rgbAliceBlue,.94,.972,1;AntiqueWhite,.98,.92,.844;Aqua,0,1,1;Aquamarine,.498,1,.83;Azure,.94,1,1;Beige,.96,.96,.864;Bisque,1,.8

Membrane Sigma Models & DFT

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In this talk

- What is the world volume analog of DFT that systematically captures all fluxes?
- What is the relation between gauge symmetry and constraints od DFT?
- Can we go beyond the strong constraint?

AKSZ sigma models

AKSZ sigma model - topological sigma models satisfying the classical master equation Alexandrov, Kontsevich, A. Schwarz, Zaboronsky '97.

In 2d Poisson sigma model is most general TFT Cattaneo, Felder '01. Quantization of this model lead to Kontsevich deformation quantization formula.

In 3d the AKSZ sigma model requires a dg-manifolds for source and target, symplectic form on a target, and a self-commuting hamiltonian of degree 3.

Theorem

A QP-2 manifold is in 1:1-correspondence with a Courant algebroid. (Roytenberg '02.)

AKSZ sigma models

Given the data of a CA, $(E, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E, \rho)$, one can uniquely construct a membrane sigma model Roytenberg '06:

$$S[X,A,F] = \int_{\Sigma_3} \left(\langle F, \mathrm{d}X\rangle + \langle A, \mathrm{d}A\rangle_E - \langle F, \rho(A)\rangle + \tfrac{1}{3}\langle A, [A,A]_E\rangle_E \right)$$

For a manifold with boundaries on can add both topological and non-topological terms Cattaneo, Felder '01; Park '00

$$S_b[X,A] = \int_{\partial\Sigma_3} \tfrac{1}{2} g_{IJ} A^I \wedge *A^J + \tfrac{1}{2}\mathcal{B}_{IJ} A^I \wedge A^J \ .$$

Courant sigma model

In local coordinates

$$S[X,A,F] = \int_{\Sigma_3} F_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\eta_{IJ}A^I \wedge \mathrm{d}A^J - \rho^i{}_I(X)A^I \wedge F_i + \tfrac{1}{6}T_{IJK}(X)A^I \wedge A^J \wedge A^K$$

$i = 1, \ldots, d$ (target space index) and $I = 1, \ldots, 2d$ (CA index).

Maps $X = (X^i) : \Sigma_3 \to M$, 1-forms $A \in \Omega^1(\Sigma_3, X^*E)$, and auxiliary 2-form $F \in \Omega^2(\Sigma_3, X^*T^*M)$.
Symmetric bilinear form of the CA $\rightsquigarrow O(d,d)$ invariant metric

$$\eta = (\eta_{IJ}) = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} ,$$

ρ and T are the anchor and twist of the CA, the latter generating a generalized Wess-Zumino term.

Gauge invariance of the Courant sigma model $\Rightarrow$ CA axioms and properties.

Gauge symmetry of Courant sigma model

Take the following gauge transformations Ikeda '02

$$\delta_\epsilon X^i = \rho^i_J \epsilon^J \ ,$$
$$\delta_\epsilon A^I = \mathrm{d}\epsilon^I + \eta^{IN} T_{NJK} A^J \epsilon^K + \eta^{IJ} \rho^i{}_J t_i \ ,$$
$$\delta_\epsilon F_m = -\epsilon^J \partial_m \rho^i{}_J F_i + \tfrac{1}{2}\epsilon^J \partial_m T_{ILJ} A^I \wedge A^L + \mathrm{d}t_m + \partial_m \rho^j{}_J A^J t_j \ ,$$

where ϵ is a gauge parameter depending on world-volume coordinates and t_i is a one-form.

We introduce $DX^i = \mathrm{d}X^i - \rho^i_J A^J$ and calculate

$$\delta_\epsilon DX^i = \epsilon^J \partial_m \rho^i{}_J DX^m - \eta^{JK} \rho^i{}_J \rho^j{}_K t_j + \epsilon^J A^K (2\rho^m{}_{[K} \partial_{\underline{m}} \rho^i{}_{J]} - \rho^i{}_N \eta^{NM} T_{MKJ}) \ .$$

Remember that $DX^i = 0$ is F_i eom, so we impose

$$\eta^{JK} \rho^i{}_J \rho^j{}_K = 0 \ , \ \ 2\rho^m{}_{[K} \partial_{\underline{m}} \rho^i{}_{J]} - \rho^i{}_N \eta^{NM} T_{MKJ} = 0.$$

This gives GG fluxes.

