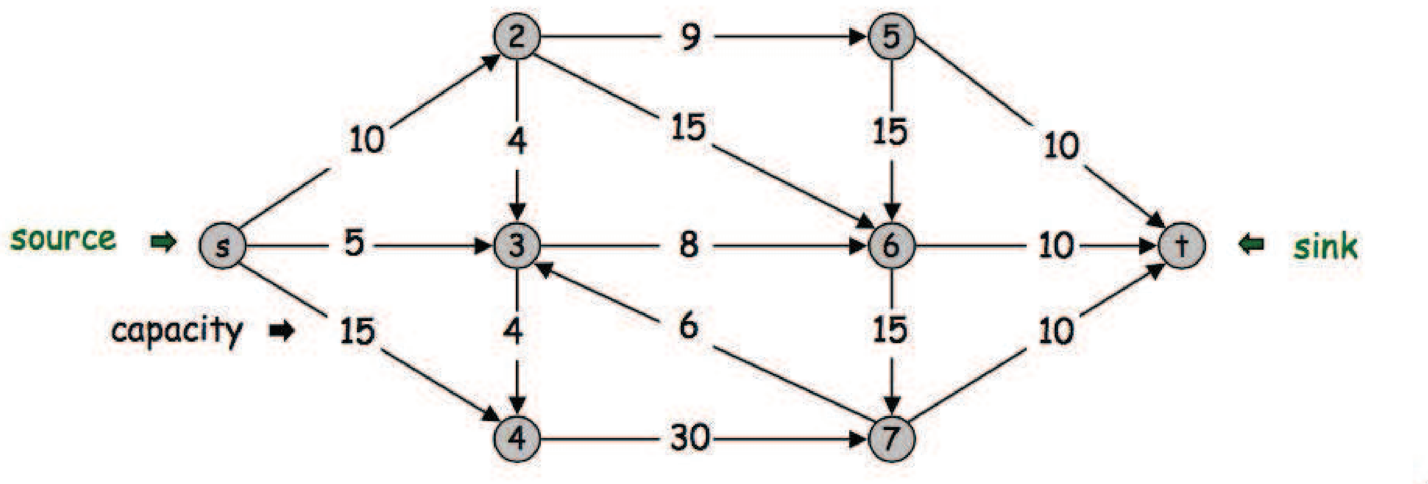
$$\lambda_i^* f_i(x^*) \ = \ 0\,, \quad \text{for each} \ \ i\,.$$

As a result

$$\begin{array}{c} f(x^*) < \ 0 \ \ \rightarrow \quad \lambda_i^* = 0 \\ \lambda_i^* \ > \ 0 \ \ \rightarrow \ \ f(x^*) = 0 \end{array}$$

