Generalized Kinematics & Dynamics

A para-Hermitian Geometry for DFT

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String Dualities and Geometry
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Question

- What is the relation between

"Doubling" the tangent space of manifold M

extended vector bundle

$$\mathbb{T}M = TM \oplus T^*M$$

Generalized Geometry

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Generalized Geometry

"Doubling" the underlying manifold M

doubled space (locally)

$$\mathcal{P} \cong M \times \tilde{M}$$

Double Field Theory

Context & Overview

Context

- Arena of string theory: doubled space

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- Coordinates for winding and momentum modes

➡ Double Field Theory [Siegel; Hull, Zwiebach]

- Phase space of string probes

➡ "Metastring Theory" [Freidel, Minic, Leigh]

Context

- Section condition - restrict coordinate dependence
- Tools from Generalised Geometry
  - ➡ Courant bracket, Dorfman derivative
- Understand nature of doubled space
- Understand relation to extended tangent bundle of GG

Key Points

- Para-Hermitian Manifold $\mathcal{P}$
- Bi-Lagrangian Splitting & Foliation
- Isomorphism between $\mathbb{T}M$ and $T\mathcal{P}$

Outline

1. Overview of GG and DFT
2. para-Hermitian Geometry
3. Foliation & Isomorphism

Overview of GG and DFT

Generalized Geometry

"Double" the tangent space of manifold M :

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Generalized Geometry

"Double" the tangent space of manifold M :

extended vector bundle

$$\mathbb{T}M = TM \oplus T^*M$$

- Generalized vectors: $X = (x, \alpha) \in \Gamma(\mathbb{T}M)$
- Natural $O(d,d)$ pairing: $\eta(X, Y) = \iota_x \beta + \iota_y \alpha$
- Dorfman bracket: $[\![X, Y]\!] = ([x, y], \mathcal{L}_x \beta - \iota_y \mathrm{d}\alpha)$

Double Field Theory

"Double" the underlying manifold M :

doubled space (locally)

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Double Field Theory

"Double" the underlying manifold M :

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- Doubled coordinates: $x^M = (x^\mu, \tilde{x}_\mu)$, $\partial_M = (\partial_\mu, \tilde{\partial}^\mu)$
- $O(d,d)$ metric on $\mathcal{P}$: $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- D-bracket:

$$[X,Y]_{\mathrm{D}}^M = X^N \partial_N Y^M - Y^N \partial_N X^M + \eta^{MN} \eta_{KL} \partial_N X^K Y^L$$

The Constraint

"Section Condition"

$$\eta^{AB}\partial_A \otimes \partial_B = 0$$

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"Section Condition"

$$\eta^{AB}\partial_A \otimes \partial_B = 0$$

Fields are independent of $\tilde{x}$

- Simple solution: $\tilde{\partial} = 0$
- If imposed: DFT reduces to generalized geometry

DFT to GG

- Want to relate $\mathbb{T}M$ and $T\mathcal{P}$

need to identify

$$\frac{\partial}{\partial \tilde{x}} \leftrightarrow \mathrm{d}x$$

- Formally possible if section condition is imposed
- C/D-brackets $\rightarrow$ Courant/Dorfman brackets

GG vs DFT

- After imposing section condition

  ➡ only depend on half the coordinates

  ➡ DFT reduces to GG

- But different ways of picking spacetime M

  ➡ Which half of $\mathcal{P}$ is base for GG?

Need extra geometric information to relate GG to DFT!

para-Hermitian Geometry

para-Hermitian Geometry

- (almost) para-Hermitian manifold $\mathcal{P}$ with (η, ω) and $K := \eta^{-1}\omega$

doubled space = para-Hermitian manifold

- metric η and symplectic form ω

$$K^2 = +1$$

- bi-Lagrangian structure $K : T\mathcal{P} = L \oplus \tilde{L}$

$$K\big|_L = +1 \qquad P = \frac{1}{2}(1 + K)$$

$$K\big|_{\tilde{L}} = -1 \qquad \tilde{P} = \frac{1}{2}(1 - K)$$

Doubled Tangent Bundles

extended vector bundle over M **:**

$$\mathbb{T}M = TM \oplus T^*M$$

tangent space of $\mathcal{P}$ **:**

$$T\mathcal{P} = L \oplus \tilde{L}$$

Doubled Tangent Bundles

extended vector bundle over M :