Gauge symmetry of Courant sigma model

Using previous results we get

$$\delta_\epsilon S \quad = \quad \int_{\Sigma_3} \tfrac{1}{2}\epsilon^L(\eta^{MN} T_{MJK} T_{ILN} + \rho^m{}_I \partial_m T_{KJL} + \tfrac{1}{3}\rho^m{}_L \partial_m T_{IJK}) A^I \wedge A^J \wedge A^K \ .$$

Courant sigma model is gauge invariant provided

$$3\eta^{MN} T_{M[JK} T_{IL]N} + 4\rho^m{}_{[I} \partial_{\underline{m}} T_{KJL]} = 0.$$

Thus we obtain GG Bianchi identities, and in total three local coordinate expressions defining CA.

Gauge symmetry of pre-Courant sigma model

Alternatively, we can define

$$\widehat{S}[X,A,F] = S[X,A,F] + \int_{\Sigma_4} \tfrac{1}{4!}\mathcal{T}_{ijkl}\mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \wedge \mathrm{d}X^\lambda \ ,$$

s.t. $\mathrm{d}\mathcal{T} = 0$.
In this case the $\widehat{S}$ is gauge invariant if

$$3\eta^{MN} T_{M[JK} T_{IL]N} + 4\rho^m{}_{[I}\partial_{\underline{m}} T_{KJL]} = \mathcal{Z}_{JKIL} \ ,$$

where

$$\mathcal{Z}_{IJKL} = \tfrac{1}{2}\rho^m{}_I\rho^n{}_J\rho^p{}_K\rho^q{}_L\mathcal{T}_{mnpq} \ .$$

Remember that the condition $\mathcal{Z}_{JKIL} = 0$ implies (modified) Jacobi identity
$\rightsquigarrow$ for $\mathcal{Z}_{JKIL} \neq 0$ $\widehat{S}$ defines pre-Courant sigma model.
Vaisman '05; Hansen and Strobl '09; Liu, Sheng, Xu '12

The DFT Membrane Sigma Model

Using the data of DFT algebroid we propose (cf. Chatzistavrakidis, Jonke, Lechtenfeld '15)

$$S[\mathbb{X}, A, F] = \int \Big(F_I \wedge \mathrm{d}\mathbb{X}^I + \eta_{IJ} A^I \wedge \mathrm{d}A^J - (\rho_+)^I{}_J A^J \wedge F_I + \tfrac{1}{3} \widehat{T}_{IJK} A^I \wedge A^J \wedge A^K \Big) \ ,$$

where $\rho_+ : L_+ \to T\mathcal{M}$ is a map to the tangent bundle and $\widehat{T}$ corresponds to DFT fluxes.

Maps $\mathbb{X} = (\mathbb{X}^I) : \Sigma_3 \to \mathcal{M}$, 1-forms $A \in \Omega^1(\Sigma_3, \mathbb{X}^* L_+)$, and auxiliary 2-form $F \in \Omega^2(\Sigma_3, \mathbb{X}^* T^*\mathcal{M})$.

Symmetric bilinear form η is $O(d,d)$ invariant metric.

Universal description of flux backgrounds

Consider a doubled torus as target of the DFT MSM and DFT structural data as $(\rho_+)^I{}_J = (\rho^i{}_j, \rho^{ij}, \rho_i{}^j, \rho_{ij})$, $T_{IJK} = (H_{ijk}, f_{ij}{}^k, Q_i{}^{jk}, R^{ijk})$,

Add symmetric term on boundary $g_{IJ} = (g_{ij}, g_i{}^j, g^i{}_j, g^{ij})$,

Expand coordinate and vector components $\mathbb{X}^I = (X^i, \widetilde{X}_i)$, $A^I = (q^i, p_i)$,

Goal here is to describe the standard T-duality chain relating geometric and non-geometric fluxes schematically through Shelton, Taylor, Wecht '05

$$H_{ijk} \xleftrightarrow{\mathrm{T}_k} f_{ij}{}^k \xleftrightarrow{\mathrm{T}_j} Q_i{}^{jk} \xleftrightarrow{\mathrm{T}_i} R^{ijk} ,$$

using DFT membrane action.