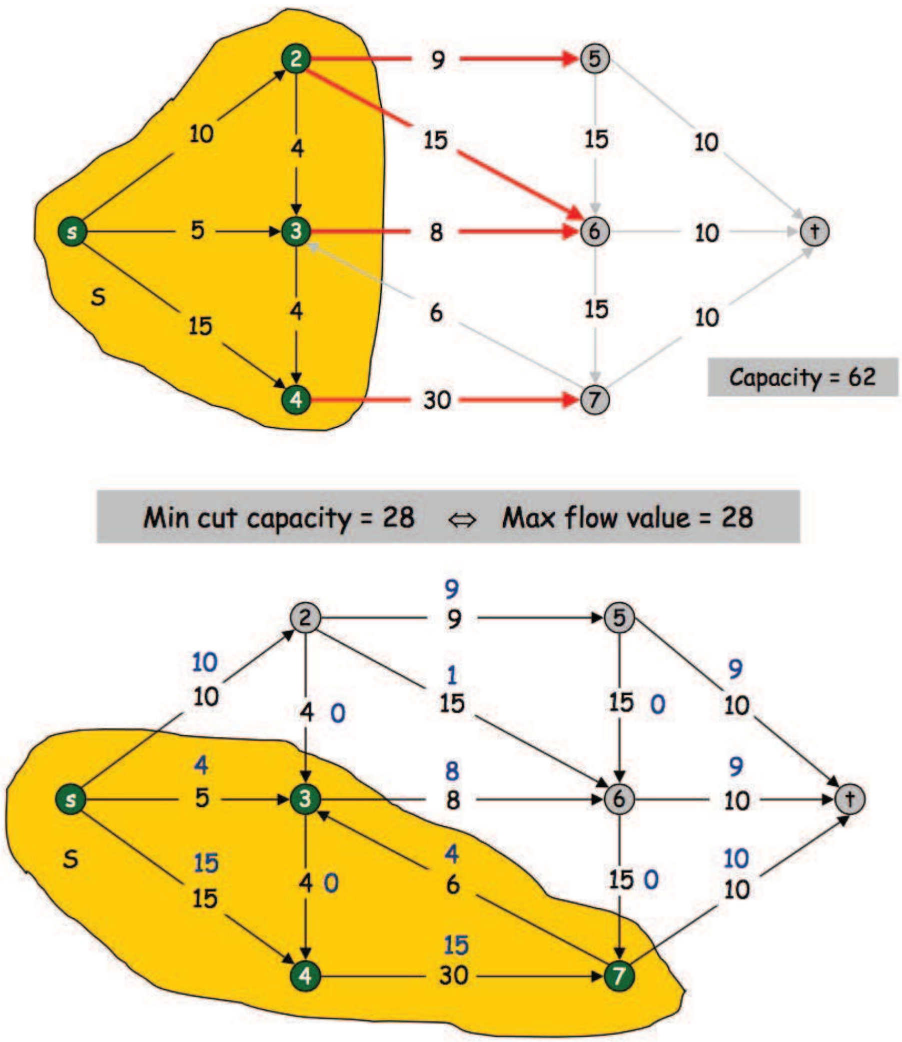
Max Flow - Min Cut Theorem

A source s a sink t, a series of nodes, oriented edges with capacities.

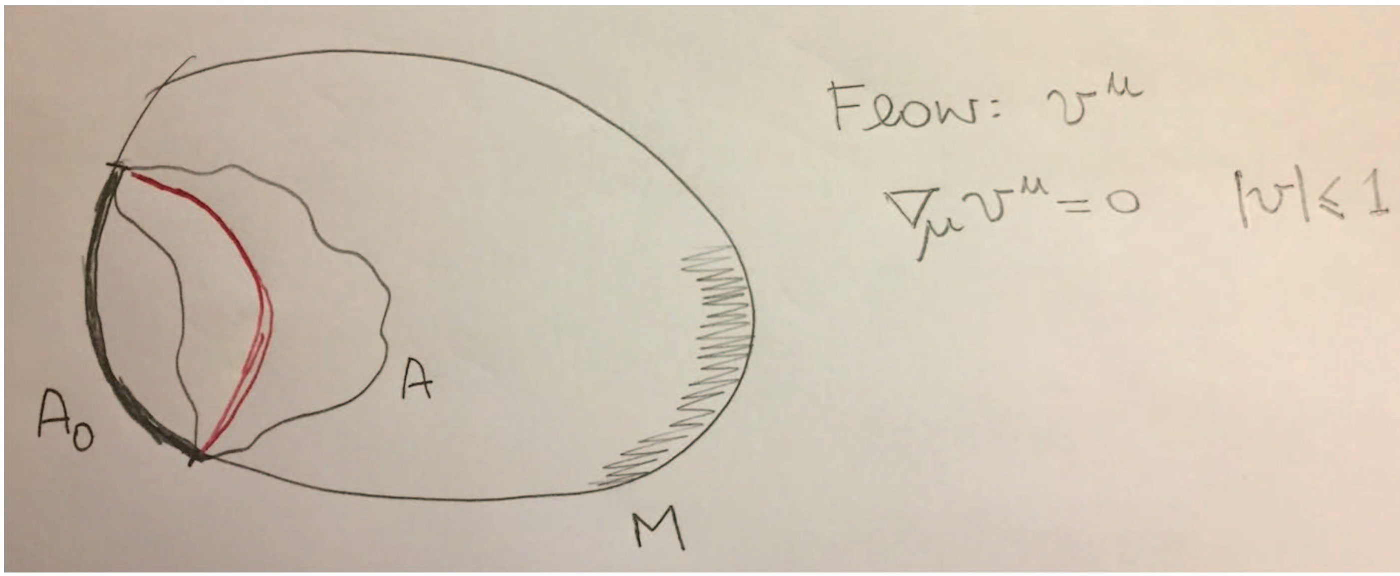


Question: What is the Max Flow one can get from source to sink?