$$\mathbb{T}M = TM \oplus T^*M$$

para-Hermitian manifold

$$\mathcal{P} \cong M \times \tilde{M}$$

tangent space of $\mathcal{P}$:

$$T\mathcal{P} = L \oplus \tilde{L}$$

Lagrangian subspaces

Doubled Tangent Bundles

extended vector bundle over M **:**

$$\mathbb{T}M = TM \oplus T^*M$$

tangent space of $\mathcal{P}$ **:**

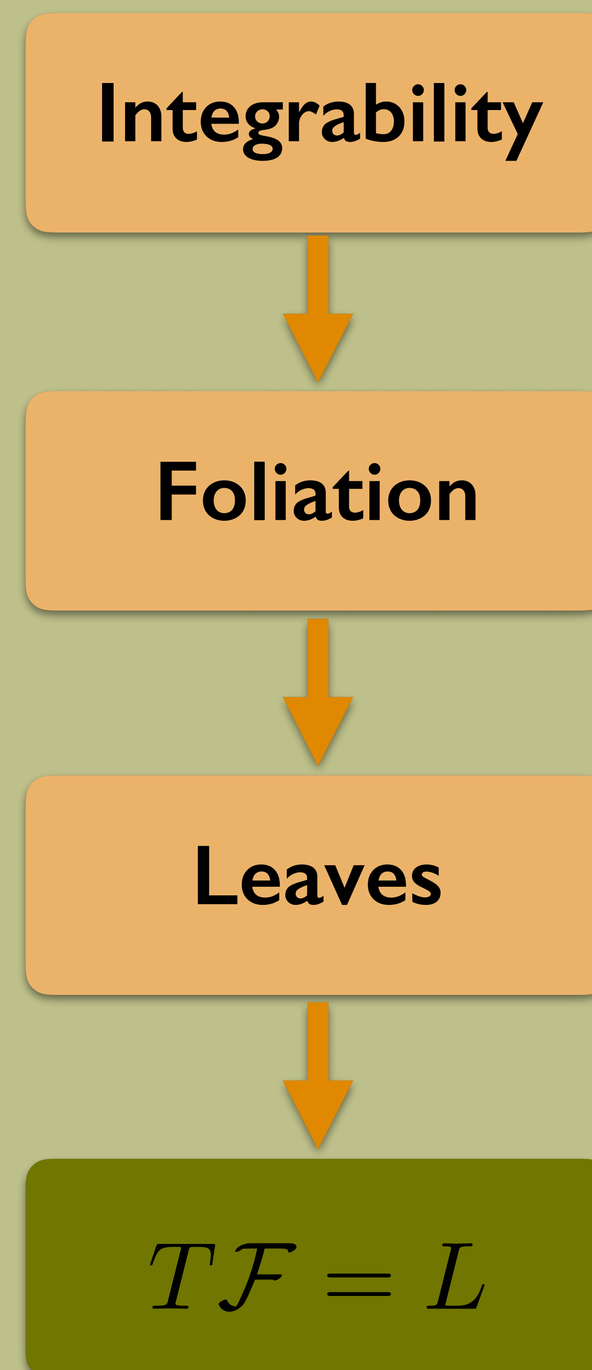
$$T\mathcal{P} = L \oplus \tilde{L}$$

What is the relation?