H flux background

Choose

$$(\rho_+)^I{}_J = (\delta^i{}_j, 0, 0, 0) \ , \quad T_{IJK} = (H_{ijk}, 0, 0, 0) \ , \quad g_{IJ} = (0, 0, 0, g^{ij}) \ .$$

Then the membrane action becomes

$$\begin{aligned} S_{\rm DFT} &= \int_{\Sigma_3} \left(F_I \wedge \mathrm{d}\mathbb{X}^I + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - q^i \wedge F_i + \tfrac{1}{6}\, H_{ijk}\, q^i \wedge q^j \wedge q^k \right) \\ &\quad + \int_{\partial\Sigma_3} \tfrac{1}{2}\, g^{ij}\, p_i \wedge *p_j \ . \end{aligned}$$

The on-shell membrane theory $\rightsquigarrow$ integrate F_I

$$q^i = \mathrm{d}X^i \qquad \text{and} \qquad \mathrm{d}\widetilde{X}_i = 0 \ .$$

The action now takes the form

$$\int_{\partial\Sigma_3} \left(p_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\, g^{ij}\, p_i \wedge *p_j \right) + \int_{\Sigma_3} \tfrac{1}{6}\, H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ ,$$

which, after integrating out p_i using $*^2 = 1$, takes precisely the desired form

$$S_H[X] := \int_{\partial\Sigma_3} \tfrac{1}{2}\, g_{ij}\, \mathrm{d}X^i \wedge *\mathrm{d}X^j + \int_{\Sigma_3} \tfrac{1}{6}\, H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k$$

f flux background

Introduce a globally defined left-invariant (inverse) vielbein as a component of the anchor map and choose

$$(\rho_+)^M{}_J = (e^\mu{}_j, 0, 0, 0) \ , \qquad T_{IJK} = (0, f_{ij}{}^k, 0, 0) \qquad \text{and} \qquad g_{IJ} = (0, 0, 0, g^{ij}) \ ,$$

Then the membrane action becomes

$$S_{\rm DFT} \ = \ \int_{\Sigma_3} F_\mu \wedge \mathrm{d}X^\mu + \widetilde{F}^\mu \wedge \mathrm{d}\widetilde{X}_\mu + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - e^\mu{}_j \, q^j \wedge F_\mu + f_{ij}{}^k \, q^i \wedge q^j \wedge p_k$$
$$+ \int_{\partial\Sigma_3} \tfrac{1}{2} \, g^{ij} \, p_i \wedge * p_j \ .$$

The field equations for $F_M = (F_\mu, \widetilde{F}^\mu)$ yield two relations

$$q^i = e^i := e^i{}_\mu \, \mathrm{d}X^\mu \qquad \text{and} \qquad \mathrm{d}\widetilde{X}_\mu = 0 \ .$$

Using the Maurer-Cartan structure equations $\mathrm{d}e^i = -\frac{1}{2} \, f_{jk}{}^i \, e^j \wedge e^k$ we obtain

$$\int_{\partial\Sigma_3} \left(p_i \wedge e^i + \tfrac{1}{2} \, g^{ij} \, p_i \wedge * p_j \right) \ ,$$

which, after integrating out p_i, takes desired form

$$S_f[X] := \int_{\partial\Sigma_3} \tfrac{1}{2} \, g_{ij} \, e^i \wedge * e^j$$

Q flux background

To describe the globally non-geometric Q-flux frame we choose

$$(\rho_+)^I{}_J = \big(\delta^i{}_j, Q_k{}^{ij}\, X^k, -\delta_i{}^j, 0\big) \qquad \text{and} \qquad T_{IJK} = \big(0, 0, Q_i{}^{jk}, 0, 0\big)\ ,$$

