Non-geometric fluxes and non-associativity in M-theory

DIETER LÜST (LMU, MPI)

String Dualities and Geometry, Bariloche, January, 2018

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M. Günaydin, D.L., E. Malek, arXiv:1607.06474
D.L., E. Malek, R. Szabo, arXiv:1705.09639
D.L., E. Malek, M. Syvari, arXiv:1710.05919

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Outline:

II) Non-associative, octonionic algebra in M-theory from non-geometric R-flux - M2 brane phase space

II) Non-associative, octonionic algebra in M-theory from non-geometric Kaluza-Klein monopoles - M-wave phase space

III) Free M-theory phase space

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Geometry in general depends on what kind of objects you test it.

Point particles in classical Einstein gravity „see“ continuous Riemannian manifolds:

- $[x^i, x^j] = 0$

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Point particles in classical Einstein gravity „see“ continuous Riemannian manifolds:

- $[x^i, x^j] = 0$

Strings, membranes may see space-time in a different way $\Rightarrow$ new string geometry.

Open strings in 2-dim. B-field background:

$$[x^i, x^j] \sim B^{ij}$$

F. Ardalan, H. Arfaei, M. Sheikh-Jabbari (1999); N. Seiberg, E. Witten (19999

Consider closed strings on three-dimensional string flux backgrounds:

Chain of three T-duality transformations:

(Hellerman, McGreevy, Williams (2002); C. Hull (2004); Shelton, Taylor, Wecht (2005); Dabholkar, Hull, 2005)

$$H_{ijk} \xrightarrow{T_i} f^i_{jk} \xrightarrow{T_j} Q^{ij}_k \xrightarrow{T_k} R^{ijk}\,, \qquad (i,j,k = 1,\ldots,3)$$

R-flux: locally non-geometric background (similar to asymmetric orbifolds).

SO(3,3) Double field theory: $R^{ijk} = 3\hat{\partial}^{[k}\beta^{ij]}$

bi-vector

Non-associative 6-dim. phase space of a probe string with 3 momenta and 3 coordinates in R-flux background:

$$[x^i, x^j] = i\frac{l_s^3}{\hbar} R^{ijk} p_k$$

R. Blumenhagen, E. Plauschinn, arXiv:1010.1263.
D.L., arXiv:1010.1361;
D. Mylonas, P. Schupp, R.Szabo, arXiv:1207.0926
I. Bakas, D.L., arXiv:1309.3172

$$[x^i, p^j] = i\hbar\delta^{ij}\,, \qquad [p^i, p^j] = 0$$

$$\Longrightarrow \quad [x^i, x^j, x^k] \equiv \frac{1}{3}\left[[x^1, x^2], x^3\right] + \text{cycl. perm.} = l_s^3 R^{ijk}$$

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This algebra can be derived from closed string CFT and applying duality symmetries.

R. Blumenhagen, A. Deser, D.L. , E. Plauschinn, F. Rennecke, arXiv:1106.0316
C. Condeescu, I. Florakis, D. L., arXiv:1202.6366
D. Andriot, M. Larfors, D.L. , P. Patalong:arXiv:1211.6437
C. Blair, arXiv:1405.2283
I. Bakas, D.L., arXiv:1505.04004

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How is the R-flux algebra related to other known non-associative algebras, in particular to the algebra of the octonions?

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Can one lift the R-flux algebra of closed strings to M-theory?

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On the mathematical side:

How is the R-flux algebra related to other known non-associative algebras, in particular to the algebra of the octonions?

Our conjecture:

the answers to these two questions are closely related

On the physics side:

Can one lift the R-flux algebra of closed strings to M-theory?

R-flux algebra from octonions:

There exist four normed division algebras over $\mathbb{R}$:

$$\mathbb{R}\,,\mathbb{C}\,,\mathbb{Q}\,,\mathbb{O}$$

Division algebra of real octonions $\mathbb{O}$: non-commutative,
non-associative