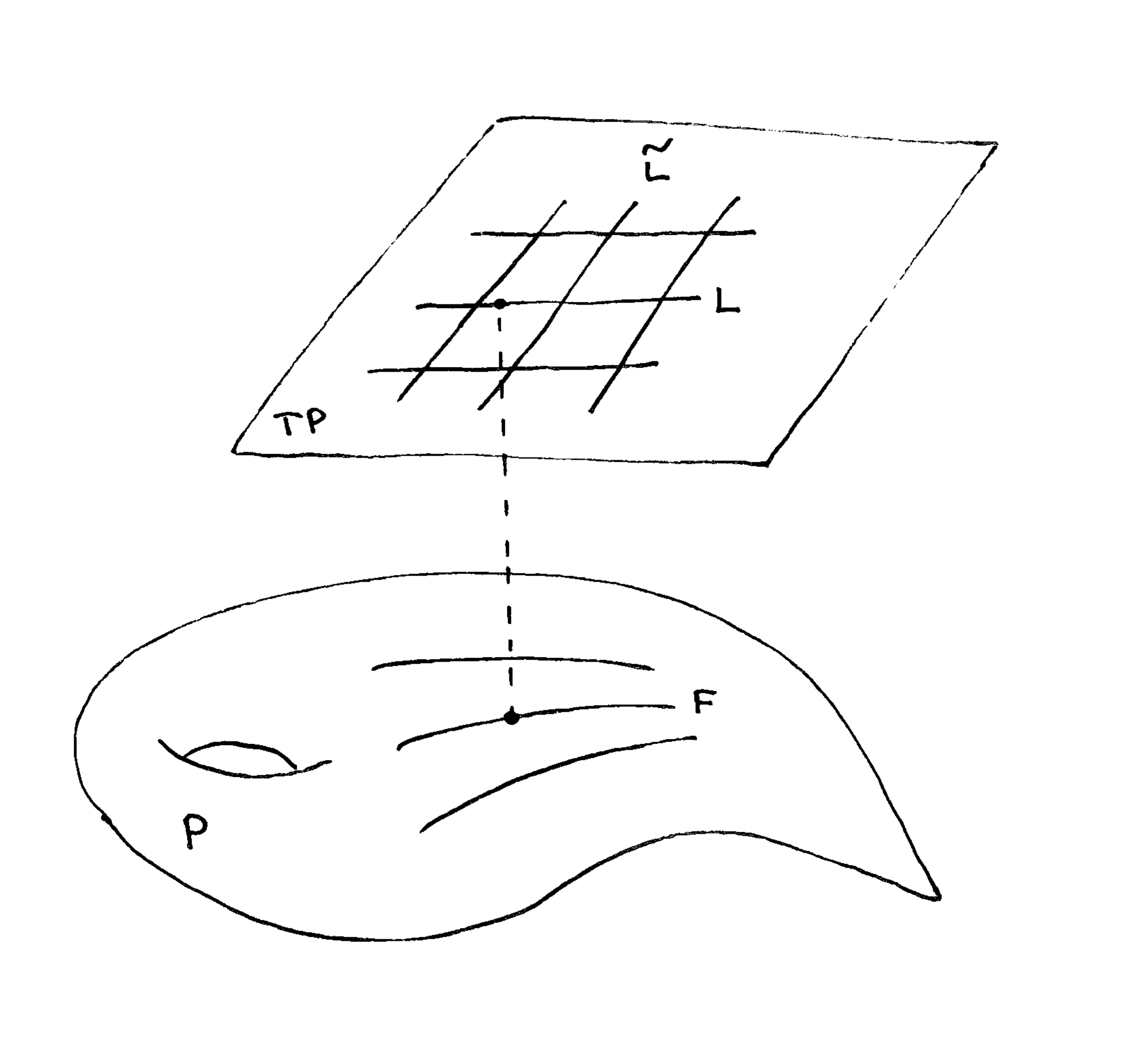
Foliation & Isomorphism

Foliation

- If L is integrable, have d-dimensional foliation $\mathcal{F}$ of $\mathcal{P}$ (Frobenius Theorem)
- $\mathcal{F}$ is partition of $\mathcal{P}$ as set of leaves with $T\mathcal{F} = L$
- For every point $p \in \mathcal{P}$ there is a unique leaf M_p in $\mathcal{F}$ that passes through p
- Foliation $\mathcal{F}$ is disjoint union of leaves $\bigsqcup M_p$