We take the only non-vanishing components of the Q-flux to be $Q_3{}^{12} = Q = -Q_3{}^{21}$, and

$$g_{IJ} = (0, h_i{}^j, 0, g^{ij}) \qquad \text{with} \quad h_i{}^j = \mathrm{diag}(0,0,1) \qquad \text{and} \qquad g^{ij} = \mathrm{diag}(1,1,0)$$

With this choice the topological part of membrane action is

$$\begin{aligned} S \quad = \quad \int_{\Sigma_3} \big(& F_I \wedge \mathrm{d}\mathbb{X}^I + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - q^i \wedge F_i + p_i \wedge F^i \\ & - Q\, X^3\, p_2 \wedge F_1 + Q\, X^3\, p_1 \wedge F_2 - Q\, p_1 \wedge p_2 \wedge q^3\big)\ . \end{aligned}$$

By integrating out the auxiliary fields F_I we obtain

$$q^m = \mathrm{d}X^m - Q_3{}^{mn}\, X^3\, p_n \qquad \text{for} \quad m,n = 1,2 \qquad \text{and} \qquad q^3 = \mathrm{d}X^3\ ,$$

and $p_i = -\mathrm{d}\widetilde{X}_i \qquad$ for $\quad i = 1,2,3$.

Q flux background, continued

Using these field equations, the three-dimensional membrane action drops to the boundary, and adding the symmetric term we get

$$\int_{\partial\Sigma_3} \left(\mathrm{d}\widetilde{X}_m \wedge \mathrm{d}X^m + Q\,X^3\,\mathrm{d}\widetilde{X}_1 \wedge \mathrm{d}\widetilde{X}_2 + \tfrac{1}{2}\,\mathrm{d}X^3 \wedge *\mathrm{d}X^3 + \tfrac{1}{2}\,\mathrm{d}\widetilde{X}_m \wedge *\mathrm{d}\widetilde{X}_m\right) .$$

Global properties? cf. Freidel, Leigh, Minic '17.
Integrating out $\widetilde{X}_m$ yields

$$\mathrm{d}\widetilde{X}_m = -\frac{1}{1+(Q\,X^3)^2}\left(*\,\mathrm{d}X^m - {Q_3}^{mn}\,X^3\,\mathrm{d}X^n\right) ,$$

and the resulting action

$$S_Q[X] = \int_{\partial\Sigma_3} \tfrac{1}{2}\,\mathrm{d}X^3 \wedge *\mathrm{d}X^3 + \tfrac{1}{2(1+(Q\,X^3)^2)}\,\mathrm{d}X^m \wedge *\mathrm{d}X^m - \tfrac{Q\,X^3}{1+(Q\,X^3)^2}\,\mathrm{d}X^1 \wedge \mathrm{d}X^2$$

Q flux background, continued

The metric and B-field in the worldsheet action are only locally defined. $\rightsquigarrow$ use open-closed field redefinition

$$\tilde{g}^{-1} + \beta = (g + B)^{-1}$$

which maps the closed string metric and B-field (g, B) to the open string bivector β and globally defined metric.

Realized within DFT membrane sigma model using

$$(\rho_+)^I{}_J = (\delta^3{}_i, 0, \delta_i{}^j, 0) \ ,$$

the remaining structure maps are as above. One obtains

$$S_Q[X, \widetilde{X}\,] = \int_{\partial\Sigma_3} \tfrac{1}{2}\, \mathrm{d}X^3 \wedge *\mathrm{d}X^3 + \tfrac{1}{2}\, \mathrm{d}\widetilde{X}_m \wedge *\mathrm{d}\widetilde{X}_m + \int_{\Sigma_3} \tfrac{1}{6}\, Q_3{}^{mn}\, \mathrm{d}X^3 \wedge \mathrm{d}\widetilde{X}_m \wedge \mathrm{d}\widetilde{X}_n \ ,$$

which is dual of the sigma-model with H-flux by exchange of X and $\widetilde{X}$.