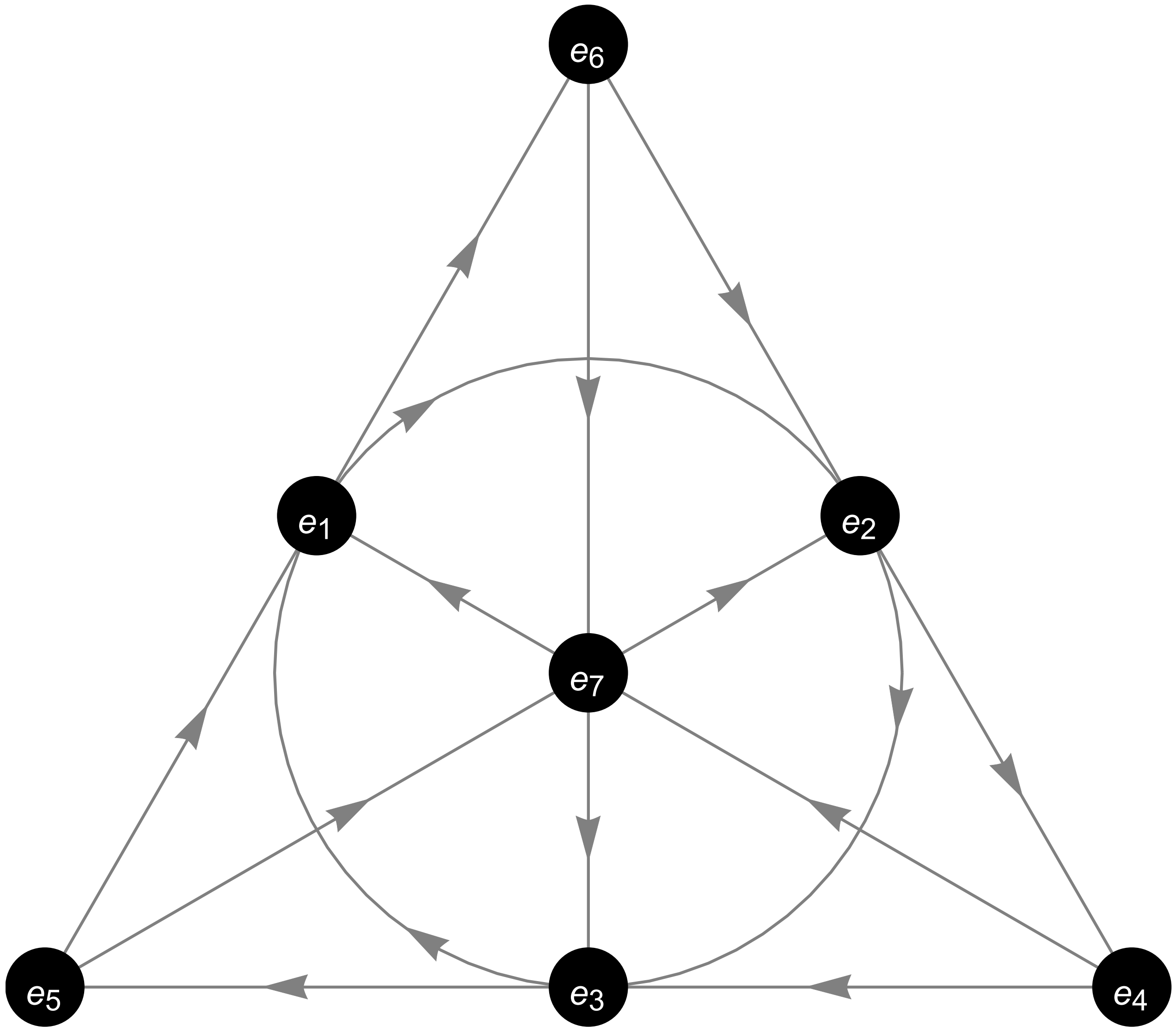
Besides the identity, there are seven imaginary units e_A

$$e_A e_B = \eta_{ABC}\, e_C \qquad (A = 1\,\ldots,7)$$

$$\eta_{ABC} = 1 \Longleftrightarrow (ABC) = (123),\ (516),\ (624),\ (435),\ (471),\ (572),\ (673)$$

This algebra is invariant under $G_2 \subset SO(7)$.

Triangle of the seven octonions:



Split indices: $e_i\,,\ \ e_{(i+3)} = f_i\,,\qquad \text{for } i = 1, 2, 3$

and e_7

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Commutators:

$$\begin{aligned}
[e_i, e_j] &= 2\epsilon_{ijk} e_k\,, & [e_7, e_i] &= 2f_i\,,\\
[f_i, f_j] &= -2\epsilon_{ijk} e_k\,, & [e_7, f_i] &= -2e_i\,,\\
[e_i, f_j] &= 2\delta_{ij} e_7 - 2\epsilon_{ijk} f_k
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\end{aligned}
$$

3-brackets:

$$
\begin{aligned}
[e_i, e_j, f_k] &= 4\epsilon_{ijk} e_7 - 8\delta_{k[i} f_{j]}\,,\\
[e_i, f_j, f_k] &= -8\delta_{i[j} e_{k]}\,,\\
[f_i, f_j, f_k] &= -4\epsilon_{ijk} e_7\,,\\
[e_i, e_j, e_7] &= -4\epsilon_{ijk} f_k\,,\\
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[f_i, f_j, e_7] &= 4\epsilon_{ijk} f_k
\end{aligned}
$$