Foliation



Integrability

Foliation

Leaves

$T\mathcal{F} = L$

Isomorphism

Construct invertible map

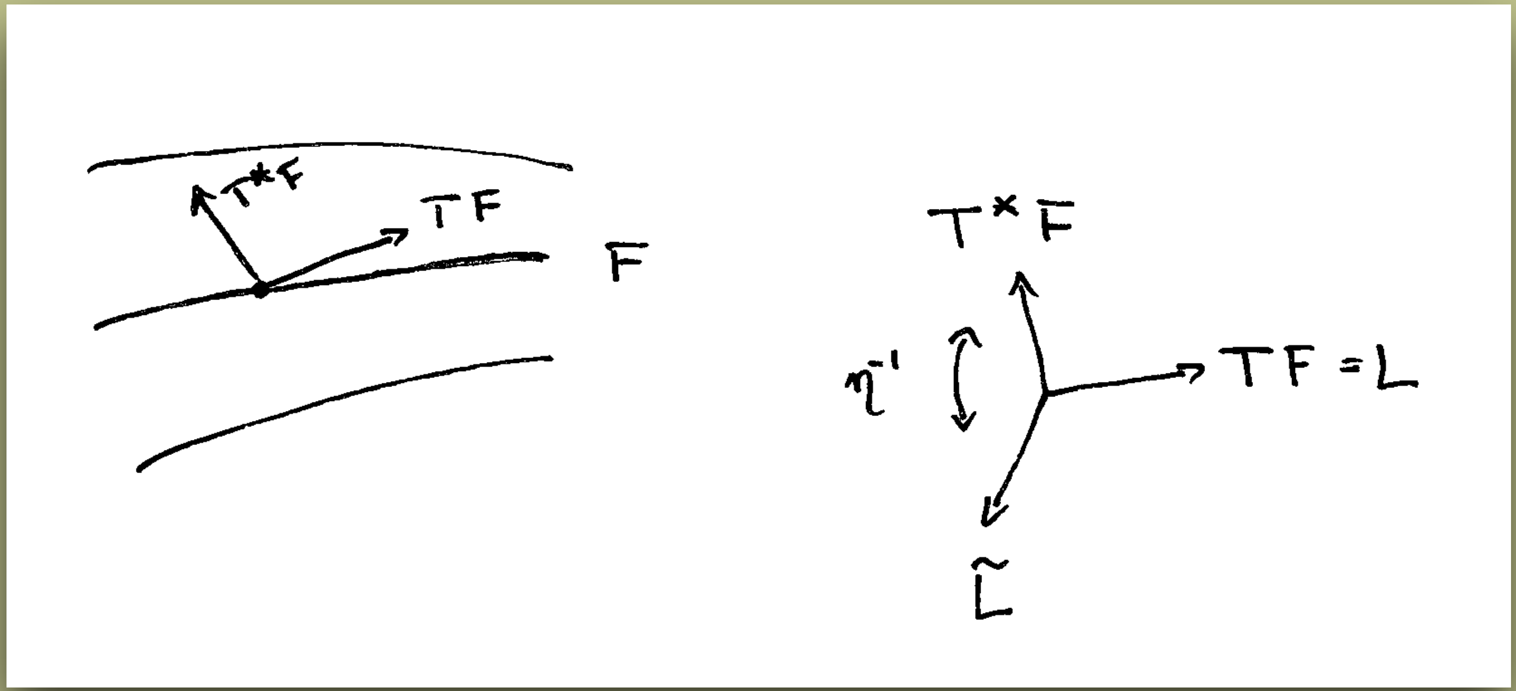
$$\rho : T\mathcal{P} \to \mathbb{T}\mathcal{F} = T\mathcal{F} \oplus T^*\mathcal{F}$$

$$\rho : X \mapsto \big(P(X), \eta(\tilde{P}(X))\big) = (x, \alpha)$$
$$\rho^{-1} : (x, \alpha) \mapsto x + \eta^{-1}(\alpha) := x + \tilde{x} = X$$

- Vector part of map is trivial
- One-form part of map uses η

Isomorphism

- Tangent and cotangent space of $\mathcal{F}$
- Mapped to L and $\tilde{L}$



Isomorphism

$$T^*\mathcal{F} = \rho(\tilde{L})$$

$$\frac{\partial}{\partial \tilde{x}} \leftrightarrow \mathrm{d}x$$

Brackets

- Map between D-bracket and Dorfman bracket

$$\rho([X,Y]_{\mathsf{D}}) = [\![\rho(X),\rho(Y)]\!]$$

D-bracket on
$T\mathcal{P} = L \oplus \tilde{L}$

Dorfman bracket on
$\mathbb{T}M = TM \oplus T^*M$

Importance of ω

- Doubled space is para-Hermitian
- Presence of symplectic structure ω gives bi-Lagrangian splitting and the map ρ
- Spacetime arises naturally

Spacetime M is leaf of foliation $\mathcal{F}$
with tangent $T\mathcal{F} = L = \mathrm{Im}\, P$.