R flux and nonassociativity

We choose

$$(\rho_+)^I{}_J = \big(\delta^i{}_j, R^{ijk}\,\widetilde{X}_k, -\delta_i{}^j, 0\big) \qquad T_{IJK} = (0,0,0,R^{ijk}) \qquad g_{IJ} = (0,0,0,g^{ij})\ .$$

The topological part of the membrane action becomes

$$\begin{aligned} S \quad = \quad \int_{\Sigma_3} \big(& F_I \wedge \mathrm{d}\mathbb{X}^I + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - q^i \wedge F_i + p_i \wedge F^i \\ & - R^{ijk}\,\widetilde{X}_k\,p_j \wedge F_i + \tfrac{1}{6}\,R^{ijk}\,p_i \wedge p_j \wedge p_k \big)\ . \end{aligned}$$

Integrating out the auxiliary fields F_I gives

$$q^i = \mathrm{d}X^i - R^{ijk}\,\widetilde{X}_k\,p_j \qquad \text{and} \qquad p_i = -\mathrm{d}\widetilde{X}_i\ ,$$

leading to

$$S_R[X,\widetilde{X}\,] = \int_{\partial\Sigma_3} \big(\mathrm{d}\widetilde{X}_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\,R^{ijk}\,\widetilde{X}_k\,\mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{X}_j + \tfrac{1}{2}\,g^{ij}\,\mathrm{d}\widetilde{X}_i \wedge *\mathrm{d}\widetilde{X}_j\big)\ ,$$

Resulting action was proposed in by Mylonas, Schupp, Szabo ’12.

R flux and nonassociativity

They defined a bivector $\Theta = \frac{1}{2}\,\Theta^{IJ}\,\partial_I \wedge \partial_J$ on phase space T^*M given by

$$\Theta^{IJ} = \begin{pmatrix} R^{ijk}\,\widetilde{X}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} .$$

It induces a twisted Poisson bracket given by

$$\{\mathbb{X}^I, \mathbb{X}^J\}_\Theta = \Theta^{IJ} ,$$

with non-vanishing Jacobiator

$$\{X^i, X^j, X^k\}_\Theta := \tfrac{1}{3}\,\{\{X^i, X^j\}_\Theta, X^k\}_\Theta + \text{cyclic} = R^{ijk} .$$

Alternatively, modify the anchor to

$$(\rho_+)^I{}_J = (0, 0, \delta_i{}^j, 0) .$$

The resulting worldsheet action

$$S_R[\widetilde{X}\,] = \int_{\partial\Sigma_3} \tfrac{1}{2}\,g^{ij}\,\mathrm{d}\widetilde{X}_i \wedge *\mathrm{d}\widetilde{X}_j + \int_{\Sigma_3} \tfrac{1}{6}\,R^{ijk}\,\mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{X}_j \wedge \mathrm{d}\widetilde{X}_k ,$$

is the same as the sigma-model action with H-flux under the duality exchanges of all fields X^i with $\widetilde{X}_i$.

Summary

In terms of the doubled space of DFT, the four T-dual backgrounds with H-, f-, Q- and R-flux all correspond to the *standard Courant algebroid* over different submanifolds of the doubled space.

This does not include the noncommutative and nonassociative models, which violate the strong constraint of DFT and therefore do not correspond to Courant sigma-models.

Global properties important additional input.

Fluxes, Bianchi Identities and Gauge Invariance

Taking a parametrization of the ρ_+ components to be $(\rho_+)^I{}_J = (\delta^i{}_j, \beta^{ij}, \delta_i{}^j + \beta^{jk} B_{ki}, B_{ij})$, and considering gauge transformations of the form (here $\hat{T} = \frac{1}{2} T$.)

$$
\begin{aligned}
\delta_\epsilon \mathbb{X}^I &= \rho^I{}_J(\mathbb{X})\epsilon^J \ , \\
\delta_\epsilon A^I &= \mathrm{d}\epsilon^I + \eta^{IJ} \hat{T}_{JKL}(\mathbb{X}) A^K \epsilon^L \ , \\
\delta_\epsilon F_K &= -\epsilon^J (\partial_K \rho^I{}_J F_I - \partial_K \hat{T}_{ILJ} A^I \wedge A^L) \ ,
\end{aligned}
$$

gives the following necessary conditions for gauge invariance of the DFT MSM action:

$$
\begin{aligned}
&\eta^{IJ} \rho^K{}_I \rho^L{}_J = \eta^{KL} \\
&2\rho^L{}_{[I} \partial_{\underline{L}} \rho^K{}_{J]} - \rho^K{}_J \eta^{JL} \hat{T}_{LIJ} = \rho_{L[I} \partial^K \rho^L{}_{J]} \\
&4\rho^M{}_{[L} \partial_{\underline{M}} \hat{T}_{IJK]} + 3\eta^{MN} \hat{T}_{M[IJ} \hat{T}_{KL]N} = \mathcal{Z}_{IJKL} \ .
\end{aligned}
$$