Associator $\quad [X, Y, Z] \equiv (XY)Z - X(YZ)$

Now we rename the octonions in the following way:

$$p_i = -i\lambda \frac{1}{2} e_i\,, \qquad x^i = i\lambda^{1/2}\frac{\sqrt{R}}{2} f_i \quad, \quad I = i\lambda^{3/2}\frac{\sqrt{R}}{2} e_7$$

Contraction of octonionic algebra: $\lambda \to 0$

$$[f_i, f_j] = -2\epsilon_{ijk} e_k \quad \Longrightarrow \quad [x^i, x^j] = iR\epsilon^{ijk} p_k$$

$$[e_i, e_j] = 2\epsilon_{ijk} e_k \quad \Longrightarrow \quad [p_i, p_j] = 0$$

$$[f_i, e_j] = -\delta^i_j e_7 + \epsilon^i{}_{jk} f_k \quad \Longrightarrow \quad [x^i, p_j] = i\delta^i_j I$$

$$[x_i, I] = 0 = [p_i, I]$$

$$[f_i, f_j, f_k] = -4\epsilon_{ijk} e_7 \quad \Longrightarrow \quad [x^i, x^j, x^k] = R\epsilon^{ijk} I$$

Agrees with non-associative R-flux algebra !

What is the role of un-contracted algebra?

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- e^7 additional M-theory coordinate x^4

$\Rightarrow$ Four coordinates: f_1, f_2, f_3, e_7

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Lift of string R-flux algebra to non-geometric M-theory background:

- e^7 additional M-theory coordinate x^4

$\Rightarrow$ Four coordinates: f_1, f_2, f_3, e_7

- but no additional momentum.

$\Rightarrow$ Three momenta: e_1, e_2, e_3

Consider IIA string on twisted torus
Heisenberg Nilmanifold N3: f^{i}_{jk}

2 T-dualities: $f^{i}_{jk} \longrightarrow R^{ijk}$

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Uplift to M-theory: add additional circle: N3 x $S^1_{x^4}$

- 3-dim IIA flux background $\Leftrightarrow$ 4-dim M-theory flux background
- two T-dualities $\Leftrightarrow$ 3 U-dualities

(Need third duality along the M-theory circle to ensure right dilaton shift.)

Using SL(5) exceptional field theory one can obtain the R-flux in M-theory.

C. Blair, E. Malek, arXiv:1412.0635.

Non-geometric R-flux background in M-theory, which is dual to twisted torus:

R-flux: $$R^{\alpha,\beta\gamma\delta\rho} = 4\hat{\partial}^{\alpha[\beta}\Omega^{\gamma\delta\rho]}$$

tri-vector

This is supposed to be still a consistent M-theory background, which can be probed by M2 branes.

Four coordinates: x^1, x^2, x^3, x^4

What are the possible conjugate momenta (or windings)?

- Consider cohomology of Heisenberg Nilmanifold:

$$H^1(N3 \times S^1, \mathbb{R}) = \mathbb{R}^3$$

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- Alternatively consider Freed-Witten anomaly:

R-Flux with probe momentum p_4 along x^4

$$\Updownarrow$$

R-Flux with D0 branes.

Dualize to IIB: H-flux with D3-branes

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This is forbidden by the Freed-Witten anomaly.

$\Rightarrow$ No momentum modes along the x^4 direction !

So we see that the phase space space of a M2-brane in the R-flux background in M-theory is seven-dimensional:

$$x^1, x^2, x^3, x^4 \, ; \, p_1, p_2, p_3$$

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Missing momentum constraint in covariant terms:

$$F \equiv p_\alpha R^{\alpha,\beta\gamma\delta\rho} = 0$$

Non-associative phase space algebra of M2-brane probes in R-flux background:

$$[x^i, x^j] = \frac{\ell_s^3}{\hbar}\, R^{4,ijk4}\, p_k \ , \qquad [x^4, x^i] = \frac{\lambda\, \ell_s^3}{\hbar}\, R^{4,1234}\, p^i \ ,$$

$$[x^i, p_j] = \hbar\, \delta^i_j\, x^4 + \hbar\, \lambda\, \varepsilon^i{}_{jk}\, x^k \ , \qquad [p_i, x^4] = \hbar\, \lambda^2\, x_i \ ,$$

$$[p_i, p_j] = -\, \hbar\, \lambda\, \varepsilon_{ijk}\, p^k \ ,$$

$$[\![x^i, x^j, x^k]\!] = -\, \ell_s^3\, R^{4,ijk4}\, x^4 \ ,$$

$$[\![x^i, x^j, x^4]\!] = \lambda^2\, \ell_s^3\, R^{4,ijk4}\, x_k \ ,$$

$$[\![p_i, x^j, x^k]\!] = -\, \lambda\, \ell_s^3\, R^{4,1234}\, \big(\delta^j_i\, p^k - \delta^k_i\, p^j\big) \ ,$$