- Basis for generalized geometry

Outlook

Outlook

- Use map to translate concepts of GG to doubled space
  - ➡ Generalized connections
  - ➡ Fluxes and twisted brackets
- Similar construction for Exceptional Geometry & EFT
  - ➡ Structures on exceptionally extended space
  - ➡ Projectors and subspaces

Summary

- Doubled space is para-Hermitian manifold
- Bi-Lagrangian splitting of tangent bundle
- Foliation of doubled space
- Isomorphism between $\mathbb{T}M$ and $T\mathcal{P}$

Extensions

The Nijenhuis Tensor

For the bi-Lagrangian structure $K = \eta^{-1}\omega$

$$
\begin{aligned}
N_K(X,Y,Z) &= \mathring{\nabla}_Y \omega(X, K(Z)) - \mathring{\nabla}_X \omega(Y, K(Z)) \\
&\quad + \mathring{\nabla}_{K(Y)} \omega(X, Z) - \mathring{\nabla}_{K(X)} \omega(Y, Z) \\
&= \mathrm{d}\omega(K(X), K(Y), K(Z)) + \mathrm{d}\omega(X, Y, K(Z)) \\
&\quad + 2\mathring{\nabla}_{K(Z)} \omega(X, Y)
\end{aligned}
$$

Spacetime Geometry

$$\{M,\ [\cdot\,,\cdot],\ g,\ \nabla\}$$

Spacetime Geometry

set of points with differentiable structure

Lie bracket on TM

$$\{M,\ [\cdot\,,\cdot],\ g,\ \nabla\}$$

Spacetime Geometry

set of points with differentiable structure

Lie bracket on TM

$$\{M,\ [\cdot\,,\cdot],\ g,\ \nabla\}$$

metric - measure of distance

connection - parallel transport

Kinematics & Dynamics

$$\{M,\ [\cdot\,,\cdot],\ g,\ \nabla\}$$

Kinematical component

$$(M, [\cdot, \cdot])$$

- structure of spacetime
- symmetry algebra (diffeomorphisms)

Dynamical component

$$(g, \nabla)$$

- invariant action
- determines curvature
- fixes torsion

Generalization

- Generalization suitable for strings in a doubled space
- Kinematical structure encoded in (η, ω)

➡ Neutral metric $\eta \in O(d,d)$

➡ Symplectic form $\omega \in Sp(2d)$

- Dynamical d.o.f. in generalized metric $\mathcal{H}(g,B)$

```
\mathbb{T}M = TM\oplus T^*M
T\mathcal{P} = L \oplus \tilde{L}
T\mathcal{F} = L
\rho: T\mathcal{P} \rightarrow \mathbb{T}\mathcal{F} = T\mathcal{F} \oplus T^*\mathcal{F}
N_K = 0


T^*\mathcal{F} = \rho(\tilde{L})
X = (x,\alpha) \in \Gamma(\mathbb{T}M)
\bigsqcup M_p

T\mathcal{F} = L = \mathrm{Im}P

\rho^{-1}: (x,\alpha)\mapsto x+\eta^{-1}(\alpha) := x+\tilde{x} = X
\rho: X\mapsto \big(P(X),\eta(\tilde{P}(X))\big) = (x,\alpha)
\rho([X,Y]_D) = [\![\rho(X),\rho(Y)]\!]
[\![X,Y]\!] = ([x,y],\mathcal{L}_x\beta - \iota_y\dd\alpha)
[X,Y]_\mathsf{D}^M = X^N\partial_N Y^M - Y^N \partial_N X^M + \eta^{MN}\eta_{KL}\partial_NX^KY^L
```

M $\mathcal{F}$ $\mathcal{P}$

ρ $T\mathcal{P}$