A cut is evaluated by adding the flows that go out of the source region:



Freedman and Headrick's reformulation of Ryu-Takayanagi via Max Flow Min Cut.



$$\mathrm{Max}_v \left[\text{Flux of } v \text{ across } A_0 \right] = \mathrm{Min}_{A \sim A_0} \, \mathrm{area}(A)$$

The Primal and the Dual Programs

Riemann surface M

Homology 1-cycles $C_\alpha \in H_1(M)$, $\alpha \in J$

Minimize the area of M with all curves in C_α have length $\geq \ell_\alpha$.

$g_{\mu\nu} = \Omega\, g^0_{\mu\nu}$, with g^0 fiducial.

Area of M is $\int_M d^2x \sqrt{g^0}\,\Omega\,.$

Parameterize curves as $y^\mu(t) : [0,1] \to M$. length$(m) = \int_m \sqrt{\Omega}|\dot{y}|_{\circ}$.

The minimal area problem,

$$\text{Minimize} \quad \int_M \sqrt{g^0}\,\Omega \quad \text{over} \quad \Omega \geq 0$$

$$\text{subject to:} \qquad \ell_\alpha - \int_m \sqrt{\Omega}\,|\dot{y}|_{\circ} \leq 0\,, \quad \forall\, \alpha \in J,\, m \in C_\alpha\,.$$

The length condition is applied to an infinite set of curves.

$$\text{Calibration one-form } u: \quad du = 0\,, \ |u| \leq 1\,.$$

The period of a calibration in a homology class C constrains the length of all curves $\gamma \in C$

$$\text{length}(\gamma) \geq \left| \int_C u \right| \qquad \text{(Min Cut = Max Flow)}$$

Introduce a calibration u^α with period ℓ_α for each C_α.

The new program, the primal is convex:

$$\text{Minimize} \quad \int_M \sqrt{g^0}\,\Omega \qquad \text{over} \quad \Omega, u^\alpha$$

$$\text{subject to:} \quad |u^\alpha|_0^2 - \Omega \leq 0\,,$$

$$du^\alpha = 0\,,$$

$$\ell_\alpha - \int_{C_\alpha} u^\alpha = 0\,, \ \forall \alpha \in J\,.$$

Systolic geodesics in C_α have length ℓ_α. On them $|u^\alpha|_0^2 = \Omega$ ($|u| = 1$) and $\widehat{u}^\alpha$ is tangent to them.

Dual program: functions φ^α on the surface and positive constants ν^α

$$\text{Maximize} \quad \left[2 \sum_\alpha \nu^\alpha \ell_\alpha - \int_{M'} d^2x \sqrt{g^0} \left(\sum_\alpha |d\varphi^\alpha|_0 \right)^2 \right] \quad \text{over} \quad \varphi^\alpha, \nu^\alpha$$

$$\text{subject to:} \qquad \Delta\varphi^\alpha|_{m_\alpha} = \nu^\alpha \,,$$

$$\varphi^\alpha|_{\partial M} = 0 \,, \quad \forall \alpha \in J \,.$$

Here $|\cdot|_0$ denotes norm in the fiducial metric

m_α is a choice of a curve in C_α.

φ^α has discontinuity ν^α across m_α.

First term in objective tries to make ν^α large;

A non-zero jump, however, forces $d\varphi^\alpha \neq 0$, making the second term larger in magnitude

First term is linear second is quadratic, so there should be a maximum.

Any trial values for ν^α and φ^α give rigorous lower bounds for the area.

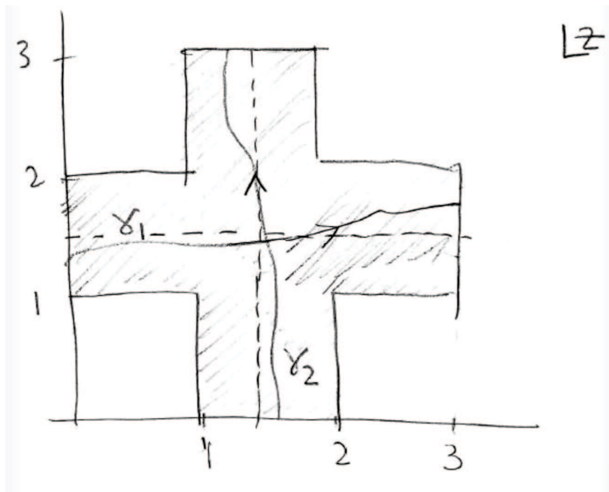
Complementary slackness and strong duality are used to show that