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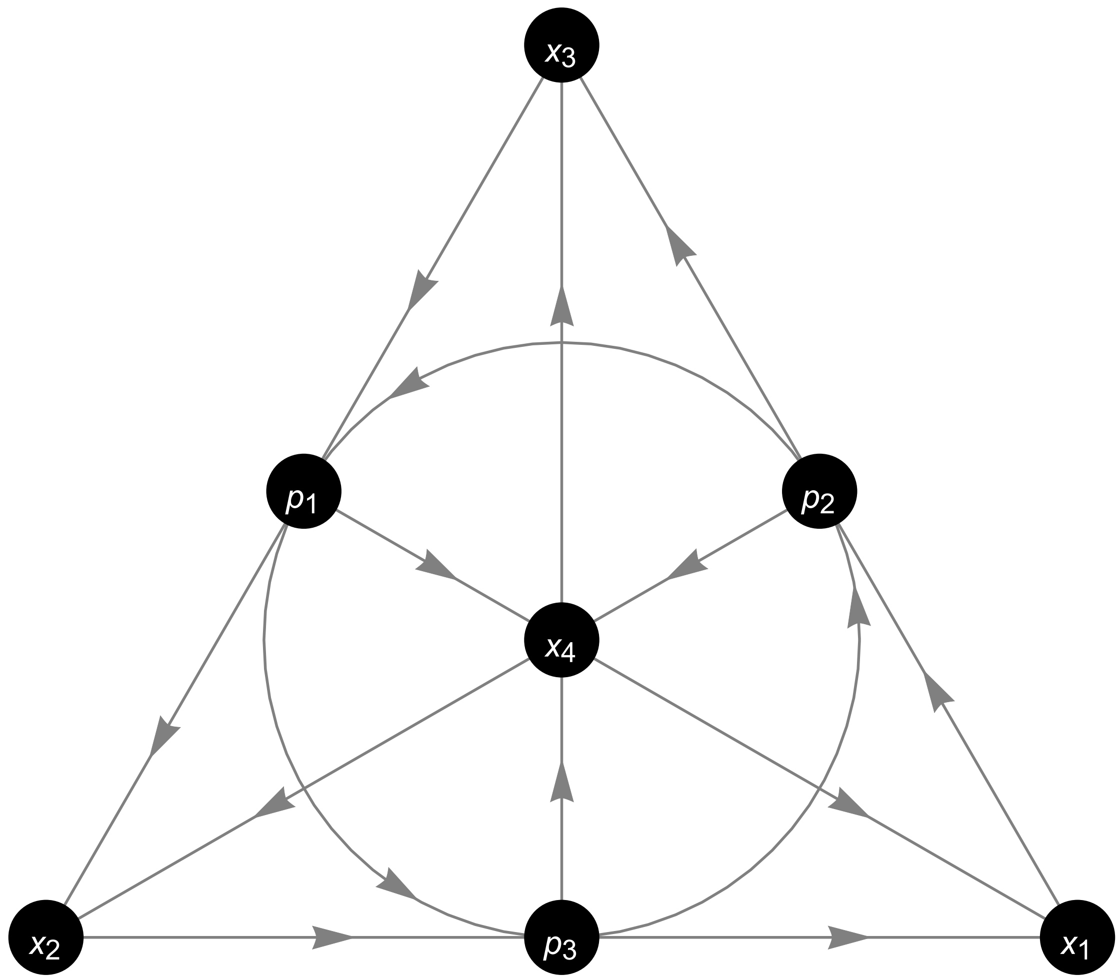
Corresponding octonionic triangle:



Natural identification of contraction parameter λ :

$$\lambda = \frac{\ell_{\mathrm{P}}}{\hbar}$$

Standard identification of M-theory parameters:

$$\ell_s^2 = \frac{\ell_{\mathrm{P}}^3}{r_{11}}\,, \qquad g_s = \left(\frac{r_{11}}{\ell_{\mathrm{P}}}\right)^{3/2}$$

(E. Witten, 1995)

Reduction to string theory

$$\ell_{\mathrm{P}} \sim r_{11}^{1/3} \longrightarrow 0\,, \qquad g_s \sim r_{11} \longrightarrow 0$$

Corresponds indeed to $\lambda \longrightarrow 0\,,$

II) Non-associative, octonionic algebra in M-theory from non-geometric Kaluza-Klein monopoles - M-wave phase space