Sufficiency also requires use of the strong constraint.

Gauge invariance continued

In particular

$$\delta_\epsilon D\mathbb{X}^I = \epsilon^J \partial_K \rho^I{}_J D\mathbb{X}^K + \rho_{KL} \partial^I \rho^K{}_M A^L \epsilon^M \ .$$

and

$$\begin{aligned} \delta_\epsilon S &= \int_{\Sigma_3} \rho_{KL} \partial^I \rho^K{}_M \epsilon^M F_I \wedge A^L + \\ &+ \epsilon^L (\eta^{MN} \hat{T}_{MJK} \hat{T}_{ILN} + \rho^M{}_I \partial_M \hat{T}_{KJL} + \tfrac{1}{3} \rho^M{}_L \partial_M \hat{T}_{IJK}) A^I \wedge A^J \wedge A^K \ . \end{aligned}$$

The last term could be absorbed in closed 4-form, as in pre-Courant example, but the first term is anomalous.

$\rightsquigarrow$ Strong constraint needed for gauge invariance of DFT sigma model.

Avoiding strong constraint?

Recall $\mathbb{A}^I_\pm = \frac{1}{2}\left(\mathbb{A}^I \pm \eta^{IJ}\,\widetilde{\mathbb{A}}_J\right)$.

What if we take η X-dependent? Freidel, Leigh, Minic '15; Freidel, Rudolph, Svoboda '17

- The twist of the C-bracket is modified:

$$[\![A,B]\!]_\eta := \mathrm{p}_+\left([\mathrm{p}_+(\mathbb{A}),\mathrm{p}_+(\mathbb{B})]_E\right) = [\![A,B]\!] + S(A,B)\ .$$

- The MSM is not modified : $\langle [\![A,A]\!]_\eta, A\rangle_{L_+} = \langle [\![A,A]\!], A\rangle_{L_+}$.
- Gauge transformation of A^I is modeified

$$\delta_\epsilon A^I = \mathrm{d}\epsilon^I + \left(\eta^{IJ}\,\hat{T}_{JKL}(\mathbb{X}) + S^I{}_{KL}(\mathbb{X})\right) A^K \epsilon^L$$

where in local coordinate form

$$S(A,B) = S^L{}_{IJ} A^I B^J e^+_L = \eta^{LK} \rho^M{}_{[I} \partial_{\underline{M}} \eta_{J]K} A^I B^J e^+_L\ .$$

Avoiding strong constraint?

The variation of $D\mathbb{X}^I$ becomes

$$\delta_\epsilon D\mathbb{X}^I = \epsilon^J \partial_K \rho^I{}_J D\mathbb{X}^K + (2\rho^K{}_{[L}\partial_K \rho^I{}_{M]} - \rho^I{}_J \eta^{JK} \hat{T}_{KLM} - \rho^I{}_J S^J{}_{LM}) A^L \epsilon^M \ .$$

$D\mathbb{X}^I$ can be exactly covariant by requiring the vanishing of the second term, which gives the relation

$$\rho^I{}_J S^J{}_{LM} = \rho_{N[L} \partial^I \rho^N{}_{M]} \ ,$$

or, equivalently

$$\rho^K{}_{[I} \partial_K \eta_{L]J} = \rho_{JK} \rho_{N[I} \partial^K \rho^N{}_{L]} \ .$$

Solutions?