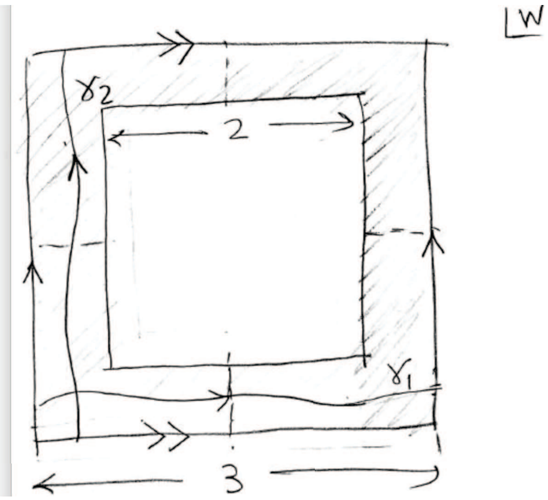
1. The α-systolic geodesics (if any) are the level sets of φ^α wherever it has non-zero gradient.

2. The area for the extremal metric is the sum of areas of *flat rectangles* of height ν^α and length ℓ_α.

$$\sum_\alpha \nu^\alpha \ell_\alpha \;=\; \text{Extremal area}\,.$$

The Results





Primal results

4.675 148 996 Nc =30

4.675 147 657 Nc=40

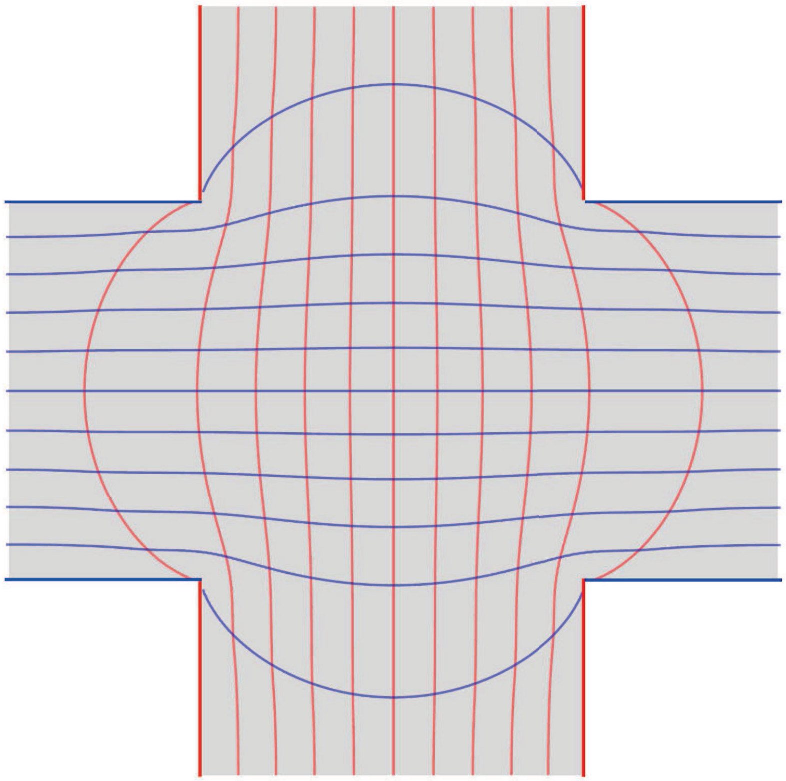
4.675 146 317 Nc=45

Extremal area = 4.675 145 ($\pm$ 1) .