Now we consider the phase space of an electron moving in the field of a magnetic monopole:

$$[x^i, x^j] = 0 \ , \qquad [x^i, p_j] = \hbar\, \delta^i_j \ , \qquad [p_i, p_j] = \frac{e\,\hbar}{c}\, \varepsilon_{ijk}\, B^k \ ,$$

$$[\![p_i \, , p_j \, , p_k]\!] = -\frac{e\,\hbar^2}{c}\, \varepsilon_{ijk}\, \nabla \cdot \vec{B}$$

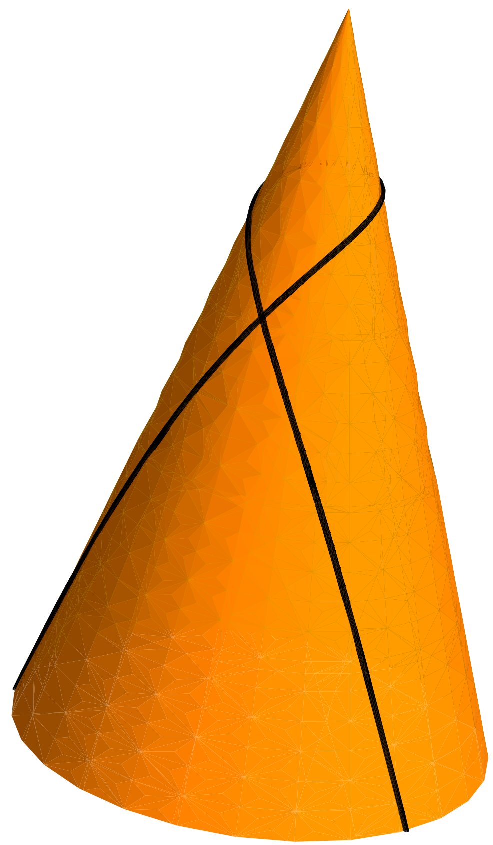
p_i : gauge invariant momenta.

R. Jackiw (1985); B. Grossman (1985); Y. Wu, A. Zee (1985); J. Mickelsson (1986), M. Günaydin, B. Zumino (1985)

B^k : magnetic field of magnetic monopole

Isolated magnetic charges: the electron avoids the non-associativity by excising these points on position space - electron never reaches the magnetic monopole:

I. Bakas, D.L. (2013)



However we like to consider a constant smeared magnetic charge density ρ : magnetic charge is uniformly distributed over space.

$\Rightarrow$ If you now wanted remove the magnetic sources from position space you would end up with empty space.

(Signal of local non-geometry just like R-flux.)

This is described by a constant magnetic charge gerbe that is realized as a family of Dirac monopole gerbes.

Phase space algebra of electron in constant magnetic charge density ρ :

$$[x^i, x^j] = 0 \ ,$$

$$[x^i, p_j] = \hbar\, \delta^i_j \ ,$$

$$[p_i, p_j] = \frac{\hbar\, e}{c}\, \rho\, \varepsilon_{ijk}\, x^k \ ,$$

$$[\![p_i, p_j, p_k]\!] = -\frac{\hbar^2\, e}{c}\, \rho\, \varepsilon_{ijk}$$

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$$[\![p_i, p_j, p_k]\!] = -\frac{\hbar^2\, e}{c}\, \rho\, \varepsilon_{ijk}$$

This algebra is isomorphic to R-flux algebra via the following (canonical) transformation:

$$x^i_R \longleftrightarrow p_{M\,i} \ , \qquad p_{R\,i} \longleftrightarrow -x^i_M \ ,$$

$$\ell_s^3 \longleftrightarrow \frac{e\,\hbar^2}{c} \ , \qquad R \longleftrightarrow \rho$$

M-theory up-lift of magnetic monopole algebra:

Conjecture: the magnetic charge algebra in M-theory is again provided by the non-associative algebra of the seven octonions.

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But now there will be four momenta and three positions!

(i) What is the M-theory background which realizes this picture?

(ii) What are the relevant M-theory probes that have this octonionic algebra as phase space?

(i) KK monopole as M-theory background:

$$ds^2 = ds^2_{M_7} + U\, d\vec{y}\cdot d\vec{y} + U^{-1}\left(dz + \vec{A}\cdot d\vec{y}\right)^2 ,$$

$$A = \vec{A}(\vec{y}\,)\cdot d\vec{y} \qquad\qquad (\vec{y}, z) \in R^3 \times S^1$$

$$F = *_3\, dU \qquad\qquad A : \text{U(1) gauge connection}$$

We need a smeared KK monopole with constant magnetic charge density.

$\Rightarrow$ No local, geometric expression for metric and A:

A in non-local, smeared form:

$$A_i(\vec{y}\,) = \frac{1}{4\pi}\int A_i^{\rm D}(\vec{y}-\vec{y}\,')\; d^3\vec{y}\,' = \frac{N}{4\pi}\,\varepsilon_{ijk}\int \frac{(y_j - y'_j)}{|\vec{y}-\vec{y}\,'|\left((y_k - y'_k) + |\vec{y}-\vec{y}\,'|\right)}\; d^3\vec{y}\,' ,$$

(ii) M-wave as M-theory probe:

We need a probe that is electrically charged under the U(1) gauge field.