Closure of gauge algebra

Closure of the gauge transformations on X:

$$\begin{aligned}(\delta_\lambda\delta_\epsilon - \delta_\epsilon\delta_\lambda)\mathbb{X}^I &= 2\rho^K{}_{[L}\partial_{\underline{K}}\rho^I{}_{J]}\lambda^L\epsilon^J = \\ &= \rho^I{}_N\left(\eta^{NS}\hat{T}_{SLJ}(\mathbb{X}) + S^N{}_{LJ}(\mathbb{X})\right)\lambda^L\epsilon^J\ .\end{aligned}$$

For gauge parameters depending only on world-sheet coordinates we have

$$[\![\lambda,\epsilon]\!]_\eta = \lambda^L\epsilon^J[\![e_L, e_J]\!]_\eta = \lambda^L\epsilon^J\left(\eta^{NS}\hat{T}_{SLJ}(\mathbb{X}) + S^N{}_{LJ}(\mathbb{X})\right)e_N\ ,$$

so we can define a new gauge parameter

$$\xi = \xi^N e_N := [\lambda,\epsilon]\ ,$$

in which case

$$(\delta_\lambda\delta_\epsilon - \delta_\epsilon\delta_\lambda)\mathbb{X}^I = \rho^I{}_N\xi^N = \delta_\xi\mathbb{X}^I\ .$$

Closure of gauge algebra - continued

Closure of the gauge transformations on A:

$$(\delta_\lambda \delta_\epsilon - \delta_\epsilon \delta_\lambda) A^I = \mathrm{d}\xi^I + C^I{}_{JK} A^K \xi^L -$$
$$-\partial_N C^I{}_{JK} \lambda^J \epsilon^K D\mathbb{X}^N + \left(-3\rho^K{}_{[N}\partial_{\underline{K}} C^I{}_{LM]} + 3C^I{}_{K[L} C^K{}_{MN]}\right) A^M \lambda^N \epsilon^L \ ,$$

where $C^I{}_{KL}(\mathbb{X}) = \eta^{IJ} \hat{T}_{JKL}(\mathbb{X}) + S^I{}_{KL}(\mathbb{X})$.

This closes only under SC. However, check $V^I = \rho^I{}_J A^J$

$$(\delta_\lambda \delta_\epsilon - \delta_\epsilon \delta_\lambda) V^I = \mathrm{d}(\rho^I{}_J \xi^J) - (2\rho^M{}_{[N}\partial_{\underline{M}}\partial_{\underline{K}}\rho^I{}_{J]} + 2\partial_M \rho^I{}_{[J}\partial_{\underline{K}}\rho^M{}_{N]}) DX^K \lambda^N \epsilon^J \ .$$

Thus, on F_I eoms the algebra of gauge transformations closes $\rightsquigarrow$ open symmetry algebra.

Conclusions

- We proposed membrane sigma model for DFT, gauge invariant under SC.
- Provides universal description of geometric and non-geometric fluxes.
- Gauge structure of the generalized membrane sigma model with $\eta(\mathbb{X})$ needs further study.
- In particular, can we realize non-associative R-flux which breaks SC of DFT?

Q flux and noncommutativity

Lüst '10

Take membrane worldvolume to be $\Sigma_3 = \Sigma_2 \times S^1$, and wrap the membrane on the target S^1

$$X^3(\sigma) = w^3\, \sigma^3$$

Dimensional reduction & integration over S^1 yields

$$S_{Q,w}[X,\widetilde{X}\,] = \int_{\Sigma_2} \big(\tfrac{1}{2}\, \mathrm{d}\widetilde{X}_m \wedge *\mathrm{d}\widetilde{X}_m + \mathrm{d}\widetilde{X}_m \wedge \mathrm{d}X^m + Q_3{}^{mn}\, w^3\, \mathrm{d}\widetilde{X}_m \wedge \mathrm{d}\widetilde{X}_n\big) \ .$$

The topological term defines a bivector inducing Poisson bracket on phase space

$$\{X^m, X^n\}_\theta = Q_3{}^{mn}\, w^3 \ , \qquad \{X^m, \widetilde{X}_n\}_\theta = \delta^m{}_n \qquad \text{and} \qquad \{\widetilde{X}_m, \widetilde{X}_n\}_\theta = 0$$

in the approach of Mylonas, Schupp, Szabo '12