Dual result

4.675 144 775 Nc=48

Systolic geodesics for the swiss cross

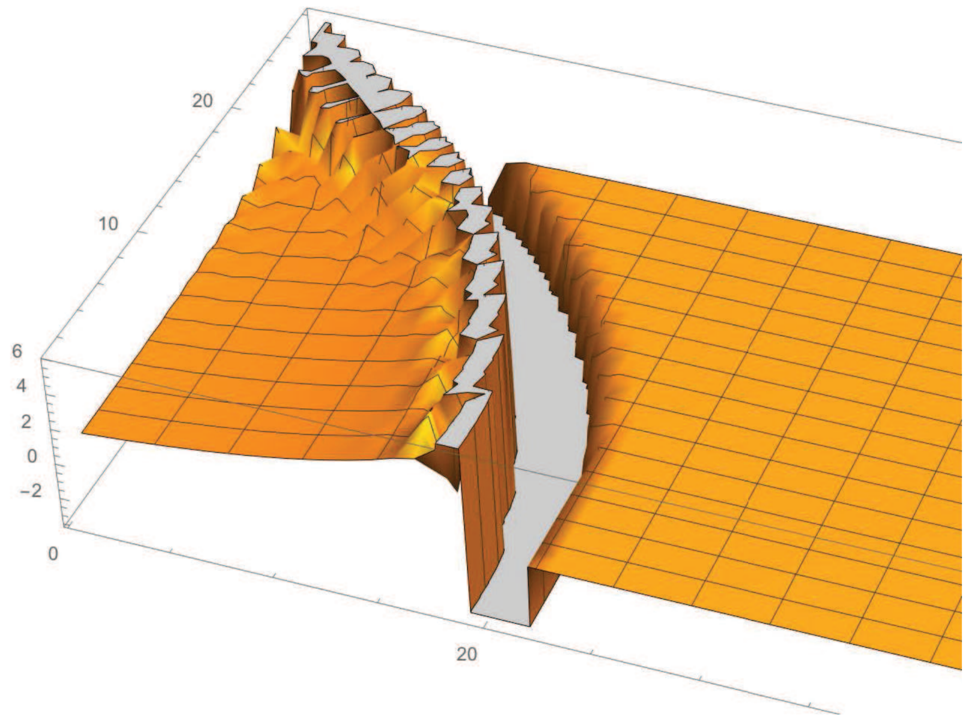


In the region with two bands of geodesics the curvature is positive.

21 In the region with one band of geodesic the metric is flat.

In the boundary between the two regions we got negative curvature.

Behavior of the curvature:



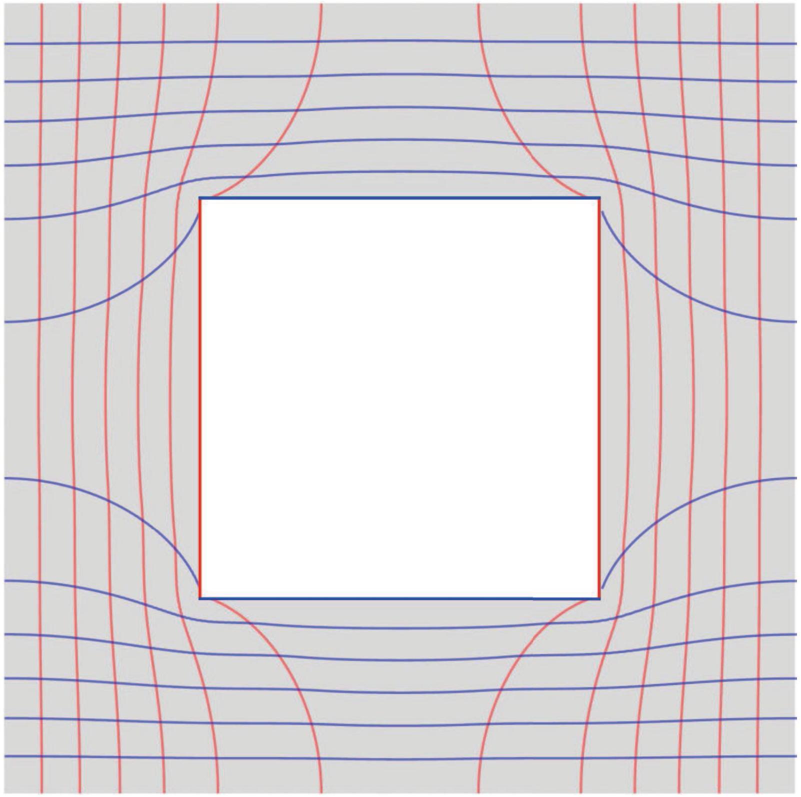
Bulk positive curvature and negative line curvature singularity!

Metric is complicated in general but area formula is simple and holds.

For a swiss cross of $\ell_s = 2$, for example, one finds $A_* = 2.806975$ and height of foliation $\nu = 0.701744$ Then,

$$A_* = 2\,\nu\ell_s = 2(0.701744) * 2 = 2.80698$$

Systolic geodesics for the torus with a boundary



Conclusions and open questions

- First example of extremal metric with crossing bands of systolic geodesics.

- The metrics may be complicated but the heights/area relation for the extremal metric is simple.

- Improve accuracy and understand better the behavior of curvature near corners

- Figure out the extension to other tori with boundary or tori with holes, as relevant to string field theory.

- Cases when more than two foliations of geodesics cross.

- Duality with inequality constraints in string theory?