The electric dual to the KK monopole NUT charge is a graviton momentum mode $\Rightarrow$ M-wave

This M-wave travels along the x^4 direction.

$\Rightarrow$ No well-defined, local position with respect to x^4.

(IIA: D6 $\leftrightarrow$ D0 , $p_4 = \dfrac{\hbar e}{r_{11}}$)

Seven-dimensional phase space

$$(x^1, x^2, x^3; p_1, p_2, p_3, p_4)$$

So we obtain as phase space algebra of a M-wave in the non-geometric KK-monopole background:

$$[x^i, x^j] = -\hbar\,\lambda\,\varepsilon^{ijk}\,x_k \ ,$$

$$[p_i, x^j] = \hbar\,\delta_i^j\,p_4 + \hbar\,\lambda\,\varepsilon_i{}^{jk}\,p_k \ , \qquad [x^i, p_4] = \hbar\,\lambda^2\,p^i \ ,$$

$$[p_i, p_j] = \frac{\hbar\,e}{c}\,\rho\,\varepsilon_{ijk}\,x^k \ , \qquad [p_4, p_i] = \frac{\hbar\,\lambda\,e}{c}\,\rho\,x_i \ ,$$

$$[x^i, x^j, x^k] = -2\,\hbar^2\lambda\,\varepsilon^{ijk}\,x^4 \ , \qquad [x^i, x^j, x^4] = \frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,x_k \ ,$$

$$[p^i, x^j, x^k] = -\frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,p_4 - \frac{\hbar^2\,\lambda}{2}\,\big(\delta^{ij}\,p^k - \delta^{ik}\,p^j\big) \ ,$$

$$[p_i, x^j, x^4] = -\frac{\hbar^2\,\lambda}{2}\,\delta_i^j\,p_4 - \frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,p_k \ ,$$

$$[p_i, p_j, x^k] = \frac{\hbar^2\,\lambda^2\,e}{2c}\,\rho\,\varepsilon_{ij}{}^k\,x^4 + \frac{\hbar^2\,\lambda\,e}{2c}\,\rho\,\big(\delta_j^k\,x_i - \delta_i^k\,x_j\big) \ ,$$

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$$[p_4, x^i, x^j] = -\frac{\hbar^2\,\lambda^3}{2}\,\varepsilon^{ijk}\,p_k \ , \qquad [p_4, x^i, x^4] = -\frac{\hbar^2\,\lambda^3}{2}\,p^i \ ,$$

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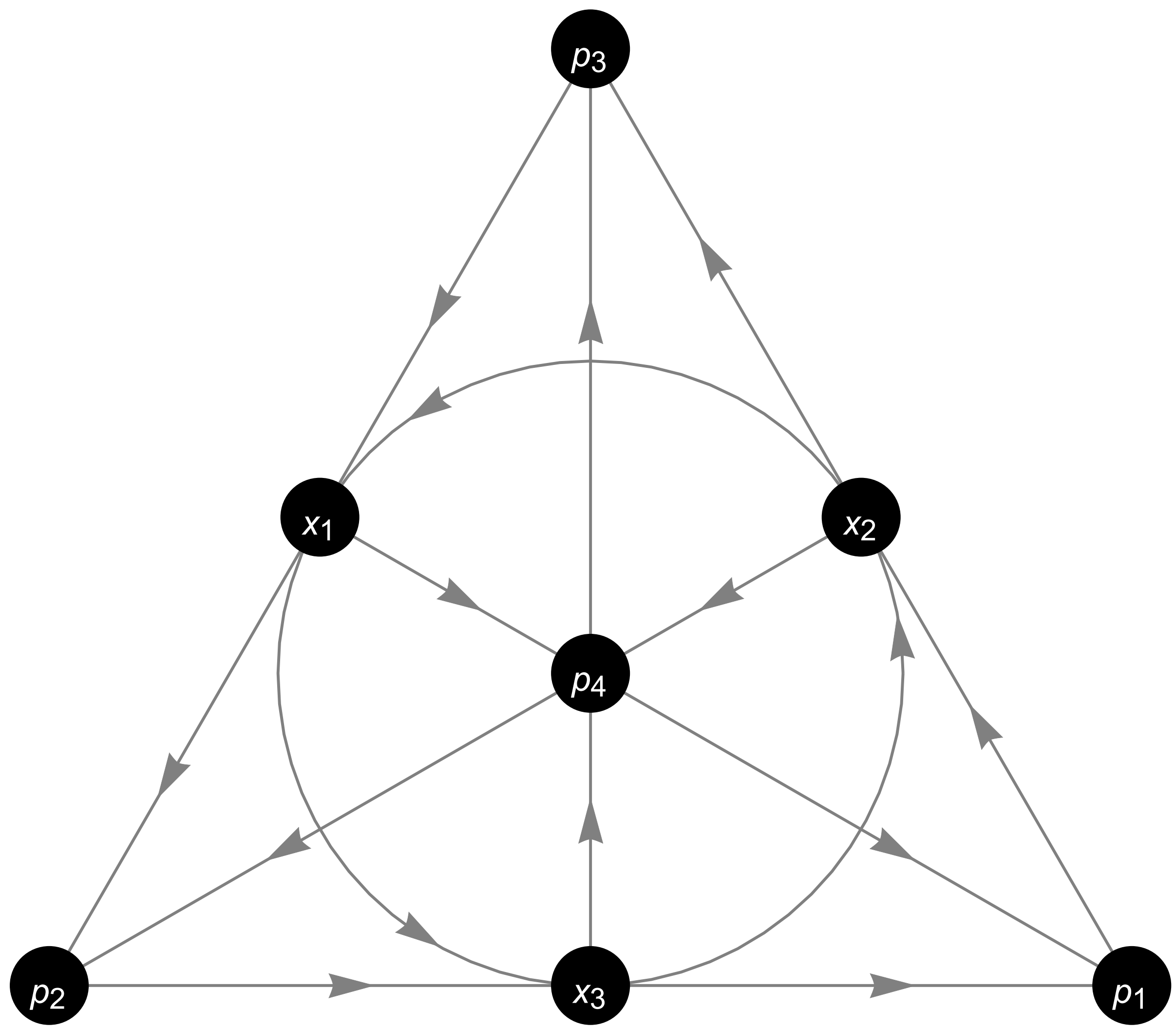
$\lambda \longrightarrow 0 \ \ (l_P, r_{11} \longrightarrow 0) \ :$ magnetic charge algebra

Corresponding octonionic triangle:



Magnetic monopoles and quantum gravity:

Restrict momenta to be on the unit sphere:

Then we obtain in the limit of $\rho \longrightarrow 0$:

$$[x^i, x^j] = -\hbar\,\lambda\,\varepsilon^{ijk}\,x_k \ ,$$

$$[x^i, q_j] = \hbar\,\sqrt{1-\lambda^2\,|\vec{q}|^2}\;\delta^i_j + \hbar\,\lambda\,\varepsilon^i{}_{jk}\,q^k \ ,$$

$$[q_i, q_j] = 0 \ .$$

L. Freidel, E. Livine (2005)

These are the commutation relations of Ponzano-Regge spin foam model of three-dimensional quantum gravity.

Octonionic magnetic algebra can be viewed as algebra of monopoles in quantum gravity.

III) Free M-theory phase space

See also: V. Kupriyanov, R.Szabo, arXiv:1701.02574

Assume no missing momentum/coordinate condition.

$p_4\,(x^4)$ corresponds to the 8th. (identity) octonion e_8 .

3-algebra: $$\left[\Xi_{\hat{A}},\Xi_{\hat{B}},\Xi_{\hat{C}}\right]=-2\,\hbar^2\,\phi_{\hat{A}\hat{B}\hat{C}\hat{D}}\,\Xi_{\hat{D}}$$

$\hat{A},\hat{B},\hat{C}=1,\dots,8$

This algebra is invariant under $Spin(7)\subset SO(8)$.

Failure of defining Nambu-Poisson bracket:

$$\begin{aligned}
[\![\Xi_{\hat{A}},\Xi_{\hat{B}},\Xi_{\hat{C}},\Xi_{\hat{D}},\Xi_{\hat{E}}]\!] &:= \tfrac{1}{12}\big(\,[[\Xi_{\hat{A}},\Xi_{\hat{B}},\Xi_{\hat{C}}],\Xi_{\hat{E}},\Xi_{\hat{D}}]+[[\Xi_{\hat{A}},\Xi_{\hat{B}},\Xi_{\hat{D}}],\Xi_{\hat{C}},\Xi_{\hat{E}}]\\
&\qquad +[[\Xi_{\hat{A}},\Xi_{\hat{B}},\Xi_{\hat{E}}],\Xi_{\hat{C}},\Xi_{\hat{D}}]+[[\Xi_{\hat{C}},\Xi_{\hat{D}},\Xi_{\hat{E}}],\Xi_{\hat{A}},\Xi_{\hat{B}}]\,\big)\\
&= \hbar^4\,\big(\,\delta_{\hat{A}\hat{C}}\,\phi_{\hat{D}\hat{E}\hat{B}\hat{F}}+\delta_{\hat{A}\hat{D}}\,\phi_{\hat{E}\hat{C}\hat{B}\hat{F}}+\delta_{\hat{A}\hat{E}}\,\phi_{\hat{C}\hat{D}\hat{B}\hat{F}}\\
&\qquad -\delta_{\hat{B}\hat{C}}\,\phi_{\hat{D}\hat{E}\hat{A}\hat{F}}-\delta_{\hat{B}\hat{D}}\,\phi_{\hat{E}\hat{C}\hat{A}\hat{F}}-\delta_{\hat{B}\hat{E}}\,\phi_{\hat{C}\hat{D}\hat{A}\hat{F}}\,\big)\,\Xi^{\hat{F}}\\
&\quad -\hbar^4\,\big(\,\phi_{\hat{B}\hat{C}\hat{D}\hat{E}}\,\Xi_{\hat{A}}-\phi_{\hat{A}\hat{C}\hat{D}\hat{E}}\,\Xi_{\hat{B}}\,\big)\ .
\end{aligned}$$

Any constraint $f(\Xi) = 0$ breaks

$$Spin(7) \longrightarrow G_2 \qquad\qquad \mathbf{8} = 7 \oplus 1$$

$p_4 = 0 \implies$ M-theory R-flux algebra.

$x^4 = 0 \implies$ M-theory monopole algebra.

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The exchange of coordinates and momenta corresponds to a Spin(7) transformation - exchange of two SU(2)'s :

$$Spin(7) \supseteq SU(2)^3 \longrightarrow SU(2)^2 \subseteq G_2$$

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Now consider the limit $R \to 0 \; (\rho \to 0)$:

Spin(7) algebra with four position and four momenta variables:

$$[p_i, p_j, x^k] = \frac{\ell_{\rm P}^2}{2}\, \varepsilon_{ij}{}^k\, x^4 + \frac{\hbar\, \ell_{\rm P}}{2}\, \big(\delta_j^k\, x_i - \delta_i^k\, x_j\big) \ , \qquad [p_i, p_j, x^4] = -\frac{\ell_{\rm P}^3}{2\hbar}\, \varepsilon_{ijk}\, x^k \ ,$$

$$[p_i, p_j, p_k] = -2\,\hbar\,\ell_{\rm P}\, \varepsilon_{ijk}\, p_4 \ , \qquad [p_4, p_i, x^j] = -\frac{\hbar\,\ell_{\rm P}}{2}\, \delta_i^j\, x^4 - \frac{\ell_{\rm P}^2}{2}\, \varepsilon_i{}^{jk}\, x_k \ ,$$

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This algebra can be considered as the free, non-associative, eight-dimensional phase space algebra of M-theory.

It becomes trivial in the limit $l_P \longrightarrow 0$.

Can one also get the full Spin(7) algebra with $R, \rho \neq 0$?

This seems to be possible in decompactification limit:

$$\frac{1}{V} R^{\mu,\nu\rho\alpha\beta} p_\mu = 0 \qquad \text{e.g.} \qquad \frac{R\, p_4}{V} = 0$$

$$V \to \infty \qquad \Rightarrow \qquad R \neq 0\,, \quad p_4 \neq 0$$

Full Spin(7) algebra:

$$[x^i, x^j, x^k] = -\frac{\ell_s^3}{2}\, R\, \varepsilon^{ijk}\, x^4 \ , \qquad [x^i, x^j, x^4] = \frac{\lambda^2\, \ell_s^3}{2}\, R\, \varepsilon^{ijk}\, x_k \ ,$$

$$[p^i, x^j, x^k] = -\frac{\lambda^2\, \ell_s^3}{2}\, R\, \varepsilon^{ijk}\, p_4 - \frac{\lambda\, \ell_s^3}{2}\, R\, \big(\delta^{ij}\, p^k - \delta^{ik}\, p^j\big) \ ,$$

$$[p_i, x^j, x^4] = -\frac{\lambda^2\, \ell_s^3}{2}\, R\, \delta_i^j\, p_4 - \frac{\lambda^2\, \ell_s^3}{2}\, R\, \varepsilon^{ijk}\, p_k \ ,$$

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V) Outlook & open questions

Non-associative algebras occur in M-theory at many places:

- Multiple M2-brane theories and 3-algebras

  J. Bagger, N. Lambert (2007)

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Generalization to higher dimensional exceptional field theory: Talk by E. Malek

(D.L., E. Malek, M. Syvari, arXiv:1710.05919)

Interesting proposal for the quantization of non-geometric M-theory background by deriving a phase space star product for the non-associative algebra of octonions.

V. Kupriyanov, R.Szabo, arXiv:1701.02574:

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Many thanks !